Bull. Austral. Math. Soc. 58E20
Vol. 58 (1998) [177–187]

THE HEAT FLOWS AND HARMONIC MAPS FROM COMPLETE MANIFOLDS INTO REGULAR BALLS

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We generalise the existence result for harmonic maps obtained by Hildebrandt-Kaul-Widman to the case where the domain manifold is complete noncompact.

1. Introduction

Let M and N be two Riemannian manifolds of dimension m and n. Suppose their metrics are given by $ds_M^2 = g_{ij}dx^i dx^j$ and $ds_N^2 = h_{\alpha\beta}du^\alpha du^\beta$. The energy density function of u is given by

$$e(u) = g^{ij}\frac{\partial u^\alpha}{\partial x^i}\frac{\partial u^\beta}{\partial x^j}h_{\alpha\beta} = |\nabla u|^2.$$

The total energy is defined by

$$E(u) = \int_M e(u)dx.$$

A mapping $u : M \longrightarrow N$ is called a harmonic map if it is a classical solution of the Euler-Lagrange equation of $E(u)$, which can be written as

$$\tau^\alpha\big(u(x)\big) = \Delta u^\alpha(x) + \Gamma^\alpha_{\beta\gamma}\big(u(x)\big)\frac{\partial u^\beta}{\partial x^i}\frac{\partial u^\gamma}{\partial x^j}g^{ij} = 0 \tag{1.1}$$

where $\tau(u)$ is called the tension field of u. The corresponding parabolic system with initial data $u_0(x)$, known as the the heat equation for harmonic maps, is as follows

$$\begin{cases} \dfrac{\partial u}{\partial t} = \tau(u) \\ u(x,0) = u_0(x) \end{cases} \tag{1.2}$$

If M is a compact Riemannian manifold with boundary ∂M, then we may consider the following Dirichlet problems:

$$\begin{cases} \tau(u) = 0 \\ u|_{\partial M} = h_0 \end{cases} \tag{1.3}$$

Received 15th December, 1997
The first author would like to express his gratitude to Professor W.-Y. Ding for his long time encouragement. He would also like to thank Dr.Y, Wang for his helpful discussions.

and

$$(1.4)\qquad \begin{cases} \dfrac{\partial u}{\partial t} = \tau(u) \\ u(x,0) = u_0(x) \\ u(\cdot,t)|_{\partial M} = h_0(\cdot) \end{cases}$$

where $u_0 \in C^1(M,N)$, $h_0 \in C^1(\partial M, N)$, $u_0|_{\partial M} = h_0$.

When M and N are compact without boundary and N has nonpositive sectional curvature, Eells-Sampson [**10**] proved that any C^1 map from M into N can be deformed to a harmonic map, by solving (1.2). The analogous version for compact manifolds with boundary was proved by Hamilton [**12**], who solved the equation (1.4).

Hildebrandt-Kaul-Widman [**15**] solved the equation (1.3) when $h_0(\partial M)$ is contained in a regular ball. Later Jost [**16**] reproved the result by the heat flow method. In [**1**] Avilés-Choi-Micallef considered the case where the domain manifold M is complete and simply connected with sectional curvature K_M satisfying $-b^2 \leqslant K_M \leqslant -a^2 < 0$. They proved that the Dirichlet problem at infinity for harmonic maps from M into a regular ball has a solution, by using the continuity method.

Ding-Lin [**8**] showed that if M and N are compact without boundary and the universal covering of N admits a strictly convex function with quadratic growth then the result of Eells-Sampson holds.

When M is a complete noncompact Riemannian manifold and N is a complete Riemannian manifold with nonpositive sectional curvature, Li-Tam [**22**] proved the long time existence result for the initial value problem of (1.2), provided the initial data u_0 has bounded energy density. If we impose some conditions on M and u_0 we can have the convergence of the heat flow $u(x,t)$ to a harmonic map from M into N (see [**9, 20, 22, 23**]), which generalises the result of Eells-Sampson to the noncompact case.

But, it is well-known that the conclusion of Eells-Sampson can not be generally achieved and one has to pose some assumptions on N to draw such a conclusion because the solution of (1.2) and the solution of (1.4) may blow up in finite time (see [**5, 6, 7**]).

In this paper, we assume that M is a complete noncompact Riemannian manifold.

To state our conclusions we introduce some definitions.

Definition 1.1: Let N be a Riemannian manifold and Ω a bounded open subset of N. We say Ω satisfies condition (B) if there exists a positive function $f \in C^2(\Omega)$ satisfying

$$-\nabla^2 f - fK_2(y)h \geqslant C_0(\Omega)h$$

$$0 < m_1(\Omega) \leqslant f(y) \leqslant m_2(\Omega) < \infty$$

and

$$\left|\nabla f(y)\right| \leqslant m_3(\Omega) < \infty$$

https://doi.org/10.1017/S0004972700032159 Published online by Cambridge University Press

for all $y \in \Omega$, where $K_2(y) = \sup\Big\{K(y,\pi), 0 \mid K(y,\pi)$ is the sectional curvature of a two plane $\pi \in T_yN\Big\}$, and $C_0(\Omega) > 0$.

DEFINITION 1.2: If Ω satisfies condition (B) and there is a nonnegative convex function f^* on Ω such that $\Omega = (f^*)^{-1}\big([0,r)\big)$, we call Ω a *generalised regular ball.*

EXAMPLE 1.3. Suppose N is a Riemannian manifold with nonpositive sectional curvature. If $\Omega \subset N$ is a open bounded subset and there is a point $y_0 \in N$ such that $\Omega \bigcap Cut(y_0) = \emptyset$, then Ω satisfies condition (B). $B_r(y_0)$ is a generalised regular ball if $B_r(y_0) \bigcap Cut(y_0) = \emptyset$.

EXAMPLE 1.4. Suppose N is a Riemannian manifold, $B_r(y_0)$ is a regular ball (see [**13**]), that is $Cut(y_0) \bigcap B_r(y_0) = \emptyset$, and $\sqrt{K}r < \pi/2$ where $K \geqslant 0$ is an upper bound of the sectional curvature of N on $B_r(y_0)$, then $B_r(yo)$ is a generalised regular ball.

In this paper we mainly consider the heat flows and harmonic maps from a complete noncompact Riemannian manifold into a regular ball.

For the heat flows we prove

THEOREM 1.5. *Let M be a complete noncompact Riemannian manifold. Suppose that Ω is a regular ball. Assume that $u_0 \in C^1(M, \Omega)$. Then (1.2) has a smooth solution $u(x,t)$ satisfying $u\big(B_R(x_0) \times [0,T)\big) \subset\subset \Omega$ for all $R > 0$ and $T > 0$.*

Such a solution is unique if M is compact or if there exists a point $x_1 \in M$ and a positive constant β such that $Vol\big(B_R(x_1)\big) \leqslant exp\big(\beta(1+R^2)\big)$for all $R > 0$.

For the convergence of the heat flow, we prove

THEOREM 1.6. *Let M be a complete noncompact Riemannian manifold. Let Ω be a regular ball. Assume that $u_0 \in C^1(M, \Omega)$, and $E(u_0) < \infty$. Then (1.2) has a smooth solution $u(x,t)$ satisfying that $u\big(B_R(x_0) \times [0,T)\big) \subset\subset \Omega$ for all $R > 0$ and $T > 0$ and that there exists a subsequence $t_v \longrightarrow \infty$ such that $u(x,t_v) \longrightarrow u_\infty(x)$, and $u_\infty(x)$ is a harmonic map.*

REMARK 1.7. If M is a complete noncompact Riemannian manifold which admits a compact convex exhaustion $\{\Omega_i\}$ ($\partial\Omega_i$ is convex). Then by applying the existence theorem in [**21**] instead of using Jost's result in the proof of Theorem 3.1, we can prove the condition in Theorem 1.5 and Theorem 1.6 that Ω is a regular ball can be replaced by the assumption that Ω is a generalised regular ball.

The main idea in the proof of our theorems is using the gradient estimates for the heat flow of harmonic maps derived in section 2 which can be seen as the parabolic version of Cheng's result [**3**] and Jost's result [**17**]. The estimates are of interest in their own right. And as a consequence we can derive a Liouville theorem for harmonic maps which is similar to a result of Hildebrandt-Jost-Widman [**14**] and generalise the results of Cheng [**3**], Choi [**4**] and Yu [**25**] (also see [**18**]).

In this paper C_m denotes various constants which depends only on m; similarly C denotes various universal constants.

2. Gradient Estimates

In this section we derive some gradient estimates for the heat flow of harmonic maps, which are essential in the proof of our main results. At the end of this section, we shall prove a Liouville theorem for harmonic maps.

Theorem 2.1. *Let M and N be Riemannian manifolds. Let $x_0 \in M$ and $r(x)$ be the distance function from x_0, and let $B_R(x_0) = \{x \in M \mid r(x) \leqslant R\}$. Suppose that $\Omega \subset N$ satisfies condition (B) and assume that the Ricci curvature of M on $B_R(x_0)$ is bounded from below by $-K_1 \leqslant 0$. If $u(x,t)$ is a solution of the equation (1.2) on $B_R(x_0) \times [0,T_1)$, $u\big(B_R(x_0) \times [0,T_1)\big) \subset \Omega$ and $B_R(x_0) \cap \partial M = \emptyset$ then*

$$\sup_{B_{R/2}(x_0)} \big|\nabla u(x,t)\big| \leqslant C_m \frac{m_2}{\sqrt{C_0 m_1}}\left(\sqrt{K_1} + \sqrt{\frac{K_1}{R}} + \frac{1}{R}\right) + C\frac{m_2 m_3}{m_1 C_0}\frac{1}{R} + \frac{m_2}{\sqrt{2C_0 m_1}}\frac{1}{\sqrt{t}} \tag{2.1}$$

and

$$\sup_{B_{R/2}(x_0)} \big|\nabla u(x,t)\big| \leqslant \frac{m_2}{m_1}\sup_{B_R(x_0)} \big|\nabla u(x,0)\big| + C_m \frac{m_2}{\sqrt{C_0 m_1}}\left(\sqrt{K_1} + \sqrt{\frac{K_1}{R}} + \frac{1}{R}\right) + C\frac{m_2 m_3}{C_0 m_1}\frac{1}{R} \tag{2.2}$$

for all $0 < t < T_1$.

To prove the theorem we introduce

$$\varphi_0(x,t) = \frac{|\nabla u(x,t)|^2}{f^2\big(u(x,t)\big)}$$

and estimate $\left(\Delta - \dfrac{\partial}{\partial t}\right)\varphi_0$. A straightforward computation gives

$$\nabla \varphi_0 = \frac{\nabla |\nabla u|^2}{f^2} - 2\frac{\nabla f\, |\nabla u|^2}{f^3} \tag{2.3}$$

$$\Delta\varphi_0 = \frac{\Delta |\nabla u|^2}{f^2} - 4\frac{\nabla f\, \nabla |\nabla u|^2}{f^3} - 2\frac{\Delta f\, |\nabla u|^2}{f^3} + 6\frac{|\nabla f|^2\, |\nabla u|^2}{f^4}$$

and

$$\frac{\partial \varphi_0}{\partial t} = \frac{\frac{\partial}{\partial t}|\nabla u|^2}{f^2} - 2\frac{\frac{\partial f}{\partial t}|\nabla u|^2}{f^3}.$$

We therefore have

$$\left(\Delta - \frac{\partial}{\partial t}\right)\varphi_0 = \frac{\left(\Delta - \frac{\partial}{\partial t}\right)|\nabla u|^2}{f^2} - 2\frac{\left(\Delta - \frac{\partial}{\partial t}\right) f\big(u(x,t)\big)\, |\nabla u|^2}{f^3}$$

$$-4\frac{\nabla f\, \nabla |\nabla u|^2}{f^3} + 6\frac{|\nabla f|^2\, |\nabla u|^2}{f^4}. \tag{2.4}$$

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Using the Weitzenböck formula we have

$$\begin{aligned}\left(\triangle - \frac{\partial}{\partial t}\right)\left|\nabla u(x,t)\right|^2 &= 2\left|\nabla du\right|^2 - 2\sum_{i,j}\left\langle R^N\left(du(e_i), du(e_j)\right)du(e_i), du(e_j)\right\rangle \\ &\quad +2\sum_i \left\langle du\left(Ric^M(e_i)\right), du(e_i)\right\rangle\end{aligned}$$

where $e_1, e_2, \cdots, e_n$ is a local orthonormal frame field. Computing directly [**3**], one has

$$\left(\triangle - \frac{\partial}{\partial t}\right) f\left(u(x,t)\right) = \nabla^2(f)(\nabla u, \nabla u).$$

Since Ω satisfies condition (B), by substituting (2.3) and the last two identities into (2.4) we have

$$\begin{aligned}\left(\triangle - \frac{\partial}{\partial t}\right)\varphi_0 \geqslant -2K_1\frac{\left|\nabla u\right|^2}{f^2} + 2C_0\frac{\left|\nabla u\right|^4}{f^3} + 2\frac{\left|\nabla du\right|^2}{f^2} \\ -2\frac{\nabla f\cdot\nabla\left|\nabla u\right|^2}{f^3} + 2\frac{\left|\nabla f\right|^2\left|\nabla u\right|^2}{f^4} - 2\nabla\varphi_0\cdot\frac{\nabla f}{f}.\end{aligned} \tag{2.5}$$

The Hölder's inequality implies

$$2\frac{\left|\nabla du\right|^2}{f^2} + 2\frac{\left|\nabla f\right|^2\left|\nabla u\right|^2}{f^4} \geqslant 4\frac{\left|\nabla du\right|\left|\nabla u\right|\left|\nabla f\right|}{f^3}$$

and

$$\left|\nabla\left|\nabla u\right|^2\right| \leqslant 2\left|\nabla du\right|\left|\nabla u\right|.$$

Substituting the last two inequalities into (2.5), we have

$$\left(\triangle - \frac{\partial}{\partial t}\right)\varphi_0 \geqslant 2C_0 m_1\varphi_0^2 - 2\nabla\varphi_0\cdot\frac{\nabla f}{f} - 2K_1\varphi_0. \tag{2.6}$$

Now we prove Theorem 2.1. For this purpose we introduce $F(x,t) = t\varphi_0(x,t)$. We obviously have from (2.6)

$$\left(\triangle - \frac{\partial}{\partial t}\right)F \geqslant 2m_1C_0\frac{F^2}{t} - \left(2K_1 + \frac{1}{t}\right)F - 2\nabla F\cdot\frac{\nabla f}{f}. \tag{2.7}$$

Let $\psi(r)$ be a C^2 function on $[0,\infty)$ such that

$$\psi(r) = \begin{cases} 1 & \text{if } r\in[0,1/2] \\ 0 & \text{if } r\in[1,\infty)\,, \end{cases}$$

$0 \leqslant \psi(r) \leqslant 1$, $\psi'(r) \leqslant 0$, $\psi''(r) \geqslant -C$ and $\left|\psi'(r)\right|^2/\left(\psi(r)\right) \leqslant C$ where C is an absolute constant. Let $g(x) = \psi(r(x)/R)$.

Assume that (x_1, t_1) is the point where gF achieves its maximum in $B_R(x_0) \times [0, T]$ $(0 < T < T_1)$. By using the argument of Calabi [2], we may assume $g(x)$ to be smooth at x_1. And we may also assume $(gF)(x_1, t_1) > 0$. By the maximum principle, at (x_1, t_1) we have

$$\nabla(gF) = 0, \tag{2.8}$$

$$\frac{\partial}{\partial t}(gF) \geqslant 0,$$

and

$$\triangle(gF) \leqslant 0.$$

Hence

$$\left(\triangle - \frac{\partial}{\partial t}\right) gF \leqslant 0. \tag{2.9}$$

Applying the Laplacian comparison theorem [11] we have

$$r \triangle r \leqslant C_m(1 + K_1 r).$$

So,

$$\frac{|\nabla g|^2}{g} \leqslant \frac{C}{R^2}, \quad \triangle g \geqslant -\frac{C_m}{R^2} - \frac{C_m K_1}{R}.$$

By (2.9) and the last two inequalities we have

$$0 \geqslant -\left(\frac{C_m}{R^2} + \frac{C_m K_1}{R}\right) F + 2\nabla g \nabla F + g\left(\triangle - \frac{\partial}{\partial t}\right) F. \tag{2.10}$$

Substituting (2.7) and (2.8) into (2.10) we have

$$\begin{aligned} 0 &\geqslant -\left(\frac{C_m}{R^2} + \frac{C_m K_1}{R}\right) F - 2\frac{|\nabla g|^2}{g} F + g\left(\triangle - \frac{\partial}{\partial t}\right) F \\ &\geqslant -\left(\frac{C_m}{R^2} + \frac{C_m K_1}{R}\right) F + 2m_1 C_0 \frac{1}{t_1} g F^2 \\ &\quad -\left(2K_1 + \frac{1}{t_1}\right) gF - m_3 \frac{|\nabla g|}{g^{1/2} t_1^{1/2}} (gF)^{1/2} F \end{aligned} \tag{2.11}$$

where we have used $\left|\nabla f\big(u(x,t)\big)\right| \leqslant m_3 |\nabla u|$. Multiplying through (2.11) by g and using $0 \leqslant g \leqslant 1$, we have

$$0 \geqslant 2m_1 C_0 \frac{1}{t_1}(gF)^2 - \left(\frac{C_m}{R^2} + \frac{C_m K_1}{R} + 2K_1 + \frac{1}{t_1}\right) gF - C m_3 \frac{1}{R}\left(\frac{gF}{t_1}\right)^{1/2} gF.$$

Using the quadratic formula one obtains

$$\begin{aligned}\left(\frac{gF}{t_1}\right)^{1/2} &\leqslant \frac{Cm_3}{2m_1C_0R}+\sqrt{\frac{1}{2m_1C_0}\left(2K_1+\frac{1}{t_1}+\frac{C_m}{R^2}+\frac{C_mK_1}{R}\right)}\\ &\leqslant C\frac{m_3}{m_1C_0R}+C_m\frac{1}{\sqrt{m_1C_0}}\left(\sqrt{K_1}+\sqrt{\frac{K_1}{R}+\frac{1}{R}}\right)+\frac{1}{\sqrt{t_1}}\frac{1}{\sqrt{2C_0m_1}},\\ (gF)^{1/2}(x_1,t_1) &\leqslant C_m\frac{1}{\sqrt{m_1C_0}}\left(\sqrt{K_1}+\sqrt{\frac{K_1}{R}+\frac{1}{R}}\right)\sqrt{T}C\frac{m_3}{m_1C_0}\frac{\sqrt{T}}{R}+\frac{1}{\sqrt{2m_1C_0}}.\end{aligned}$$

So

$$\begin{aligned}&\sup\left\{t^{1/2}\left|\nabla u(x,t)\right|\,\middle|(x,t)\in B_{R/2}(x_0)\times[0,T]\right\}\\ &\qquad\leqslant C_m\frac{m_2}{\sqrt{m_1C_0}}\left(\sqrt{K_1}+\sqrt{\frac{K_1}{R}+\frac{1}{R}}\right)\sqrt{T}+C\frac{m_2m_3}{m_1C_0}\frac{\sqrt{T}}{R}+\frac{m_2}{\sqrt{2m_1C_0}}\end{aligned}$$

and so

$$\sup_{B_{R/2}(x_0)} T^{1/2}\left|\nabla u(x,T)\right|\leqslant C_m\frac{m_2\sqrt{T}}{\sqrt{m_1C_0}}\left(\sqrt{K_1}+\sqrt{\frac{K_1}{R}+\frac{1}{R}}\right)+C\frac{m_2m_3}{m_1C_0}\frac{\sqrt{T}}{R}+\frac{m_2}{\sqrt{2m_1C_0}},$$

(2.1) follows.

To prove (2.2) we set $F(x,t)=\varphi_0(x,t)$. If gF achieves its maximum in $B_{R/2}(x_0)\times[0,T]$ for $0<T<T_1$ at $(x_1,0)$, then we have

$$\sup_{B_{R/2}(x_0)}\left|\nabla u(x,t)\right|\leqslant\frac{m_2}{m_1}\sup_{B_R(x_0)}\left|\nabla u(x,0)\right|. \tag{2.12}$$

If gF achieves its maximum at (x_1,t_1) $(t_1>0)$, then by an argument similar to the one used in the proof of (2.1) we have

$$\sup_{B_{R/2}(x_0)}\left|\nabla u(x,t)\right|\leqslant C_m\frac{m_2}{\sqrt{m_1C_0}}\left(\sqrt{K_1}+\sqrt{\frac{K_1}{R}+\frac{1}{R}}\right)+C\frac{m_2m_3}{m_1C_0}\frac{1}{R}, \tag{2.13}$$

(2.2) follows from (2.12) and (2.13).

If instead of considering the function of gF we consider F directly, we have the following theorems.

Theorem 2.2. *Let M be a compact Riemannian manifold, let N be a Riemannian manifold, and let $\Omega\subset N$satisfy condition (B). If $u(x,t)$ is a solution of the equation (1.2) on $M\times[0,T_1)$ and $u\big(M\times[0,T_1)\big)\subset\Omega$ then*

$$\left|\nabla u(x,t)\right|\leqslant C_m\frac{m_2\sqrt{K_1}}{\sqrt{m_1C_0}}+\frac{m_2}{\sqrt{2C_0m_1}}\frac{1}{\sqrt{t}} \tag{2.14}$$

and

$$|\nabla u(x,t)| \leqslant \frac{m_2}{m_1}\sup_M|\nabla u(x,0)| + C_m\frac{m_2\sqrt{K_1}}{\sqrt{C_0}m_1} \tag{2.15}$$

for all $(x,t) \in M \times (0,T_1)$, *where* $-K_1 \leqslant 0$ *is the lower bound of the Ricci curvature of* M.

For harmonic maps, as corollaries of Theorem 2.1 and Theorem 2.2 we have the following gradient estimates.

THEOREM 2.3. *Suppose that* M, N, $B_R(x_0)$, Ω *satisfy the hypotheses of Theorem 2.1. If* $u(x)$ *is a harmonic map from* $B_R(x_0)$ *into* Ω *then*

$$\sup_{B_{R/2}(x_0)}|\nabla u(x)| \leqslant C_m\frac{m_2}{\sqrt{C_0}m_1}\left(\sqrt{K_1}+\sqrt{\frac{K_1}{R}}+\frac{1}{R}\right)+C\frac{m_2m_3}{m_1C_0}\frac{1}{R}.$$

THEOREM 2.4. *Suppose that* M, N, Ω *satisfy the hypotheses of Theorem 2.2. If* $u(x)$ *is a harmonic map from* M *into* Ω *then*

$$\sup_M|\nabla u(x)| \leqslant C_m\frac{m_2\sqrt{K_1}}{\sqrt{C_0}m_1}.$$

By applying Theorem 2.3 we can obtain a Liouville theorem for harmonic maps.

THEOREM 2.5. *Let* M *be a complete Riemannian manifold with nonnegative Ricci curvature, and let* N *be a Riemannian manifold. Suppose that* $\Omega \subset N$ *satisfies condition (B). If* $u(x)$ *is a harmonic map from* M *into* Ω, *then* u *is constant.*

3. Heat Flows

In this section we consider the global existence of the heat flow of harmonic maps from complete noncompact manifolds into regular balls.

THEOREM 3.1. *Let* M *be a complete noncompact Riemannian manifold. Suppose that* Ω *is a regular ball. Assume* $u_0 \in C^1(M,\Omega)$. *Then (1.2) has a smooth solution* u *defined on* $M \times [0,\infty)$ *satisfying the properties*

(a) $u\big(B_R(x_0)\times[0,T)\big) \subset\subset \Omega$

(b) $\sup_{B_R(x_0)\times[0,T)}|\nabla u| < \infty$ *for all* $R > 0$ *and* $T > 0$.

Such a solution is unique if there exists a point $x_1 \in M$ *and a positive constant* β *such that* $Vol\big(B_R(x_0)\big) \leqslant exp\big(\beta(1+R^2)\big)$ *for all* $R > 0$.

PROOF: Choose a sequence of compact smooth domains Ω_i such that $\Omega_i \subset \Omega_{i+1}$ for all $i = 1,2\ldots$ and $B_i(x_0) \subset \Omega_i$ for some fixed point $x_0 \in M$. By Jost's result **[16]** we can get a solution $u_i(x,t)$ of the following equation for each i.

$$\begin{cases} \dfrac{\partial u_i}{\partial t} = \tau(u_i) \\ u_i(x,0) = u_0(x) \\ u_i(\cdot,t) = u_0(\cdot)|_{\partial\Omega_i} \text{ for all } t > 0 \end{cases}$$

satisfying $u_i\big(\Omega_i \times [0,T)\big) \subset\subset \Omega$ and $\sup\limits_{\Omega_i\times[0,T)}\big|\nabla(u_i)\big| < \infty$ for all i and $T > 0$.

For any $R > 0$, there exists an integer $i(R) > 0$ such that $B_{2R}(x_0) \subset \Omega_i$ for all $i \geqslant i(R)$. Theorem 2.1 yields

$$\sup_{B_R(x_0)\times[0,T]}\big|\nabla u_i(x,t)\big| \leqslant \frac{m_2}{m_1}\sup_{B_{2R}(x_0)}|\nabla u_0| + C_m\frac{m_2}{\sqrt{C_0 m_1}}\left(\sqrt{K_1}+\sqrt{\frac{K_1}{R}}+\frac{1}{R}\right)+C\frac{m_2 m_3}{C_0 m_1}\frac{1}{R}$$

for all $i \geqslant i(R)$ and $T > 0$, where $-K_1 \leqslant 0$ is the lower bound of the Ricci curvature of M on $B_{2R}(x_0)$.

By standard interior estimates for the equation (1.2) and the diagonal subsequence argument, there exists a subsequence u_j and $u \in C^{2,1}\big(M\times[0,\infty),\Omega\big)$ such that $u_j \longrightarrow u$ in $C^{2,1}\big(B_R(x_0)\times[0,R],\Omega\big)$for any $R > 0$. Clearly $u(x,t)$ is a solution of (1.2) satisfying (a) and (b), and $\lim\limits_{t\longrightarrow 0} u(x,t) = u_0(x)$ uniformly in $B_R(x_0)$ for any $R > 0$.

If v is another solution of (1.2) satisfying (a), then we consider the distance function $d(u,v)$. We set $\phi(x,t) = d^2(u,v)$. Clearly [**24**] $\phi(x,t)$ is smooth on $M\times[0,\infty)$ and satisfies $\left(\triangle - \dfrac{\partial}{\partial t}\right)\phi \geqslant -K_2\phi$ where $K_2 = \sup\limits_{y\in\Omega} K_2(y)$. By the maximum principle [**19**] we know that $\phi(x,t) \leqslant e^{K_2 t}\phi(x,0) = 0$ if there exists a point $x_1 \in M$ and a positive constant β such that $Vol\big(B_R(x_1)\big) \leqslant exp\big(\beta(1+R^2)\big)$ for all $R > 0$. This establishes the theorem. ∎

4. Harmonic Maps

In this section we consider the convergence of the heat flow to a harmonic map from a complete noncompact Riemannian manifold to a generalised regular ball.

Theorem 4.1. *Let M be a complete Riemannian manifold. Let Ω be a generalised regular ball. Suppose that $u_0 \in C^1(M,\Omega)$ and $E(u_0) < \infty$. If $u(x,t)$ is a solution of the equation (1.2) satisfying $u\big(B_R(x_0)\times[0,T)\big) \subset \Omega$ for all $R > 0$ and $T > 0$, then there exists a subsequence $t_v \longrightarrow \infty$ such that $u(x,t_v) \longrightarrow u_\infty(x)$ in $C^2\big(B_R(x_0),\Omega\big)$ for any $x_0 \in M, R > 0$, and $u_\infty(x)$ is a harmonic map.*

Proof: It is clear that $u(x,t)$ satisfies

$$E\big(u(\cdot,t)\big) + 2\int_0^t\int_M |u_t|^2\,dxdt \leqslant E(u_0)$$

which yields that

$$\int_0^\infty\int_M u_t^2 dxdt \leqslant E(u_0).$$

Therefore there exists a subsequence $t_v \longrightarrow \infty$ such that

$$u_t(x,t_v) \longrightarrow 0 \tag{4.1}$$

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weakly in $L^2(M)$. Theorem 2.1 yields

$$\sup_{B_R(x_0)\times[0,\infty)} |\nabla u| \leqslant C(m, R, M, \Omega, u_0). \tag{4.2}$$

By (4.2), and standard interior estimates for the equation (1.1) we may assume

$$u(x, t_v) \longrightarrow u_\infty \tag{4.3}$$

in $C^2\big(B_R(x_0), \Omega\big)$ for any $x_0 \in M, R > 0$.

(4.1) and (4.3) imply that u_∞ is a weak harmonic map. Since $u_\infty \in C^2(M, \Omega)$, u_∞ is a harmonic map. ∎

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