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VOLUME INEQUALITIES USING SECTIONS OF CONVEX SETS

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We give a general result for the lower bound of the volume of a compact convex set K in E^d in terms of the volumes of orthogonal sections of K.

1. INTRODUCTION

Let K be a compact convex body in E^d, and let $\mathcal{H} = \{H_1, \dots, H_d\}$ be a family of d pairwise orthogonal hyperplanes in E^d. If S is an r-dimensional body in E^d, we use $|S|$ to denote the r-dimensional volume of S. In [2], Meyer gives a lower bound for the volume $|K|$ of K in terms of the volumes of the sections $K_i = K \cap H_i$ of K. He shows:

LEMMA 1.

$$[d!|K|]^{d-1} \geqslant [(d-1)!]^d \prod_{i=1}^{d} |K_i|.$$

Equality occurs for example when K is a cross polytope (generalised octahedron).

Let $l_i = \cap_{j \neq i} H_j$ $(1 \leqslant i \leqslant d)$, and for each i define $Y_i = K \cap l_i$. The following result is a weak version of Theorem 4 of [3]:

LEMMA 2.

$$d!|K| \geqslant \prod_{i=1}^{d} |Y_i|.$$

It is easy to see (and we shall show) that equality occurs here for example when K is a cross polytope.

The purpose of this note is to show that Lemmas 1 and 2 are special cases of a wider family of such inequalities involving the volumes of orthogonal sections.

By suitable choice of coordinate system, we may assume that for each i, the hyperplane H_i of $\mathcal{H}$ has equation $x_i = 0$. For $1 \leqslant r < d$, let $V_{t_1 t_2 \dots t_r}$ $(1 \leqslant t_1 < t_2 < \dots < t_d \leqslant d)$ denote the r-dimensional subspace, occurring as the intersection of the distinct hyperplanes $H_{t_1}, \dots, H_{t_r}$, and let $S_{t_1 t_2 \dots t_r} = K \cap V_{t_1 t_2 \dots t_r}$, the section of K by $V_{t_1 t_2 \dots t_r}$. These are the only types of section that will be considered. We shall prove:

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THEOREM 1. *For* $d \geqslant 2$,

$$[d!|K|]^{d-1} \geqslant (r!)^d \prod_{\text{cyclic}} |S_{12\ldots r}| \cdot \left(\prod_{i=1}^{d} |Y_i|\right)^{d-r-1}, \tag{1}$$

where the first product is the product of the d *terms obtained from the given term by cyclically permuting the subscripts* $1, 2, \ldots, d$. *Equality occurs for example when* K *is a cross polytope.*

THEOREM 2. *For* $d \geqslant 2$,

$$[d!|K|]^{d-1} \geqslant (r!)^d \prod |S_{t_1 t_2 \ldots t_r}|^{d/\binom{d}{r}} \cdot \left(\prod_{i=1}^{d} |Y_i|\right)^{d-r-1},$$

where the first product is taken over the $\binom{d}{r}$ *terms obtained by choosing* $1 \leqslant t_1 < t_2 < \ldots < t_r \leqslant d$. *Equality occurs for example when* K *is a cross polytope.*

We observe that Lemmas 1 and 2 are special cases of Theorems 1 and 2, with $r = d - 1$ and $r = 1$ respectively.

2. Proof of Theorem 1

The proof will rely on Steiner symmetrisation, so we shall need:

LEMMA 3. *Let* S *be a section of the convex body* K *formed by the intersection of* K *with some subset of* $\mathcal{H}$, *and suppose that* S, K *map to* S', K' *respectively under Steiner symmetrisation about the hyperplane* $H_j : x_j = 0$. *Then*

(a) $|K'| = |K|$;
(b) $|S'| = |S|$ *if* S *lies in a hyperplane perpendicular to* H_j;
(c) $|S'| \geqslant |S|$ *otherwise.*

The properties are well-known and simple to prove (see for example [1]).

We now symmetrise K about each of the hyperplanes $H_1, \ldots, H_d$ in turn to obtain the symmetral K^*. Suppose that section S maps to S^* under this symmetrisation. According to Lemma 3,

$$|K^*| = |K|, \; |S^*| \geqslant |S|.$$

It will thus be sufficient to establish our results for symmetrised bodies, and henceforth we assume that K is symmetric about each hyperplane of $\mathcal{H}$.

Let Q denote the closed positive orthant $\{(x_1, \ldots, x_d) : x_i \geqslant 0, \; 1 \leqslant i \leqslant d\}$. Set $k = K \cap Q$, $y_i = Y_i \cap Q$, $s_{t_1 \ldots t_r} = S_{t_1 \ldots t_r} \cap Q$. By the symmetry of K we have

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$|K| = 2^d|k|$, $|Y_i| = 2|y_i|$, and $|S_{t_1\ldots t_r}| = 2^r|s_{t_1\ldots t_r}|$. Thus to obtain inequality (1), it will be enough to show

$$(2) \qquad [d!|k|]^{d-1} \geqslant (r!)^d \prod_{\text{cyclic}} |s_{12\ldots r}| \cdot \left(\prod_{i=1}^{d} |y_i|\right)^{d-r-1}$$

We progressively build up a $(d-r-1)$-fold pyramid with basis $s_{12\ldots r}$ by taking

$$\begin{aligned} p_{r+1} &= \operatorname{conv}\{s_{12\ldots r} \cup y_{r+1}\}, \\ p_{r+2} &= \operatorname{conv}\{p_{r+1} \cup y_{r+2}\}, \\ &\ldots \\ p = p_{d-1} &= \operatorname{conv}\{p_{d-2} \cup y_{d-1}\}, \end{aligned}$$

at each step choosing a segment y_j to increase the dimension by 1. The resulting pyramid p lies in the hyperplane H_d. Now for any point M in k, the cone $C_d = \operatorname{conv}\{M, p\}$ with vertex M and basis p is contained in k. Cyclically permuting the numbers $1, 2, \ldots, d$ gives rise to d such cones, all with vertex M. Since for each j the basis of the cone C_j lies in the coordinate hyperplane H_j, the intersection $C_i \cap C_j$ of distinct cones C_i, C_j itself lies in a hyperplane — the hyperplane determined by M and the $(d-2)$-space $H_i \cap H_j$. It follows that for all $i \neq j$ we have $|C_i \cap C_j| = 0$.

Thus, for any point $M = (x_1, \ldots, x_d)$ in k we have

$$(3) \qquad |k| \geqslant |C_1| + |C_2| + \cdots + |C_d| = \sum_{\text{cyclic}} \frac{r!}{d!}(|s_{12\ldots r}||y_{r+1}| \cdots |y_{d-1}|)x_d,$$

using the same cyclic notation on the subscripts as before.

Because M is an arbitrary point of k, it follows that k is contained in the simplex L:

$$L = \Big\{(x_1, \ldots, x_d) \in Q : \sum_{\text{cyclic}} \big(|s_{12\ldots r}||y_{r+1}| \ldots |y_{d-1}|\big)x_d \leqslant \frac{d!}{r!}|k|\Big\}.$$

Since the volume of L is given by $(1/d!)$ times the product of the axial intercepts,

$$|L| = \frac{1}{d!}\left[\frac{d!|k|}{r!}\right]^d \left(\prod_{\text{cyclic}} |s_{12\ldots r}| \cdot \left[\prod_{i=1}^{d} y_i\right]^{d-r-1}\right)^{-1}$$

Substituting $|k| \leqslant |L|$ and simplifying gives

$$[d!|k|]^{d-1} \geqslant (r!)^d \prod_{\text{cyclic}} |s_{12\ldots r}| \cdot \left(\prod_{i=1}^{d} y_i\right)^{d-r-1}$$

This establishes (2), and Theorem 1 follows. Equality is obtained when k coincides with the simplex L; this occurs for example when K is a cross polytope.

3. Proof of Theorem 2

It is clear that the argument used in proving Theorem 1 is not restricted to sections whose subscripts are cyclic permutations of $(1,2,\dots,r)$. Let T denote the set of (unordered) r-tuples obtained by cyclically permuting the elements of $(1,2,\dots,d)$. Thus

$$T = \{(1,2,\dots,r),(2,3,\dots,r+1),\dots,(d,1,2,\dots,r-1)\}.$$

Geometrically, we can think of each of these r-tuples as a set of r connected vertices of a regular d-gon. It follows that the group of automorphisms of T is the dihedral group D_d, having order $2d$. Now the symmetric group S_d of order $d!$ can be partitioned into the $d!/(2d)$ left cosets of D_d. Mapping the set T using the permutations of S_d gives rise to $d!/(2d)$ sets of r-tuples, (call these *d-sets* say). In the listing of these sets the total number of r-tuples derived from T is $d!/(2d)\times d$. All possible r-tuples appear, and by symmetry, all appear the same number of times. Since there are $\binom{d}{r}$ possible distinct r-tuples, each appears $(1/2)d!\binom{d}{r}$ times.

Collecting together the sections with r-tuple subscripts lying in each of the above d-sets, and using the argument of the proof of Theorem 1 on each such collection, we obtain $d!/(2d)$ inequalities of the form (1), except that in each inequality the subscripts for the sections S form the r-tuples of a d-set. Multiplying these inequalities together, and taking the $2d/(d!)$-th power gives:

$$[d!|K|]^{d-1} \geqslant (r!)^{d-1}\left(\prod |S_{t_1t_2\dots t_d}|\right)^{d/\binom{d}{r}}\left(\prod_{i=1}^{d}|Y_i|\right)^{d-r-1},$$

where the first product is taken over the $\binom{d}{r}$ terms obtained by selecting $1\leqslant t_1 < t_2 < \dots < t_r \leqslant d$. Equality is given for example when K is a cross polytope, as for Theorem 1 .

4. Final Remarks

We conclude by observing that further (more complicated!) inequalities can be obtained using this method. Thus in expression (3), in the cyclic summation, the representative term contains the volume of the section $s_{12\dots r}$ multiplied by $|y_{r+1}|,\dots,|y_d|$, and x_d. By fixing $s_{12\dots d}$ and permuting the suffixes $r+1,\dots,d$, we obtain $d-r$ terms which can be used to contribute towards a suitable sum for estimating the volume of K. This leads to an inequality in the situation where the space E^d occurs as a direct sum of the vector subspaces $V_{t_1t_2\dots t_r}$ which determine the sections $S_{t_1t_2\dots t_r}$.

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