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WEAK NORMALITY AND RELATED PROPERTIES

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In [5], Zenor stated the definition of weakly normal. In the main, since weak normality does not imply either normality or regularity, various properties related to either normality or regularity will be considered in the context of weak normality.

Throughout this paper the word "space" will mean topological space. The closure of a point set M will be denoted by $\mathrm{cl}(M)$. The closure of a point set M with respect to the subspace K will be denoted by $\mathrm{cl}(M, K)$.

Definition 1. A space S is weakly normal provided that if $\{H_i\}_{i=1}^{\infty}$ is a monotonically decreasing sequence of closed sets in S with no common part and H is a closed set in S not intersecting H_1, then there is a positive integer N and an open set D such that $H_N \subset D$ and $\mathrm{cl}(D)$ does not intersect H.

It was pointed out to the author by Dr. R. Briggs that if Ω is the first ordinal preceded by uncountably many ordinals and Ω' denotes all ordinals which precede Ω, then $[\Omega' + \{\Omega\}] \times [\Omega']$ is a non-normal, limit point compact space. Every limit point compact space is weakly normal.

For future use the following is stated as a lemma.

LEMMA 1 [3]. *A space S is countably paracompact if and only if for every monotonically decreasing sequence of closed point sets $\{H_i\}_{i=1}^{\infty}$ with no common part, there is a sequence of open sets $\{D_i\}_{i=1}^{\infty}$ such that $D_i \supset H_i$ and $\bigcap_{i=1}^{\infty} \mathrm{cl}(D_i)$ does not exist.*

THEOREM 1. *If S is a weakly normal space such that some monotonically decreasing sequence of closed point sets $\{H_i\}_{i=1}^{\infty}$ with no common part has a sequence of open sets $\{D_i\}_{i=1}^{\infty}$ such that $D_i \supset H_i$ and $\bigcap_{i=1}^{\infty} D_i$ does not exist, then there is a sequence of open sets $\{R_i\}_{i=1}^{\infty}$ such that $R_i \supset H_i$ and $\bigcap_{i=1}^{\infty} \mathrm{cl}(R_i)$ does not exist.*

Proof. For each D_i there is some $j = j(i)$ and some open set R_j such that $R_j \supset H_j$ and $\mathrm{cl}(R_j) \subset D_j$. It may be assumed that $j(i + 1) > j(i)$. For other values of j, take any open set R_j containing H_j, say $R_j = S$. Then

$$\bigcap_{i=1}^{\infty} \mathrm{cl}\,(R_j) \subset \bigcap_{i=1}^{\infty} D_i.$$

COROLLARY 1 [5]. *If S is a weakly normal G_δ-space, then S is countably paracompact.*

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Dowker in [**1**] proved that countable paracompactness is equivalent to countable pointwise paracompactness in a normal space.

COROLLARY 2. *If S is a weakly normal space, then S is countably paracompact if and only if S is countably pointwise paracompact.*

Proof. Using the property that S is countably pointwise paracompact, it can be shown that the conditions in the hypothesis of Theorem 1 are met for every monotonically decreasing sequence of closed point sets with no common part. It then follows from Lemma 1 that S is countably paracompact.

It is well known that countable paracompactness implies countable pointwise paracompactness.

Moore [**4**] defined a property which he called D. He then showed that in a space which had property D that if a subset M was limit point compact, then $\mathrm{cl}(M)$ was limit point compact. He also showed that every normal space has property D.

Definition 2. A space S is said to have property D if, when M is a countably infinite point set with no limit point, there is a collection G of mutually exclusive open sets such that

(1) G covers M and each element of G contains one and only one point of M and

(2) if K is a point set covered by G and each element of G contains one and only one point of K, then K has no limit point.

THEOREM 2. *If S is a countably paracompact* T_2*-space, then S has property* D.

Proof. Let $M = \{p_1, p_2, p_3, \ldots\}$ be a countably infinite set of distinct points with no limit point. Let $M_i = M - \{p_i\}$ and let $U_i = S - M_i$. The covering $\{U_i\}$ has a locally finite refinement $\{V_i\}$ with $V_i \subset U_i$. Each p_i has an open neighbourhood N_i contained in V_i and meeting only a finite number of V_j. The family $\{N_i\}$ is locally finite and each N_i meets at most a finite number of other N_j.

Since X is a T_2-space, for each pair p_i, p_j there are disjoint open sets W_{ij}, W_{ji} with $p_i \in W_{ij}$, $p_j \in W_{ji}$. Let $G_i = N_i \cap [\bigcap_j W_{ij}]$ where the intersection is for the finite number of j for which $N_i \cap N_j$ exists. Then $p_i \in G_i$ and $\{G_i\}$ is a locally finite collection of disjoint open sets.

COROLLARY 3. *If S is a* T_2*-space and either weakly normal and G_δ or weakly normal and countably pointwise paracompact, then S satisfies property* D.

Hodel [**2**] defined the following three axioms.

Axiom 1. If X is any space such that every open set has property P, then every subset of X has property P.

Axiom 2. If X is any space having property P, and U is an F_σ subset of X, then U has property P.

Axiom 3. If X is any space, and $\{V_\alpha | \alpha \text{ in } A\}$ is a locally finite open cover of X such that for all α in A, $\mathrm{cl}(V_\alpha)$ has property P, then X has property P. It is well known that if property P is replaced by normal, then each axiom is true.

THEOREM 3. *If every open subset of the space S is weakly normal, then every subset of S is weakly normal.*

Proof. The technique used in the proof of this theorem follows closely the technique used to show that if every open subset of a space is normal, then every subset of that space is normal.

THEOREM 4. *There is a weakly normal space such that not every open subset is weakly normal.*

Proof. Let Ω be the first ordinal preceded by uncountably many ordinals and let S be all ordinals preceding Ω. Let ω be the first ordinal preceded by countably many ordinals, and let ω' be the ordinals preceding ω. It is well known that $T = [S + \{\Omega\}] \times [\omega' + \{\omega\}]$ is normal and hence weakly normal, but $T - (\Omega, \omega)$ is an open set in S which is not weakly normal.

THEOREM 5. *If S is the union of a sequence of disjoint open subsets S_i, where each S_i is weakly normal but not normal, then S is not weakly normal.*

Proof. Let E_i and F_i be disjoint closed sets of S_i which do not have disjoint neighbourhoods. It is sufficient to take $H_i = \bigcup_{j>i} E_j$ and $H = \bigcup_{i=1}^{\infty} F_i$.

THEOREM 6. *If property* P *is replaced by weakly normal in the axioms of Hodel, then Axioms* 2 *and* 3 *are false.*

Proof. The example of S of Theorem 5 contradicts Axiom 3. Let X be formed from S by adding one point p so that a basic neighbourhood of p consists of p together with all but a finite number of S_i. Then X is weakly normal, but S is an open F_σ subset which is not weakly normal. This contradicts Axiom 2.

THEOREM 7. *There is a weakly normal* T_2-*space which is not regular.*

Proof. Let S be defined as in Theorem 4. Let $T = S \times [0, 1]$, and let p be a point not in T. A base G for $X = T + \{p\}$ is defined as follows:

(1) If $q \in X - [S \times \{0\} + \{p\}]$, then an element of G containing q is a degenerate point set;
(2) If $q \in S \times \{0\}$, then an element of G containing q is an element of the base of S in the order topology crossed with $[0, 1/n]$, where $n = 1, 2, 3, \ldots$;
(3) An element of G containing p is a final segment of S crossed with $(0, 1/n)$, where $n = 1, 2, 3, \ldots$ plus p.

Now X is readily seen to be a T_2-space and every point is a G_δ set. Also every open set which contains p has uncountably many limit points in the closed set $S \times \{0\}$; hence X is not regular.

Since S is normal, it follows that $X - \{p\}$ is normal. If $\{H_i\}$ is a decreasing sequence of closed sets of X with no common part, then, for some i, H_i does not contain p and does not meet $S \times \{0\}$. If a basic neighbourhood R of p does not meet H_i, then its closure $\mathrm{cl}(R)$ does not meet H_i. If H is closed and does not meet H_i, then $H - R$ and H_i have disjoint neighbourhoods. It follows that X is weakly normal.

Definition 3. S is said to be completely weakly normal if, when $\{H_i\}_{i=1}^{\infty}$ is a monotonically decreasing sequence of sets such that $\bigcap_{i=1}^{\infty} \mathrm{cl}(H_i)$ does not exist, and H is a set mutually separated from H_1, there is an open set D and a positive integer N such that $D \supset H_N$ and $\mathrm{cl}(D) \cap H$ does not exist.

Definition 4. S is said to be completely semi-normal if, when $\{H_i\}_{i=1}^{\infty}$ is a monotonically decreasing sequence of sets with no common part and H is a set mutually separated from H_1, there is an open set D and a positive integer N such that $D \supset H_N$ and $\mathrm{cl}(D) \cap H$ does not exist.

THEOREM 8. *If S is completely semi-normal, then every subset of S is weakly normal.*

THEOREM 9. *If every subset of S is weakly normal, then S is completely weakly normal.*

It is known that countable paracompactness does not imply normal. It can be easily shown that there is a countably paracompact space which is not weakly normal. It then seems reasonable to ask if some property may be added to countable paracompactness to arrive at weak normality.

Definition 5. A space is said to be w-w-normal if and only if $\{T_i\}_{i=1}^{\infty}$ and $\{H_i\}_{i=1}^{\infty}$ are two monotonically decreasing sequences of closed point sets such that $\bigcap_{i=1}^{\infty} T_i$ and $\bigcap_{i=1}^{\infty} H_i$ do not exist, and $T_1 \cap H_1$ does not exist. Then there are positive integers s and r and mutually exclusive open sets D_s and D_r such that $D_s \supset T_s$ and $D_r \supset H_r$.

THEOREM 10. *There is a space which is* w-w-*normal but not weakly normal.*

Proof. Consider the open set defined in Theorem 4 as a space. It is not weakly normal, but is w-w-normal.

THEOREM 11. *If S is* w-w-*normal and countably paracompact, then S is weakly normal.*

Proof. This follows easily from Lemma 1.

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