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# INJECTIVE AND PROJECTIVE BOOLEAN-LIKE RINGS

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**Abstract**

A Boolean-like ring $R$ is a commutative ring with unity in which $2x = 0$ and $xy(1 + x)(1 + y) = 0$ hold for all elements $x, y$ of the ring $R$. It is shown in this paper that in the category of Boolean-like rings, $R$ is injective if and only if $R$ is a complete Boolean ring and $R$ is projective if and only if $R = \{0, 1\}$.

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## Introduction

The determination of the injective and projective members of a category is usually a challenging problem and adds to the knowledge of the category. Many successful attempts have been made in this direction. For example, R. Sikorski (1948) has shown that in the category of all Boolean rings the injective objects are the complete Boolean rings. Projective and injective distributive lattices, and projective and injective Heyting Algebras were determined respectively by R. Balbes (1966) and R. Balbes and A. Horn (1970). Recently R. Cignoli (1975) has characterized injective DeMorgan and Kleene Algebras and David C. Haines (1974) has determined injectives in the category of $p$-rings. The purpose of this paper is to find out projective and injective Boolean-like rings. A. L. Foster (1946) introduced the concept of Boolean-like rings as a generalisation of Boolean rings. A Boolean-like ring is a commutative ring with unity in which $2x = 0$ and

$xy(1+x)(1+y)=0$ hold for all elements $x$, $y$ of the ring. In this paper, we show that the injective objects in the category of Boolean-like rings are complete Boolean rings and the projective objects are the 2 element Boolean ring. Thus we see that in the categories of distributive lattices, bounded distributive lattices, relatively complemented distributive lattices, Boolean algebras, Heyting algebras and Boolean-like rings, the injective objects are precisely complete Boolean rings even though the projective objects differ in the different categories. Throughout this paper, $\mathfrak{A}$ stands for the equational category with objects Boolean-like rings and with morphisms the usual ring homomorphisms and $\mathfrak{B}$ stands for the equational category of all Boolean rings with morphisms as ring homomorphisms. We follow for the various definitions of terms in the categories those given in the book 'Distributive Lattices' by R. Balbes and P. Dwinger (1974). Also, if $r$ belongs to a Boolean-like ring $R$, then $r_B$ denotes the idempotent part of $r$ and $r_N$ denotes the nilpotent part of $r$ in $R$ as in A. L. Foster (1946).

## 1

LEMMA 1.1. *$\mathfrak{B}$ is a reflective full subcategory of $\mathfrak{A}$ and the reflector $F$: $\mathfrak{A} \to \mathfrak{B}$ defined by $F(A) = A_B$ (the Boolean subring of $A$) and $F(f) = f \mid A_B$, where $f$: $A \to A'$ is a morphism in $\mathfrak{A}$, preserves monomorphisms.*

PROOF. Routine.

REMARK 1.1.1. $F$ does not in general reflect monomorphisms.

COROLLARY 1.1.2. *By Theorem* I.18.6 *of R. Balbes and P. Dwinger* (1974), *we get that the injectives in $\mathfrak{B}$, (which are complete Boolean rings) are also injectives in $\mathfrak{A}$.*

THEOREM 1.2. *The injective objects in $\mathfrak{A}$ are the complete Boolean rings.*

PROOF. Let $R$ be an injective object in $\mathfrak{A}$. Let $B_1, B_2 \in \operatorname{Ob} \mathfrak{B}$ and $B_1 \subset B_2$. Let $g$: $B_1 \to R_B$ be any homomorphism. Then $g$ can be considered as a homomorphism from $B_1$ into $R$. As $B_1, B_2 \in \operatorname{Ob} \mathfrak{A}$ and as $R$ is injective in $\mathfrak{A}$, there exists a homomorphism $h$: $B_2 \to R$ such that $h \mid B_1 = g$. But $B_2 \in \operatorname{Ob} \mathfrak{B}$ implies that $h(B_2) \subset R_B$. Hence, $h$: $B_2 \to R_B$ and $h \mid B_1 = g$. This shows that $R_B$ is injective in $\mathfrak{B}$. Since injective objects in $\mathfrak{B}$ are complete Boolean rings, we get that $R_B$ is a complete Boolean ring. Suppose there exists a nonzero nilpotent element $n$ in $R$. Let $R_1$ be the Boolean-like ring $\{0, a_1, a_2, 1\} \oplus \{0, n_1, n_2, n_1+n_2\}$ with $a_1 n_1 = n_1$, $a_2 n_2 = n_2$ and $M$ its subring $\{0, 1, n_1, n_2, n_1+n_2, 1+n_1, 1+n_2, 1+n_1+n_2\}$ (see example 4.5 of V. Swaminathan (1980)). Let $f$: $M \to R$ be the map defined by $f(0) = 0, f(1) = 1, f(n_1) = n, f(n_2) = n, f(n_1 + n_2) = 0, f(1 + n_1) = 1 + n, f(1 + n_2) = 1 + n, f(n_1 + n_2 + 1) = 1$. Then $f$ is a homomorphism of $M$ into $R$. As $R$ is injective, there exists a homomorphism $f_1$: $R_1 \to R$ such that $f_1 \mid M = f$. Now, $f_1(a_1(n_1 + n_2)) = f_1(a_1)f_1(n_1 + n_2) = f_1(a_1)f(n_1 + n_2) = f_1(a_1) \cdot 0 = 0$ and $f_1(a_1(n_1 + n_2)) = f_1(n_1) = f(n_1) = n \neq 0$, which gives a contradiction. Therefore, the nilradical of $R$ is $\{0\}$. This implies $R$ is a complete Boolean ring. Combining with Corollary 1.1.2, we get that the injective objects in $\mathcal{C}$ are the complete Boolean rings.

REMARK 1.2.1. $\mathcal{C}$ does not have enough injectives. In fact, if $R$ is any Boolean-like ring with nonzero nilradical, then there exists no injective object in $\mathcal{C}$ containing $R$.

## 2

In this section, we determine the projective objects in $\mathcal{C}$. We first prove that there are no proper epic subrings in $\mathcal{C}$ which ensures that every epimorphism in $\mathcal{C}$ is onto. For this, we require the following lemmas on extension of homomorphisms.

LEMMA 2.1. *Let $R$, $R_1$, $R_2 \in \mathrm{Ob}\,\mathcal{C}$ with $R$ a subring of $R_2$, $a \in R_2 - R$, $b \in R_1$ and $a$, $b$ are idempotents. Suppose $f$: $R \to R_1$ is a homomorphism. Then a necessary and sufficient condition in order that $f$ can be extended to a homomorphism $g$: $(R \cup \{a\})_{R_2} \to R_1$ with the property that $g(a) = b$ is that*

$$\text{(i)} \quad ra = 0 \text{ implies } f(r)b = 0 \quad \text{for all } r \in R \quad \text{and}$$

$$\text{(ii)} \quad ra = r \text{ implies } f(r)b = f(r) \quad \text{for all } r \in R.$$

NOTE. If $S$ is a subset of a Boolean-like ring $R$, then $(S)_R$ denotes the subring generated by $S$ in $R$.

PROOF. If $f$ can be extended to a homomorphism $g$: $(R \cup \{a\})_{R_2} \to R_1$ with the property that $g(a) = b$, then obviously (i) and (ii) hold. Conversely, let (i) and (ii) hold. We observe that $(R \cup \{a\})_{R_2} = \{r_1 + r_2 a \mid r_1, r_2 \in R\}$. Define $g$: $(R \cup \{a\})_{R_2} \to R_1$ by $g(r_1 + r_2 a) = f(r_1) + f(r_2)b$. $g$ is well-defined for, suppose $r_1, r_2, r_3, r_4 \in R$, $r_1 + r_2 a = r_3 + r_4 a$. Then $r_1 + r_3 = (r_2 + r_4)a$. As $a$ is an idempotent, $(r_1 + r_3)(1 + a) = 0$ which gives $r_1 + r_3 = (r_1 + r_3)a$. Hence by condition (ii) we get $f(r_1 + r_3) = f(r_1 + r_3)b$. This implies $f(r_1) + f(r_3) = (f(r_1) + f(r_3))b$. Also, $(r_1 + r_2 + r_3 + r_4)a = 0$ and hence by (i) $(f(r_1 + r_2 + r_3 + r_4))b = 0$. Therefore, $(f(r_1) + f(r_2) + f(r_3) + f(r_4))b = 0$. That is $(f(r_1) + f(r_3))b = (f(r_2) + f(r_4))b$. Hence we get $f(r_1) + f(r_3) = (f(r_2) + f(r_4))b$. That is, $f(r_1) + f(r_2)b = f(r_3) + f(r_4)b$. This gives $g(r_1 + r_2 a) = g(r_3 + r_4 a)$. Hence $g$ is well-defined. That $g$ is a homomorphism can be easily checked. Also $g \mid R = f$ and $g(a) = b$.

COROLLARY 2.1.1. *By restricting the above result to* $\mathfrak{B}$, *we get Corollary* V.2.2 *of Balbes and Dwinger* (1974).

LEMMA 2.2. *Let* $R, R_1, R_2 \in \operatorname{Ob} \mathcal{C}$ *with* $R$ *a subring of ring* $R_2$, $n \in R_2 - R$, $n_1 \in R_1$ *and* $n, n_1$ *are nilpotent elements. Suppose* $f\colon R \to R_1$ *is a homomorphism. Then a necessary and sufficient condition in order that* $f$ *can be extended to a homomorphism* $g\colon (R \cup \{n\})_{R_2} \to R_1$ *with the property that* $g(n) = n_1$ *is that* $b, n_0 \in R$, *b being idempotent and* $n_0$ *nilpotent and* $bn = n_0$ *implies* $f(b)n_1 = f(n_0)$.

PROOF. If $f$ can be extended to a homomorphism $g\colon (R \cup \{n\})_{R_2} \to R_1$ with $g(n) = n_1$, then obviously the condition holds. Conversely let the condition hold. We observe that $(R \cup \{n\})_{R_2} = \{r_1 + r_2 n \mid r_1, r_2 \in R\}$. Define $g\colon (R \cup \{n\})_{R_2} \to R_1$ by $g(r_1 + r_2 n) = f(r_1) + f(r_2)n_1$. $g$ is well-defined for, let $r_1 + r_2 n = r_3 + r_4 n$ with $r_1, r_2, r_3, r_4 \in R$. This implies $r_1 + r_3 = (r_2 + r_4)n$.That is $(r_1 + r_3)_B = 0$. Hence $(r_1)_B = (r_3)_B$. Therefore, $(f(r_1))_B = (f(r_3))_B$. Also, $(r_1)_N + (r_3)_N = ((r_2)_B + (r_4)_B)n$. Now, $(r_2)_B + (r_4)_B \in R$ and is an idempotent in $R$ and $((r_2)_B + (r_4)_B)n \in R$. Hence, by the conditon of the theorem, $f((r_2)_B + (r_4)_B)n_1 = f((r_1)_N + (r_3)_N)$. This gives $(f(r_2) + f(r_4))_B n_1 = (f(r_1))_N + (f(r_3))_N$. That is, $(f(r_1))_N + (f(r_2))_B n_1 = (f(r_3))_N + (f(r_4))_B n_1$. Adding both sides $(f(r_1))_B$ which is the same as $(f(r_3))_B$ we get that $f(r_1) + (f(r_2)_B)n_1 = f(r_3) + (f(r_4))_B n_1$. Hence $f(r_1) + f(r_2)n_1 = f(r_3) + f(r_4)n_1$. Therefore $g(r_1 + r_2 n) = g(r_3 + r_4 n)$. Thus $g$ is well-defined. It can be easily verified that $g$ is a homomorphism, $g(n) = n_1$ and $g \mid R = g$.

Combining Lemmas 2.1 and 2.2 we get the following

LEMMA 2.3. *Let* $R, R_1, R_2 \in \operatorname{Ob} \mathcal{C}$ *with* $R$ *a subring of* $R_2$, $x \in R_2 - R_1$, $y \in R_1$ *and* $y_B = 0$ *if and only if* $x_B = 0$. *Suppose* $f\colon R \to R_1$ *is a homomorphism. Then a necessary and sufficient condition that* $f$ *can be extended to a homomorphism* $g\colon (R \cup \{x\})_{R_2} \to R_1$ *with the property that* $g(x) = y$ *is that*

(i) $rx_B = 0$ *implies* $f(r)y_B = 0$ *for all* $r \in R$,

(ii) $rx_B = r$ *implies* $f(r)y_B = f(r)$ *for all* $r \in R$, *and*

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(iii) $b, n_0 \in (R \cup \{x_B\})_{R_2}$, *b being idempotent*, $n_0$ *being nilpotent and* $bx_N = n_0$ *implies* $[f(r_1) + (f(r_2))y_B]y_N = f(r_3) + f(r_4)y_B$ *where* $b = r_1 + r_2x_B$, $n_0 = r_3 + r_4x_B$ *and* $r_1, r_2, r_3, r_4 \in R$.

LEMMA 2.4. *Let* $R_1, R \in \operatorname{Ob} \mathcal{Q}$, *with* $R_1$ *a subring of* $R$. *Let* $I$ *be a maximal ideal in* $R_1$. *Then, there exists a maximal ideal* $J$ *in* $R$ *such that* $J \cap R_1 = I$. ($J$ *is called an extension of* $I$.)

PROOF. The proof easily follows from the fact that $1 \notin \langle I \rangle_R$ (Note. If $I$ is a subset of a Boolean-like ring $R$ then $\langle I \rangle_R$ denotes the ideal generated by $I$ in $R$.).

COROLLARY 2.4.1. *Let* $R_1, R \in \operatorname{Ob} \mathcal{Q}$ *with* $R_1$ *a subring of* $R$. *Then every homomorphism* $f\colon R_1 \to \{0, 1\}$ *can be extended to a homomorphism* $g\colon R \to \{0, 1\}$.

PROOF. The proof follows from Lemma 2.4 and the fact that $f$ is determined by the maximal ideal $f^{-1}(0)$.

LEMMA 2.5. *Let* $R_1, R \in \operatorname{Ob} \mathcal{Q}$ *with* $R_1$ *a subring of* $R$. *Every maximal ideal* $I$ *of* $R_1$ *has a unique extension if and only if* $R_1 \subset R_B$ *in which case* $\langle I \cup N \rangle_R$ *is the unique extension of* $I$ *where* $N$ *is the nilradical of* $R$.

PROOF. Let $R_1 \supset R_B$. Let $I$ be a maximal ideal of $R_1$. Let $J_1, J_2$ be two maximal ideals of $R$ containing $I$. Then $J_1 \cap R_B$ and $J_2 \cap R_B$ are maximal ideals in $R_B$ containing $I \cap R_B$. But $I \cap R_B$ is maximal in $R_B$ since $R_1 \supset R_B$. Hence $J_1 \cap R_B = J_2 \cap R_B = I \cap R_B$. As $J_1, J_2$ are maximal, we get that $J_1 = J_2$. Conversely, let every maximal ideal of $R_1$ have a unique extension. Let $I$ be a maximal ideal in $(R_1)_B$. We claim that $I$ has unique extension in $R_B$. For if not, let $J_1, J_2$ be two distinct maximal ideals in $R_B$ which are extensions of $I$. Then $J_1 \cap (R_1)_B = J_2 \cap (R_1)_B = I$. If $N_1$ is the nilradical of $R_1$, we can easily verify that $\langle I \cup N_1 \rangle_{R_1}$ is a maximal ideal of $R_1$ and $\langle J_1 \cup N \rangle_R$, $\langle J_2 \cup N \rangle_R$ are distinct maximal ideals of $R$ containing $\langle I \cup N_1 \rangle_{R_1}$ which contradicts the hypothesis that every maximal ideal of $R_1$ has a unique extension. Hence every maximal ideal in $(R_1)_B$ can be uniquely extended to $R_B$. From K. P. S. Bhaskara Rao and M. Baskara Rao (1979), Lemma 7.3, we get that $(R_1)_B = R_B$. Hence $R_1 \supset R_B$.

LEMMA 2.6. *Let* $R_1, R \in \operatorname{Ob} \mathcal{Q}$ *with* $R_1$ *a subring of* $R$. *Let* $I$ *be a submaximal ideal of* $R_1$ (*an ideal* $I$ *of* $R_1$ *is called submaximal provided it is covered by a maximal ideal of* $R_1$).

(i) *If* $I$ *contains the nilradical of* $R_1$ *then there exists a submaximal ideal* $J$ *of* $R$ *such that* $J \cap R_1 = I$.

(ii) *If $I$ does not contain the nilradical of $R_1$, then there exists a submaximal ideal $J$ of $R$ such that $J \cap R_1 = I$ if and only if $n \in$ nilradical of $R_1$, $n \notin I$ implies $n \notin \langle I \rangle_R$.*

(iii) *If $I$ does not contain the nilradical of $R_1$, and if $(R_1)_B = R_B$ then there exists a submaximal ideal $J$ of $R$ such that $J \cap R_1 = I$.*

PROOF. (i) Let $I$ contain the nilradical of $R_1$. Since $I$ is submaximal, it can be easily proved that $I = J_1 \cap J_2$ where $J_1$, $J_2$ are distinct maximal ideals of $R_1$. By Lemma 2.4, there exists maximal ideals $J_1'$, $J_2'$ of $R$ such that $J_1' \cap R_1 = J_1$, $J_2' \cap R_1 = J_2$. Consider $J_1' \cap J_2'$. This is a submaximal ideal of $R$. Further, $J_1' \cap J_2' \cap R_1 = J_1 \cap J_2 = I$.

(ii) $I$ is a submaximal ideal of $R_1$ and $I$ does not contain the nilradical of $R_1$. Let $n \in$ nilradical of $R_1$, $n \notin I$ imply that $n \notin \langle I \rangle_R$. Then as $I$ does not contain the nilradical of $R_1$, there exists $n$ in the nilradical of $R_1$ such that $n \notin I$. Hence $n \notin \langle I \rangle_R$. Considering $\Sigma = \{J \subset R \mid J$ is an ideal of $R$ containing $\langle I \rangle_R$ and $n \notin J\}$ and using Zorn's lemma, we get a submaximal primary ideal $J$ of $R$ such that $n \notin J$. Now $J \cap R_1 \supset \langle I \rangle_R \cap R_1 \supset I$. As $n \notin J \cap R_1$, $J \cap R_1$ is not a maximal ideal of $R_1$ and hence $I \subset J \cap R_1 \subsetneq$ a maximal ideal of $R_1$. As $I$ is submaximal not containing the nilradical of $R_1$, we get that $I$ is covered by a unique maximal ideal and hence $I = J \cap R_1$. Conversely, if $J$ is a submaximal ideal of $R$ such that $J \cap R_1 = I$, then $n \in$ nilradical of $R_1$, $n \notin I$ implies $n \notin J \cap R_1$ which gives $n \notin J$. As $J \supset I$ we get that $J \supset \langle I \rangle_R$ and hence $n \notin \langle I \rangle_R$.

(iii) Let $I$ be a submaximal ideal of $R_1$ not containing the nilradical of $R_1$. Let $(R_1)_B = R_B$. We claim that if $n \in$ nilradical of $R_1$ and $n \notin I$, then $n \notin \langle I \rangle_R$. For, if not, let $n \in \langle I \rangle_R$. Then $n = \sum_{j=1}^k r_j i_j$ where $r_j \in R$ and $i_j \in I$. That is,

$$n = \sum_{j=1}^{k} (r_j)_N (i_j)_B + \sum_{j=1}^{k} (r_j)_B (i_j)_N.$$

As $(R_1)_B = R_B$ we get that $n_1 = \sum_{j=1}^k (r_j)_B (i_j)_N \in I$. Hence $n = n_1 + \sum_{j=1}^k (r_j)_N (i_j)_B$ where $n_1 \in I$.

$$n\left(\bigvee_{j=1}^{k} (i_j)_B\right) = (n_1)\left(\bigvee_{j=1}^{k} (i_j)_B\right) + \sum_{j=1}^{k} (r_j)_B (i_j)_B.$$

As $\bigvee_{j=1}^k (i_j)_B \in I$ and $n \in R_1$ we get that $n(\bigvee_{j=1}^k (i_j)_B) \in I$. Therefore $\sum_{j=1}^k (r_j)_N (i_j)_B \in I$, which implies $n \in I$, a contradiction. Hence $n \in$ nilradical of $R$, $n \notin I$ implies $n \notin \langle I \rangle_{R_1}$. Therefore there exists a submaximal ideal of $J$ of $R$ such that $J \cap R_1 = I$.

COROLLARY 2.6.1. *If $R_1, R \in \mathrm{Ob}\,\mathfrak{B}$ and $R_1$ is a subring of $R$, then every submaximal ideal of $R_1$ can be extended to a submaximal ideal of $R$.*

COROLLARY 2.6.2. *Let* $R_1, R \in \mathrm{Ob}\,\mathcal{Q}$ *with* $R_1$ *a subring of* $R$. *Then every epimorphism* $f: R_1 \to \{0, a_1, a_2, 1\}$ *can be extended to an epimorphism* $g: R \to \{0, a_1, a_2, 1\}$ *where* $\{0, a_1, a_2, 1\}$ *is the four element Boolean ring.*

COROLLARY 2.6.3. *Let* $R_1, R \in \mathrm{Ob}\,\mathcal{Q}$ *with* $R_1$ *a subring of* $R$. *Let* $(R_1)_B = R_B$. *Then every epimorphism* $f: R_1 \to \{0, 1, n, 1+n\}$ *can be extended to an epimorphism* $g: R \to \{0, 1, n, 1+n\}$.

REMARK 2.6.4. The submaximal primary ideal $I = \{0, n_1 + n_2\}$ in the subring $R_1 = \{0, 1, n_1, n_2, n_1 + n_2, 1 + n_1, 1 + n_2, 1 + n_1 + n_2\}$ of the ring $R = \{0, a_1, a_2, 1\} \oplus \{0, n_1, n_2, n_1 + n_2\}$ with $a_1 n_1 = n_1$, $a_1 n_2 = 0$, $a_2 n_1 = 0$, $a_2 n_2 = n_2$ has no extension to a submaximal ideal. Hence the homomorphism $f: R_1 \to \{0, 1, n, 1+n\}$ defined by $f(0) = 0$, $f(1) = 1$, $f(n_1) = f(n_2) = n$, $f(1 + n_1) = f(1 + n_2) = 1 + n$, $f(n_1 + n_2) = 0$, $f(1 + n_1 + n_2) = 1$ has no extension to an epimorphism from $R$ to $\{0, 1, n, 1+n\}$.

LEMMA 2.7. $\mathcal{Q}$ *does not have the congruence extension property* (*see definition* I.7.2 *of R. Balbes and P. Dwinger* (1974)). *But the full subcategory* $\mathcal{Q}_1$, *of all Boolean-like rings whose nonzero nilpotent elements are all atoms has the congruence extension property.*

(Note. If $R$ is a Boolean-like ring and if $n$ is a nonzero nilpotent element of $R$, then $n$ is called an atom if $bn = 0$ or $n$ for any $b \in R_B$.)

PROOF. Consider the subring $R_1 = \{0, 1, n_1, n_2, n_1 + n_2, 1 + n_1, 1 + n_2, 1 + n_1 + n_2\}$ of the Boolean-like ring $R = \{0, a_1, a_2, 1\} \oplus \{0, n_1, n_2, n_1 + n_2\}$ with $a_1 n_1 = n_1$, $a_1 n_2 = 0$. Then $\{0, n_1 + n_2\}$ is an ideal in $R_1$ and there exists no ideal $I$ in $R$ such that $I \cap R_1 = \{0, n_1 + n_2\}$.

Let $R$ be a Boolean-like ring such that every nonzero nilpotent of $R$ is an atom. Let $R_1$ be a subring of $R$ and $I$ an ideal of $R_1$. Then $I \cap (R_1)_B$ is an ideal of $R_B$. Let $I_1 = I \cap (R_1)_B$. Then $\langle I_1 \rangle_{R_B} \cap (R_1)_B = I_1$. If $J = \langle I_1 \rangle_{R_B}$, then it can be easily verified that $\langle J \cup I_N \rangle_R \cap R_1 = I$ where $I_N$ is the set of all nilpotent elements of $I$. Thus, there exists an ideal $J$ of $R$ such that $J \cap R_1 = I$.

Suppose $R$ contains a nonzero nilpotent $n$ which is not an atom. Then, there exists an idempotent $b \in R$ such that $bn = n_1$ and $n_1 \neq 0$, $n_1 \neq n$. Let $R_1 = \{0, 1, n, n_1, n + n_1, 1 + n, 1 + n_1, 1 + n + n_1\}$. $R_1$ is a subring of $R$. Let $I = \{0, n\}$. $I$ is an ideal of $R_1$ but any ideal of $R$ containing $I$ will also contain $n_1$ and hence $R$ does not have congruence extension property.

REMARK 2.7.1. $\mathscr{C}_1$ is not an equational category even though it is closed under formation of subrings and homomorphic images.

We recall the following definition from R. Balbes and P. Dwinger (1974).

DEFINITION 2.7. Let $\mathscr{C}$ be an equational category and $A \in \operatorname{Ob} \mathscr{C}$. A subalgebra $B$ of $A$ is called epic in $A$ if the inclusion map $1_{B,A}$ is an epimorphism. Notice that, $B$ is epic in $A$ if and only if $f \mid B = g \mid B$ implies $f = g$ for each pair $f, g \in [A, C]$. Also every epimorphism is onto if and only if there are no proper epic subalgebras.

THEOREM 2.8. *There are no proper epic subrings in* $\operatorname{Ob} \mathscr{C}$.

PROOF. Suppose $R_1, R \in \operatorname{Ob} \mathscr{C}$ and $R_1$ is a proper subring of $R$.

*Case* 1. $(R_1)_B \neq R_B$. In this case there exists an idempotent $a \in R$ such that $a \notin R_1$. Let $S = \{x \in R_B \mid x \geqslant a\}$. Then $S$ is a multiplicatively closed subset of $R_B$ and $1 \in S$. Hence $S$ is a multiplicatively closed subset of $R$. Let $S_1 = S \cap R_1$. Then $S_1$ is a multiplicatively closed subset of $R_1$ and $1 \in S_1$. $\langle a \rangle_R \cap R_1$ is a proper ideal of $R_1$ and $\langle a \rangle_R \cap R_1 \cap S_1 = \varnothing$. Hence there exists a maximal ideal $I$ in $R_1$ such that $I \supset \langle a \rangle_R \cap R_1$ and $I \cap S_1 = \varnothing$. Define $f_I$: $R_I \to \{0, 1\}$ by $f_I(x) = 0$ if $x \in I$ and $f_I(x) = 1$ if $x \notin I$. Then $f_I$ is a homomorphism of $R_1$ onto $\{0, 1\}$. Setting $b = 0$ and using Lemma 2.1, we see that the conditions of the Lemma 2.1, are satisfied and hence there exists a homomorphism $g_1$: $\langle R_1 \cup \{a\} \rangle_R \to \{0, 1\}$ such that $g_1(a) = 0$ and $g_1 \mid R_1 = f_I$. Setting $b = 1$, suppose $r \in R_1$ and $ra = 0$. Then $r_B a = 0$. That is, $(1 + r_B)a = a$. Therefore $1 + r_B \geqslant a$ and $1 + r_B \in R_1$. So $1 + r_B \in R_1 \cap S = S_1$. If $1 + r_B \in I$, then $1 + r_B \in I \cap S_1$. But $I \cap S_1 = \varnothing$. Thus $1 + r_B \notin I$. Hence $r_B \in I$. As $I$ is maximal in $R_1$, $r_N \in I$ and $r \in I$. Therefore $f_I(r) = 0$. This gives $f_I(r)b = 0.1 = 0$. Also, if $r \in R_1$, $ra = r$ then $f_I(r)b = f_I(r).1 = f_I(r)$. Hence, by Lemma 2.1, there exists a homomorphism $g_2$: $(R_1 \cup \{a\})_R \to \{0, 1\}$ such that $g_2(a) = 1$ and $g_2 \mid R_1 = f_I$. Obviously $g_1 \neq g_2$. By Corollary 2.4.1, there exist epimorphisms $g_1'$: $R \to \{0, 1\}$, $g_2'$: $R \to \{0, 1\}$ such that $g_1' \mid (R_1 \cup \{a\})_R = g_1$ and $g_2' \mid (R_1 \cup \{a\})_R = g_2$. Thus $g_1' \mid R_1 = g_2' \mid R_1 = f_I$ and $g_1' \neq g_2'$. Hence $R_1$ is not an epic subring of $R$.

*Case* 2. Let $(R_1)_B = R_B$. In this case, there exists a nonzero nilpotent element $n \in R$ such that $n \notin R_1$. Let $B_1 = \{b \in R_B \mid bn \in R_1\}$. Then $B_1$ is a proper ideal of $R_B$. As $R_B$ is a subring of $R_1$, there exists a maximal ideal $I$ in $R_1$ such that

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$I \cap R_B \supset B_1$. Consider $f_I$: $R_1 \to \{0, 1, n_1, 1 + n_1\}$ such that $f_I(x) = 0$ if $x \in I$ and $f_I(x) = 1$ if $x \notin I$. Then $f_I$ is a homomorphism of $R_1$ into $\{0, 1, n_1, 1 + n_1\}$. Suppose $b, n_0 \in R_1$, $b$ being idempotent and $n_0$ nilpotent and $bn = n_0$. Then $b \in B_1 \subset I \cap R_B$. Hence $b \in I$ and therefore $f_I(b) = 0$. Also, as $I$ is maximal ideal in $R_1$, $n_0 \in I$ and hence $f_I(n_0) = 0$. Thus, using Lemma 2.2, we get homomorphisms $g_1, g_2$: $(R_1 \cup \{n\})_R \to \{0, 1, n_1, 1 + n\}$ such that $g_1(n) = 0$, $g_2(n) = n_1$, $g_1 \mid R_1 = g_2 \mid R_1 = f_I$. Obviously $g_1 \neq g_2$. Since $g_1^{-1}(0)$ is maximal ideal in $(R_1 \cup \{n\})_R$ we get that there exists a homomorphism $g_1'$: $R \to \{0, 1, n, 1 + n\}$ such that $g_1' \mid (R_1 \cup \{n\})_R = g_1$. As $(R_1)_B = ((R_1 \cup \{n\})_R)_B = R_B$ and $g_2$: $(R_1 \cup \{n\})_R \to \{0, 1, n_1, 1 + n_1\}$ is an epimorphism, from Corollary 2.6.3 we get that there exists a homomorphism $g_2'$: $R \to \{0, 1, n_1, 1 + n_1\}$ such that $g_2' \mid (R_1 \cup \{n\})_R = g_2$. Thus $g_1' \neq g_2'$ and $g_1' \mid R_1 = g_2' \mid R_1 = f_I$. Hence $R_1$ is not an epic subring of $R$.

Corollary 2.8.1. *Every epimorphism in $\mathcal{C}$ is onto.*

Theorem 2.9. *The only projective in $\mathcal{C}$ is the two element Boolean ring.*

Proof. We recall from Balbes and Dwinger (1974) I.20.14, that if $\mathcal{C}$ is a nontrivial equational category in which every epimorphism is onto, then $A \in \text{Ob}\,\mathcal{C}$ is projective if and only if $A$ is a retract of an $\mathcal{C}$-free algebra. In $\mathcal{C}$, $\{0, 1\}$ is the only free object and hence the only projective in $\mathcal{C}$ is $\{0, 1\}$.

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