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A NOTE ON POSITIVE SEMIGROUPS

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A theorem of T. Ando, R. Nagel and H. Uhlig on the positivity of generators of some positive semigroups in Banach lattices can not be generalized to general ordered Banach spaces.

This note is concerned with a property, found by Ando (unpublished note) and Nagel-Uhlig [3], of an ordered Banach space B equipped with a closed and proper positive cone B_+. They have proved that, when B is a Banach lattice and $G \in L(B)$, the condition $e^{tG}(B_+) \subset B_+$ for all $t \in (0,\infty)$ implies $(G + ||G||)(B_+) \subset B_+$. An ordered Banach space B will be said to have the ANU-property if this statement is valid for all $G \in L(B)$.

A related property is found in [2]. When M is a von Neumann algebra on a Hilbert space H and there is a cyclic and separating vector $\xi_0 \in H$ for M, the condition $e^{tG}(H_+) = H_+$ for all $t \in (-\infty,\infty)$ is equivalent to the condition that $G = x + JxJ$ for some $x \in M$, where H_+ is the self-dual cone $P^\natural_{\xi_0}$ and J is the modular conjugation associated with ξ_0.

We shall first prove that, for such a von Neumann algebra, the ordered Hilbert space H has the ANU-property if and only if M is abelian. Next, we shall consider a localized form of the ANU-property

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and give a condition for an ordered Banach space to have this property.

§1. The case of von Neumann algebras.

Let M be a von Neumann algebra on a Hilbert space H and we suppose that there is a cyclic and separating vector $\xi_0 \in H$ for M. We denote tha modular operator and the modular conjugation, associated with ξ_0, by Δ and J respectively. We equip the Hilbert space H with the self-dual positive cone:

$$H_+ = P^\natural_{\xi_0} = \overline{\{ \Delta^{1/4} a\xi_0 : a \in M_+ \}}$$

(See [1], § 2.5 or [5].)

THEOREM 1. *The ordered Hilbert space H has the ANU-property if and only if M is abelian.*

Proof. We shall prove that M is abelian if H has the ANU-property; the converse is included in the theorem of Ando and Nagel-Uhlig cited above. Since $e^{t(x + JxJ)}(H_+) = H_+$ for all $t \in (-\infty, \infty)$ and $x \in M$, the ANU-property implies

$$((x + JxJ) + ||x + JxJ||)(H_+) \subset H_+ \quad \text{for all} \quad x \in M.$$

In particular, since $||(a/||a||) - 1|| \leq 1$ for every $a \in M_+$ ([1], Lemma 2.2.9), it follows from the above formula with $x = (a/||a||) - 1$ that

$$(a + JaJ)(H_+) \subset H_+ \quad \text{for all} \quad a \in M_+ .$$

Furthermore, if $a \in M_+$ and $b \in M_+$, we have

$$\Delta^{1/4}(ab + ba)\xi_0 = (\Delta^{1/4}a\Delta^{-1/4} + J\Delta^{1/4}a\Delta^{-1/4}J)\Delta^{1/4}b\xi_0 \in H_+ ,$$

and $\Delta^{1/4}a\Delta^{-1/4} \in M$ if a is analytic. Therefore, $ab + ba \in M_+$ if $a \in M_+$, $b \in M_+$ and a is analytic. Since H_+ is closed, we can conclude that $ab + ba \in M_+$ for all $a \in M_+$ and $b \in M_+$. Hence, M is abelian

§2. A general case.

Let B be an ordered Banach space equipped with a closed and proper positive cone B_+. In order to consider a localized form of the ANU-property, we need some preparatory results about the faces of B_+.

A hereditary subcone of B_+ is called a face of B_+. For $a \in B_+$, the smallest face of B_+ containing a is denoted by F_a. Therefore, $F_a = \{ x \in B_+ : x \leq \alpha a$ for some $\alpha \geq 0 \}$.

Let N be the canonical half-norm associated with B_+:

$$N(x) = inf \{ ||x + y|| : y \in B_+ \} \text{ for all } x \in B .$$

For a subset C of B_+, we have defined in [4] the N-closure C^N of C by

$$C^N = \{ x \in B_+ : N(x - x_n) \to 0 \text{ for some } x_n \in C \} .$$

When F is a face of B_+, it follows from [4], Theorem 8, that $F^N = F^{\perp\perp}$, where

$$F^{\perp\perp} = \{x \in B_+ : (f,x) = 0 \text{ if } f \geq 0 \text{ and } (f,y) = 0 \text{ for all } y \in F \} .$$

It follows immediately that F^N is a closed face of B_+. The N-closure $(F_a)^N$ of F_a will be denoted by CF_a. Obviously,

$$CF_a = \{ x \in B_+ : (f,x) = 0 \text{ if } f \geq 0 \text{ and } (f,a) = 0 \}$$

We shall call an element $a \in B_+$ central if there exists a linear operator $P_a \in L(B)$ such that $0 \leq P_a = P_a^2 \leq 1$ and $P_a(B_+) = CF_a$.

When B is a Banach lattice, it follows from [5], Corollary 10, that $CF_a = \overline{F}_a$. Therefore, if B is a σ-complete Banach lattice and its norm is order continuous, we have $P_a(B^+) = CF_a$ for the band projection P_a on $a^{\perp\perp}$. Hence, every element is central.

When B is the Hilbert space H with $H_+ = P^\natural_{\xi_0}$ considered in §1, let $a \in H_+$. Then, by [3], Theorem 4.2, there exists a projection $e \in M$ such that $\overline{F}_a = eJeJH_+ = CF_a$ and $P_a = eJeJ$ is the projection on the closed linear subspace spanned by F_a, which is positive. It follows from [2], Proposition 4.10, that $P_a \leq 1$, that is, a is central in the sense defined above, if and only if the projection e is central.

To ensure that the norm of P_a is one in general cases, we need another property. An ordered Banach space is said to have the Robinson property if

$$||u|| = \sup \{ ||u(x)|| : x \in B_+ , ||x|| \leq 1 \}$$

for every positive continuous linear operator $u : B \to B$. For more details about this property, see [6]. If the norm on B and its dual norm are absolutely monotone, then B has the Robinson property. Therefore, all Banach lattices, the hermitian parts of C^*-algebras and the preduals of W^*-algebras have this property. The fact that $||P_a|| = 1$ follows immediately from $0 \leq P_a = P_a^2 \leq 1$ when B has the Robinson property.

THEOREM 2. *Let B be an ordered Banach space with the Robinson property and let the norm be monotone. Then, if $G \in L(B)$ and $a \in B_+$ such that $e^{tG}(F_a) \subset B_+$, the following conditions are equivalent.*

1. $Ga + ||G||a \in B_+$.
2. *There exist $b \in B_+$, $c \in B_+$ and $\varepsilon \geq 0$ such that*
   $Ga + (1 + \varepsilon)||G||a = b - c$, *$c$ is central and $P_c b = 0$.*

Proof. When the condition 1 is satisfied, the condition 2 holds with $c = 0$. Conversely, suppose that the condition 2 is satisfied and $||G|| = 1 - \varepsilon$. The following proof is the same as the Nagel-Uhlig's proof for the case when B is a Banach lattice. Suppose that $c \neq 0$. Then, $P_c a \neq 0$ and

$$0 > P_c Ga + P_c a = P_c G P_c a + P_c G(1 - P_c)a + P_c a .$$

However,

$$P_c G(1 - P_c)a = \lim_{t \to 0+} t^{-1}(P_c e^{tG}(1 - P_c)a) = 0 ,$$

because $(1 - P_c)a \epsilon F_a$ and $0 \leq P_c \leq 1$. Hence,

$$0 \leq P_c a \leq -P_c G P_c a \ .$$

Since the norm is monotone, $||P_c|| = 1$ and $||G|| < 1$, we have $||P_c a|| = ||GP_c a|| < ||P_c a||$, a contradiction.

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