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REPRESENTING RANK COMPLETE CONTINUOUS RINGS

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Given a suitable regular ring R, we construct a sheaf-like representation for R as a ring of continuous sections from a completely regular space to an appropriately toplogized disjoint union of factor rings corresponding to "extremal" pseudo-rank functions. Applied to rings which are complete with respect to a rank function this representation is an isomorphism, the completely regular space is extremally disconnected and compact, and the "stalks" are the simple factor rings. These factor rings are discrete exactly if they are artinian, so the construction is not generally a sheaf. In particular, this yields an isomorphic representation for continuous geometries complete with respect to a (lattice) valuation, in terms of the simple homomorphic images.

Let R denote a (von Neumann) regular ring. A *pseudo-rank function* [**6**] N, on R is a function $N : R \rightarrow [0, 1]$, satisfying

$$N(1) = 1$$
$$N(rs) \leqq N(r), N(s)$$
$$N(e+f) = N(e) + N(f) \quad \text{if } e, f \text{ are orthogonal idempotents.}$$

As a consequence, if $aR \lesssim bR$ (for $a, b \in R$, as right R-modules) then $N(a) \leqq N(b)$; in general $N(r+s) \leqq N(r) + N(s)$, so N induces a uniform topology given by the pseudo-metric d_N, $d_N(x, y) = N(x - y)$. If $\{N\}$ is a family of pseudo-rank functions on R, we shall abuse our terminology and refer to the uniform topology with gauge $\{d_N\}$, as the topology induced by $\{N\}$.

A pseudo-rank function N is called a *rank function* if d_N is a metric, i.e. $N(r) = 0$ implies $r = 0$.

We denote by $\mathbf{P}(R)$, the collection of pseudo-rank functions on R. $\mathbf{P}(R)$ is a subset of $[0, 1]^R$. Endowed with the relative product topology (equivalently, the point-open topology) $\mathbf{P}(R)$ becomes a compact, convex subset of $[0, 1]^R$ ([**6**, Lemma 7]). We denote by $E(R)$ or E, the collection of extremal points of $\mathbf{P}(R)$. If $\mathbf{P}(R)$ is non-empty, the Krein-Milman theorem tells us $\mathbf{P}(R)$ is the closure of the convex hull of $E(R)$, so in particular $E(R)$ is non-empty.

If $N \in \mathbf{P}(R)$, then $\ker N = \{r \in R | N(r) = 0\}$ is a two-sided ideal (since $N(r+s) \leqq N(r) + N(s)$; $N(rs) \leqq N(r), N(s)$). We define R_N to be $^{R}/_{\ker N}$. Then N induces a *rank* function on R_N ([**6**, Lemma 5]), which we shall also call N.

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If R is a regular ring with a rank function N, the completion of R at N (i.e. at the metric d_N) is a regular ring ([**8**]). If $N \in \mathbf{P}(R)$, $N \in E(R)$ if and only if the completion of R_N at N is simple ([**6**, Corollary 20]).

Let us form the disjoint union $A = \bigcup_{N \in E} R_N$. Each r in R induces a function $\hat{r} : E \to A$, defined by $\hat{r}(N) = r + \ker N \in R_N$. We want to topologize A so the relative topology on R_N is the N-induced topology, so R_N becomes a topological ring in the relative topology, and so that $\hat{r}$ is continuous. To do so, we first define a (metric) topology on R.

Each r in R gives rise to a function $\tilde{r} : \mathbf{P}(R) \to [0, 1]$, $\tilde{r}(N) = N(r)$. Now $\mathbf{P}(R)$ is compact, so it seems reasonable to impose the topology of uniform convergence on R.

At this point, it is necessary to impose additional conditions on R.

1) R is unit-regular (for all $r \in R$, there exists a unit u with $rur = r$; see [**8**] for the effect this assumption has). In particular this gurantees $\mathbf{P}({}^{R}/_{M}) \neq \emptyset$ for any proper two-sided ideal M of R.
2) Given nonzero r in R, there exists $N \in \mathbf{P}(R)$ with $N(r) > 0$. This is equivalent to the map $r \to \hat{r}$ being one to one; as we are planning to represent R via ^, this assumption is necessary. All known regular rings satisfying 2) are unit regular.

Define the function $N^* : R \to [0, 1]$ ([**11**]) by

$$N^*(r) = \operatorname{Inf}\,\{m/n | n(rR) \precsim mR, n > 0\}$$

(we are using the convention that if M is a module, then nM denotes a direct sum of n copies of M; if M' is another module $M \precsim M'$ indicates M is isomorphic to a submodule of M').

By [**8**, Lemma 3.1, Theorem 3.2], for all r in R, $N^*(r) = \sup_{N \in \mathbf{P}(R)} N(r)$, and there exists $N_r \in \mathbf{P}(R)$ with $N_r(r) = N^*(r)$. It follows from the supremum formula, that $N^*(r + s) \leqq N^*(r) + N^*(s)$, $N^*(rs) \leqq N^*(r)$, $N^*(s)$, and $N^*(1) = 1$. Thus N^* is a (pseudo-) *lower rank function* ([**11**]), and it induces a pseudo-metric d^*, with respect to which R becomes a topological ring, $d^*(x, y) = N^*(x - y)$.

A collection of pseudo-rank functions on R, $U \subset \mathbf{P}(R)$ is said to be a *Hausdorff family* ([**7**]) if for all r nonzero in R, there exists $N \in U$ with $N(r) > 0$. Obviously R satisfies assumption 2) above if and only $\mathbf{P}(R)$ is itself a Hausdorff family. By [**8**, Theorem 5.1], unit regular R possesses a Hausdorff family (equivalently, 2) is satisfied) if and only if $N^*(r) > 0$ for all nonzero r in R; i.e. d^* is a metric.

LEMMA 1. *Let R be a unit regular ring, and K a subset of $\mathbf{P}(R)$ that is also a Hausdorff family.*

(a) *The topology of uniform convergence on K is the same as the topology of uniform convergence on the closure of the convex hull of K.*
(b) *If K is compact, the function $N_K{}^*$ defined by $N_K{}^*(r) = \sup_{N \in K} N(r)$ induces a metric d_K, $d_K(x, y) = N_K{}^*(x - y)$, which metrizes the topology*

of uniform convergence on K. Finally, given r in R, there exists $N \in K$ with $N(r) = N_K{}^*(r)$.

(c) *If K is both compact and convex, then if $E(K)$ is the collection of extremal points of K, $N_K{}^*(r) = \sup_{N \in E(K)} N(r)$, and if $E(K)$ is itself compact, for all $r \in R$ there exists $N \in E(K)$ with $N(r) = N_K{}^*(r)$.*

Proof. (a) is trivial.

(b) Each r in R induces $\tilde{r} : K \to [0, 1]$ continuous. So we have a collection of functions in $C(K, [0, 1])$ topologized by uniform convergence. As is well-known, this topology is the same as the sup-norm topology, which here is re-interpreted as $N_K{}^*$.

Choose r in R. The function $\tilde{r} : K \to [0, 1]$, $\tilde{r}(N) = N(r)$ is a continuous function from a compact space to $[0, 1]$. Thus the supremum is obtained, i.e. there exists $N_r \in K$ with $\tilde{r}(N_r) = \sup_{N \in K} \tilde{r}(N)$; i.e. $N_r(r) = N_K{}^*(r)$.

(c) By the Krein-Milman Theorem, K is the closed convex hull of $E(K)$. Pick r in R. There exists $N \in K$ with $N(r) = N_K{}^*(r)$. If $\sup_{N \in E(K)} N(r) \leqq N_K{}^*(r) - \epsilon$, for some $\epsilon > 0$, then for all $M \in \operatorname{cvx} E(K)$, $M(r) \leqq N_K{}^*(r) - \epsilon$. Now $V = \{M \in K | M(r) > N_k{}^*r) - \epsilon/2\}$ is a (relatively) open neighbourhood of N in K, but $V \cap \operatorname{cvx} E(K) = \emptyset$, contradicting the Krein-Milman theorem. Thus $\sup_{N \in E(K)} N(r) \geqq N_K{}^*(r)$ which by (b) gives equality. The final statement is a consequence of the final statement of (b).

COROLLARY 2. *If the regular ring R possesses a Hausdorff family of pseudo-rank functions, then the topology of uniform convergence (as functions $\mathbf{P}(R) \to [0, 1]$) is metrizable, and determined by the lower rank function N^*.*

It is clear that R is a topological ring in the N^* metric. Finally, we may put a topology on $A = \bigcup_{N \in E(R)} R_N$. There is a map

$$R\,x\,E \xrightarrow{\alpha} A$$

$$(r, N) \to r + \ker N \in R_N.$$

Put the product topology on $R \times E$, and the strong α topology on A (the strongest topology on A which allows α to be continuous). It follows automatically that for each r in R, the function $E \to A$, $N \to r + \ker N$ is continuous, and for each $N \in E(R)$, the map $R \to A$, $r \to r + \ker N$ is continuous. It is clear that the image of R under this latter map, R_N, has the N-induced metric topology as its relative topology.

Let $\Gamma = \Gamma(E, A)$ denote the *ring* of sections (i.e. continuous maps $s : E \to A$ such that $s(N) \in R_N$ for all $N \in E$). The map $R \to \Gamma$ is continuous by the topology on A. $(r(N) = r + \ker N \in R_N)$.

COROLLARY 3. *Let R be unit regular. The map $R \to \Gamma$ is one to one if and only if R possesses a Hausdorff family of pseudorank functions.*

We assume R does have a Hausdorff family, and identify R with a subring of Γ.

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Now Γ is itself a topological ring: every $N \in E(R)$ extends uniquely to Γ in the obvious manner $\gamma \rightarrow \gamma(N)$. We may define an N^*-type norm on Γ, namely

$$N^*(\gamma) = \sup_{N \in E} \gamma(N).$$

It is clear that this N^* restricted to (the image of) R is just the original N^*, so there is no ambiguity in the notation.

Of course we want to determine when $R = \Gamma$. There are two difficulties: the first is that E need not be compact, and so the representation would not be very satisfactory (however, it turns out that if R is N^*-dense in Γ, and E is totally disconnected, then E *is* compact). In this case, some sort of representation is available via the Stone-Čech compactification.

The second problem is that $E(R)$ need not be totally disconnected. There is, however, a situation where these problems are overcome.

Let R be a unit regular ring with a Hausdorff family of pseudo-rank functions. Suppose R satisfies

(*) for all $N, M \in E(R)$, $N \neq M$, there exists a central element r in R with $r \notin \ker N$, $r \in \ker M$.

This condition is satisfied, if for example, R is directly finite and self-injective, or if R is strongly regular. We will show, if R satisfies (*), then E is compact, totally disconnected, and R is N^*-dense in Γ.

LEMMA 4. (effectively in [**7**, Theorem 6.5]) *Let R be a unit regular ring. Let $B(R)$ denote the (Boolean algebra of) central idempotents of R, and $X \equiv X(R)$ the Stone space of $B(R)$. Then there is a map*

$$E(R) \xrightarrow{\beta} X(R)$$

$$N \rightarrow \ker N \cap B(R).$$

This map is continuous and onto. If E is compact, β is open.

Proof. Clearly $\ker N \cap B(R)$ is an ideal of $B(R)$ (even though the ring operations in $B(R)$ are distinct from those of R). The map $R \rightarrow R_N = {}^R/_{\ker N}$ induces a nonzero Boolean algebra map ${}^{B(R)}/_{\ker N \cap B(R)} \rightarrow B(R_N)$. Now R_N is a prime ring, so $B(R_N) = \{0, 1\}$, whence $\ker N \cap B(R)$ is maximal.

X has a basis of clopen sets of the form

$$U_e = \{x \in X | e \notin X\} \quad (e \in B(R)).$$

Then $\beta^{-1}(U_e) = \{N \in E(R) | N(e) > 0\}$ (observe that if $e \in B(R)$, then $e \rightarrow 1 \in R_N$); this is open in the point-open topology, so β is continuous.

To show β is onto, choose a maximal ideal W of $B(R)$. Then we may find a maximal ideal M of R containing WR. As R is unit regular, ${}^R/_M$ is a simple unit regular ring. By [**8**; Corollary 3.7] ${}^R/_M$ has rank functions and therefore by the

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Krein-Milman Theorem, possesses an extremal pseudo-rank function N'. Since ${}^R/_M$ is simple, N' is a rank function. Now the composition of the maps

$$R \to {}^R/_M \xrightarrow{N'} [0, 1],$$

gives a pseudo-rank function N with $\ker N = M$. As N' is extremal, it easily follows that N is extremal. Obviously $\ker N \cap B(R) = M \cap B(R) = W$. The last statement is completely trivial.

To obtain results utilizing (*), we must recall some results from [**8**]. If R is unit regular, the Grothendieck group of R, $K_0(R)$, has the structure of partially ordered abelian group, and pseudo-rank functions correspond to isotone (order-preserving) group homomorphisms to the additive group of the reals, sending $[R]$ to 1, called *functionals.*

$FP(R)$ willl denote the collection of finitely generated projective right modules. If $P \in FP(R)$, $[P]$ will denote the image of P in $K_0(R)$. $\{[P] | P \in FP(R)\}$ is the positive cone of $K_0(R)$.

A subgroup K of a partially ordered group G is *convex* if $a, b \in K$, $c \in G$, $a \leqq c \leqq b$ implies $c \in K$. K is *directed* if for all a in K, there exist b, c in the positive cone of K with $a = b - c$.

Let I be a two-sided ideal of a unit regular ring. We define G_I to be the (directed) subgroup of $K_0(R)$ having $\{\sum n_i[e_iR] | e_i = e_i^2 \in I,\ n_i > 0\}$ as positive cone. $FP(I)$ will denote the collection of finitely generated projective modules P with $P \simeq \oplus\, e_iR$, $e_i \in I$. Obviously, the image of $FP(I)$ is just the positive cone of G_I.

The map $R \to {}^R/_I$ induces a map $K_0(R) \to K_0({}^R/_I)$ as partially ordered abelian groups. We will show that ${}^{K_0(R)}/_{G_I} \simeq K_0({}^R/_I)$ (with the quotient ordering on the left term).

LEMMA 5. *Let e, f be idempotents in a unit regular ring R, and suppose for a two-sided ideal I of R, $eR \otimes {}^R/_I \simeq fR \otimes {}^R/_I$ as right ${}^R/_I$ modules. Then there exist idempotents m_1, m_2 in I with $eR \oplus m_1R \simeq fR \oplus m_2R$.*

Proof. Set $T = {}^R/_M$, and denote the image of $r \in R$ in T by $\bar{r}$. Since R is regular, $\bar{e} \cdot T \simeq eR \otimes T$. Since T is unit regular, and we have $\bar{e}T \simeq \bar{f}T$, there exists a unit u in T with $u\bar{e}u^{-1} = \bar{f}$, [**10**, Theorem 2]. The map from the units of R to the units of ${}^R/_M$ is onto, ([**12**]) so there exists invertible $x \in R$ with $\bar{x} = u$, and so $xex^{-1} - f = m \in M$. We have

$$mR + fR = mR + xex^{-1}R.$$

Thus we may find idempotents $m_i \in mR \subset I$ with

$$m_2R \oplus fR = m_1R \oplus xex^{-1}R.$$

Now $xex^{-1}R \simeq eR$, so we have the desired result.

COROLLARY 6. *Let R be unit regular, I a two-sided ideal of R, and $P, Q \in FP(R)$.*

(a) *If* $P \otimes {}^R/_I \simeq Q \otimes {}^R/_I$ *as* ${}^R/_I$*-modules, there exist* $A_1, A_2 \in FP(I)$ *such that* $P \oplus A_1 \simeq Q \oplus A_2$.

(b) *If* $P \otimes {}^R/_I \lesssim Q \otimes {}^R/_I$ *as* ${}^R/_I$*-modules, there exists* $A \in FP(I)$ *with* $P \lesssim Q \oplus A$.

Proof. P, Q may be regarded as principal right ideals of M_nR for suitably large n. As M_nR is unit-regular [**13**], and there is a natural bijection between ideals of R and those of M_nR, Lemma 5 applies, and so (a) is proved. (b) Given an idempotent $z \in T$, there exists an idempotent e in R with $\overline{eR} = zT$. Thus $FP(R) \to FP({}^R/_I)$ is onto, so there exists $P' \in FP(R)$ such that

$$(P \oplus P') \otimes {}^R/_I \simeq (P \otimes {}^R/_I) \oplus (P' \otimes {}^R/_I) \simeq Q \otimes {}^R/_I.$$

Now apply (a).

PROPOSITION 7. *Let* R *be a unit regular ring, and* I *a two-sided ideal. The map* $R \to {}^R/_I$ *induces an isomorphism (as partially ordered abelian groups)*

$$^{K_0(R)}/_{G_I} \simeq K_0({}^R/_I).$$

G_I *is a convex directed group, and the ordering on* $^{K_0(R)}/_{G_I}$ *is the quotient ordering.*

Proof. The map

$$K_0(R) \xrightarrow{\alpha} K_0({}^R/_I)$$

$$[P] - [Q] \to [P \otimes T] - [Q \otimes T]$$

is onto, isotone, and α maps the positive cone of $K_0(R)$ *onto* the positive cone of $K_0({}^R/_I)$. Obviously $G_I \subset \ker \alpha$.

We now show $G_I = \ker \alpha$, and G_I is convex. Suppose $[P] - [Q] \in \ker \alpha$. Then $P \otimes T \simeq Q \otimes T$, so there exist (by Lemma 6) A, A' in $FP(I)$ with

$$P \oplus A \simeq Q \oplus A'.$$

Thus $[P] + [A_1] = [Q] + [A_2]$, so $[P] - [Q] = [A_1] - [A_2]$. Thus $\ker \alpha \subset G_I$, so equality has been shown. Now if $[P] - [Q] \leqq [P_2] - [Q_2] \leqq [P_1] - [Q_1]$, and the left and right terms belong to $\ker \alpha$, we obtain

$$[A] - [A'] \leqq [P_2] - [Q_2] \leqq [A_1] - [A_1'],$$

the A's in $FP(I)$. Thus $A \oplus Q_2 \lesssim P_2 \oplus A'$; tensoring with T, we get $\alpha[Q_2] \leqq \alpha[P_2]$. Using the other inequality, $\alpha[P_2] \leqq \alpha[Q_2]$, so $\alpha[P_2] = \alpha[Q_2]$. Thus $[P_2] - [Q_2] \in \ker \alpha$, so $\ker \alpha$ is indeed convex, whence G_I is. Now apply [**3**; Theorem 7, p. 21] (we have: α is an 0-epimorphism), so the isomorphism holds as ordered groups (with the quotient ordering on $^{K_0(R)}/_{G_I}$).

PROPOSITION 8. *Let* R *be a unit regular ring, and* I *be a two-sided ideal of* R *generated by central elements. For an idempotent* e *in* R, *there exist functionals on* $K_0(R)$, f, g *such that*

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(a) ${}^{g}/_{G_I}, {}^{f}/_{G_I} = 0$
(*b*) $f([eR]) = \text{Inf } \{m/n | neR \cdot E \lesssim m(ER), n > 0$, *some central* $E = E^2 \notin I\}$
(c) $g([eR]) = \sup \{m/n | mR \cdot E \lesssim neRE, n > 0$, *some central* $E = E^2 \notin I\}$.

This is similar to [**7**, Theorem 6.4]; however it requires more tedious computation.

Proof. As in the proof of the Extension Lemma [**8**, Theorem 3.2, Lemma 3.1], if suffices to show that f, g can be so defined on

$$\mathbf{Z} \cdot [R] + \mathbf{Z} \cdot [e] + \mathbf{Z} \cdot [FP(I)] \subseteq K_0(R).$$

There exist functionals $\bar{f}$, $\bar{g}$ on $K_0({}^R/_I)$ such that (cf. [**8**, Lemma 4.1])

$$\bar{f}([\bar{e}T]) = \text{Inf } \{m/n | n\bar{e}T \lesssim mT, n > 0\}$$
$$\bar{g}([\bar{e}T]) = \sup \{m/n | mT \lesssim n\bar{e}T, n > 0\}.$$

We obtain functionals f_0, g_0 on $K_0(R)$ by composing

$$K_0(R) \longrightarrow {}^{K_0(R)}/_{G_I} \simeq K_0({}^R/_I) \xrightarrow{\bar{f}, \bar{g}} (\mathbf{R}, +).$$

So ${}^{f_0}/_{G_I} = 0$, and

$$f_0([eR]) = \text{Inf } \{m/n | n(eR \otimes T) \lesssim mT, n > 0\}.$$

Clearly $f_0([eR]) \leqq f([eR])$. ($E \to 1$ under $R \to {}^R/_I$). Suppose $n(eR \otimes {}^R/_I) \lesssim {}^{mR}/_I$. Then by Lemma 6, there exists $A \in FP(I)$ with $n(eR) \lesssim mR \oplus A$. Now I is generated by central idempotents, so there exists a central idempotent $1 - E \in I$ with $A \lesssim p(1 - E)R$, for suitable p. Thus $AE = (0)$; as $1 - E \in I, E \notin I$, and we have after tensoring with $E \cdot R$, $n(eR \cdot E) \lesssim mRE$. Hence $f_0([eR]) = f([eR])$. Thus such an f may be defined, and similarly with g. This completes the proof.

Now $A = \bigcup R_N$ has been topologized by the strong $R \times E \xrightarrow{\alpha} A$ topology. There is an obvious function

$$\beta : \bigcup R_N \to [0, 1]$$
$$\beta(r_N) = N(r_N) \quad \text{if } r_N \in R_N.$$

Recall that a function β to $[0, 1]$ is *upper semicontinuous* if $\beta^{-1}[0, a)$ is open for all $a \in (0, 1]$.

LEMMA 9. *The function* $\beta : A \to [0, 1]$ *defined by* $\beta(r_N) = N(r_N)$ *is upper semicontinuous.*

Proof. It suffices to show $\beta \circ \alpha : E \to [0, 1]$ is upper semi continuous. Now $(\beta\alpha)^{-1}[0, a) = \{(r, N) \in RxE | N(r) < a\} = U_E$ say. Now $R \times E$ has the relative topology from $R \times \mathbf{P}(R)$. Define

$$U = \{(r, N) \in R \times \mathbf{P}(R) | N(r) < a\}.$$

Choose $(r, N) \in U$. Then $N(r) < a - \epsilon$ for some $\epsilon > 0$. The set V defined by

$$V = \{t \in R | M(r - t) < \epsilon \text{ for all } M \in \mathbf{P}(R)\} \times \{M \in \mathbf{P}(R) | M(r) < a - \epsilon\}$$

is a product of open sets in R, $\mathbf{P}(R)$, so V is open in $R \times \mathbf{P}(R)$. Clearly $(r, N) \in V$. If $(t, M) \in V$,

$$M(t) \leqq M(r - t) + M(r) < a,$$

so $V \subset U$. Thus U is open, and as $U_E = U \cap (R \times E)$, U_E is open, so $\beta\alpha$ is upper semicontinuous. As A possesses the strong α topology, β is upper semicontinuous.

Finally, we get to the main point.

THEOREM 10. *Let R be a unit regular ring having a Hausdorff family of pseudo-rank functions. For the following conditions we have* (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d):

(a) *for all N, $M \in E(R)$, $N \neq M$, there exists r central in R with $r \in$* ker N, $r \notin$ ker M;

(b) *the map $E(R) \to X(R)$ (= the Stone space of $B(R)$) is one to one;*

(c) *the map $E(R) \to X(R)$ is a homeomorphism;*

(d) *R is N^*dense in Γ.*

Proof. Obviously (a) $\Rightarrow$ (b) (if $r \in R$ and $eR = rR$, where r is central and e is an idempotent, then e is central, and $N(r) = N(e)$ for any $N \in \mathbf{P}(R)$).

(b) $\Rightarrow$ (c). We have $E(R) \to X(R)$ is continuous and onto by Lemma 4. It suffices to show the map is open. Choose $N \in E(R)$; set $x = \ker N \cap B(R)$ $(x \in X)$. By one to oneness, N is the *unique* extremal pseudorank function whose kernel contains x. Consider $S = {}^{R}/_{\ker N}$ and $T = R/xR$; S and T are unit regular, and $T \to S$ is onto. If $M_1 \in E(T)$, then M_1 is induced by an M in $\mathbf{P}(R)$, and clearly $M \in E(R)$. If $M \neq N$, $\ker M \cap B(R) \neq \ker N \cap B(R)$; but $x \in \ker M$, so M must equal N, whence $|E(T)| = 1$. Thus, $|\mathbf{P}(T)| = 1$, so T has a unique pseudorank function N, that induced by N. By Proposition 8,

$$N(r) = \operatorname{Inf} \{m/n | nrR \cdot E \lesssim mER, n > 0, \text{ some } E = E^2 \in B(R) - x\}.$$

Now proceed as in the proof of openness in Theorem 6.5 of [**7**].

(c) $\Rightarrow$ (d). We have $E(R)$ is homeomorphic to the compact totally disconnected (Boolean) space X. Choose $s \in \Gamma$, and $\epsilon > 0$. For each $N \in E(R)$, there exists $r^N \in R$ such that $r^N(N) = s(N)(r^N(N) = r^N + \ker N$; we have identified R with its image in Γ) as $R \to R_N$ is onto. Define the function γ to be $\beta \circ (r^N - s) : E \to [0, 1]$, where β is the function of Lemma 9. As $r^N - s$ is continuous, γ is upper semi-continuous. Thus $U_N = \gamma^{-1}[0, \epsilon)$ is open, and $N \in U_N$. Now $\bigcup_{N \in E(R)} U_N$ is an open covering for E so by compactness and the partition property of totally disconnected spaces, there exist finitely many U_i, corresponding N_i, and $r^i \in R$, and $V_i \subset U_i$, V_i disjoint clopen and

$\dot{U} V_i = E(R)$. As V_i is clopen, the function $e_i : E \to A$,

$$e_i(N) = \begin{cases} 1_N & N \in V_i \\ 0_N & N \notin V_i \end{cases}$$

is continuous, and so $e_i \in \Gamma$. Because of the homeomorphism $E \leftrightarrow X$, in fact $e_i \in R$. The e_i are central idempotents. Define $t = \sum e_i r_i$. Then $t \in R$, and $(t - s)(N) < \epsilon$ for all $N \in E(R)$. Thus $N^*(t - s) < \epsilon$, so R is N^*-dense in Γ.

Observe that if E is merely totally disconnected, then $R \otimes_{C(R)} C(\Gamma)(C(\ ) =$ centre) is dense in Γ. Is $R \otimes C(\Gamma)$ regular? It suffices to show, if R is regular with centre *a* field F, and K is an overfield of F, then $R \otimes_F K$ is regular.

For self-injective (directly finite) regular rings $10(c)$ was proven in [**7**, Theorem 6.5].

LEMMA 11 [**7**, Theorem 6.4]. *Let R be direcly finite regular and self-injective. Then R satisfies condition* (a) *of Theorem* 10.

Proof. By [**10**, Corollary 7] R is unit regular. Pick $N \in E(R)$. Then $R/_{\ker N}$ has an extremal rank function, so is prime. R_N satisfies, therefore, comparability [**4**; Lemma 5], so R_N is simple and has a unique rank function [**11**, Proposition 7]. Now choose $M \neq N \in E(R)$. If $x_N = (\ker N) \cap B(R)$, $R/_{x_N R}$ is prime regular with its ideals totally ordered. In particular $R/_{x_N R}$ has exactly one maximal ideal which must therefore be ker N. As $M \neq N$, and R_N has only one pseudo-rank function, ker $M \neq$ ker N, so ker N is the *unique* maximal ideal sitting atop x_N.

COROLLARY 12. *If R is directly finite regular self-injective, then R is N^*dense in* Γ.

LEMMA 13. *Suppose R is a unit regular ring with a Hausdorff family of pseudo-rank functions, and R is contained in Γ densely. Let K be a compact subset of* $\mathbf{P}(R)$ *that is a Hausdorff family, and suppose $\bar{R}$ is the completion with the respect to the metric*

$$d_K(x, y) = \sup_{N \in K} N(x - y).$$

Then there is a unique uniformly continuous map $\Gamma \to \bar{R}$ such that

$$\begin{array}{l} R \to \Gamma \\ \downarrow \\ \bar{R} \end{array}$$

commutes.

Proof. If N is a pseudo-rank function, $N^* \geqq N$, and then $N^* \geqq N_K{}^*$. This is routine.

COROLLARY 14. *Let R be a regular ring that is complete with respect to a rank function. Then $R = \Gamma$, so R is isomorphic to the ring of continuous sections $E(R) \to \bigcup R_N$, where $E(R)$ is extremally disconnected, compact. Each R_N is a right and left self-injective simple regular ring, with its rank-metric topology.*

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Proof. R is right and left self-injective regular [**6**, Theorem 14], obviously has a Hausdorff family of pseudo-rank functions ($\{N\}$, if N is the rank function), and R is complete with respect to $K = \{N\}$, so by Lemma 13, $R \simeq \Gamma$. Now $E(R) \simeq X(R)$; as the centre of R is self-injective, it is complete, so $X(R)$ is extremally disconnected. Any simple factor ring of a self-injective ring is self-injective, and it easily follows that R_N is complete.

In [**2**], Dauns and Hoffmann showed that every biregular ring (for all r in R, there exists $e = e^2 \in B(R)$ such that $RrR = eR$) is represented as a sheaf of simple rings over a Boolean space, and every ring so represented is biregular. Here we have a representation with "stalks" simple, but the thing is *not* a sheaf: the map $\bigcup R_N \rightarrow E$, $\Pi(r_N) = N$, is not a local homeomorphism, i.e. R_N is not (generally) discrete.

If R is already biregular, and the stalks are discrete, the N^* topology is discrete, so this representation yields the biregular representation. In particular this applies to strongly regular rings.

The question arises, if R is right and left self-injective (regular), is $R = \Gamma$? It would suffice to show R is N^* complete (Corollary 12) but I have not been able to show this.

Given a regular ring R with a Hausdorff family of pseudo-rank functions, the function N^* induces a metric topology and by [**11**, Proposition 14], the completion, R^*, is regular, and of course N^* extends to R^* (same definition). There is a strong relationship between $\mathbf{P}(R)$ and $\mathbf{P}(R^*)$: they are affinely homeomorphic, and so $E(R)$ and $E(R^*)$ are homeomorphic. In particular, R^* possesses no "new" pseudo-rank functions.

PROPOSITION 15. *Let R be a regular ring with a Hausdorff family of pseudo-rank functions, let R^* denote the N^* completion of R. Then R^* is regular, and the inclusion $R \rightarrow R^*$ induces affine homeomorphisms*

$$\mathbf{P}(R) \leftarrow \mathbf{P}(R^*)$$

$$E(R) \leftarrow E(R^*).$$

Proof. R^* is regular by [**11**, Proposition 14]. The map $R \rightarrow R^*$ induces the restriction map $\mathbf{P}(R) \xleftarrow{\beta} \mathbf{P}(R^*)$. This is continuous by [**6**], and is obviously affine. If $N \in \mathbf{P}(R)$, $N \leqq N^*$, so N extends uniquely (and uniformly continuously) to a pseudorank function on $\mathbf{P}(R^*)$, say $\tilde{N}$. Thus β is onto, and by the uniqueness of the extension, β is also one to one. It remains to show β is open; but any onto, continuous map of a compact space to a Hausdorff space is open.

As the map is affine, the last statement also holds.

Now what is the relation between the stalks of R and those of R^* ? Because we need plenty of central idempotents to give a satisfactory answer, we change the problem to the case of R dense in Γ. Let $\Gamma = \Gamma(E, A)$ for a specific regular ring R (i.e. $E = E(R)$, R is unit regular and has a Hausdorff family of pseudo-rank functions). Now $A = \bigcup R_N$. Denote by $\bar{R}_N$ the completion of R_N at N,

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its extremal rank function. Thus $\bar{R}_N$ is simple self-injective. Define $A^* = \bigcup \bar{R}_N$ with the topology to be determined shortly. We will see $\Gamma^* = \Gamma(E, A^*)$ if E has a basis of clopen sets.

Let $B(\Gamma)$ denote the collection of central idempotents in Γ; these correspond to clopen sets in $E(R)$.

LEMMA 16. *With the notation and conventions of the preceding two paragraphs, if $E = E(R)$ has a basis of clopen sets, then we have: given $r, s \in R$, $P \in E(R)$ with $P(r - s) < \epsilon$, for some $\epsilon < 0$, there exist r_1, s_1 belonging to the subring of Γ generated by $B(\Gamma)$ and R such that $r_1 - r$, $s_1 - s$ belong to* ker P, *and $N^*(r_1 - s_1) < 2\epsilon$ (P, N^* having been extended to Γ in the obvious manner).*

Proof. Define $U = \{N \in E(R) | N(r - s) < \epsilon\}$. U is open in $E(R)$, so there exists a clopen K with $P \in K \subset U$. Let $e \in B(\Gamma)$ be the central idempotent corresponding to K, i.e. $e(N) = 1_N$ if $N \in K$, $e(N) = 0_N$ if $N \notin K$. Define $r_1 = er$, $s_1 = es$. If $N \notin K$, $N(r_1 - s_1) = 0$; if $N \in K$, $N(r_1 - s_1) = N(r - s) < \epsilon$, as $K \subset U$. Thus

$$N^*(r_1 - s_1) = \sup_{N \in E(R)} N(r - s) \leqq \epsilon < 2\epsilon.$$

PROPOSITION 17. *Suppose R is unit regular with a Hausdorff family of pseudo-rank functions, and suppose $E = E(R)$ has a basis of clopen sets and R is N^*-dense in Γ. Then we have:*

(a) *E, $E(\Gamma)$, $E(\Gamma^*)$ are homeomorphic naturally;*

(b) *the stalks of Γ^* are exactly $\{\bar{R}_N\}_{N \in E(R)}$;*

(c) *$\Gamma^* = \Gamma(E, A^*)$, where $A^* = \bigcup_{N \in E(R)} \bar{R}_N$;*

(d) *$E(R)$ is compact, and homeomorphic to the Stone space of $B(\Gamma^*)$.*

Proof. We already have $E(\Gamma) \simeq E(\Gamma^*)$. As $R^* = \Gamma^*$ (since R is N^* dense in Γ), $E(R) \simeq E(\Gamma^*)$ again by 16.

(b) Identify $E(R)$ with $E(\Gamma^*)$. Clearly $R_N \subset (\Gamma^*)_N$. If $\gamma + \ker N \in (\Gamma^*)_N$, some $\gamma \in \Gamma^*$, there exists r in R with $N^*(r - \gamma) < \epsilon$, so $N(r - \gamma) < \epsilon$, whence R_N is N-dense in $(\Gamma^*)_N$. By Lemma 16, however, it is clear that $(\Gamma^*)_N$ is already N-complete, whence $(\Gamma_N)^* = \bar{R}_N$.

(c) Follows from (a), (b).

(d) We show the map $E(\Gamma^*) \to X(\Gamma^*)$ described in Theorem 10 is one to one, whence, by that theorem, is a homeomorphism. Choose distinct N, M in $E(\Gamma^*)$. As $E \simeq E(\Gamma^*)$, there exist clopen disjoint neighbourhoods containing N, M respectively; then the corresponding idempotents satisfy condition (a) of Theorem 10.

The representation obtained in [**14**] for continuous rings is simply the usual Pierce sheaf; the stalks are the prime factors corresponding to the *minimal* prime ideals (in [**14**], "local" takes on the meaning of possessing a unique maximal two-sided ideal).

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