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LETTERS TO THE EDITOR

A NOTE ON EXPLOSIVENESS OF MARKOV BRANCHING PROCESSES

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Abstract

It is shown that no condition of the form $E[g(N)]=\infty$, where N denotes a typical family size, is sufficient for explosiveness of a Markov branching process.

EXPLOSION CRITERION; MOMENT CONDITION

1. Introduction

A continuous-time Markov chain is called *explosive* if there is positive probability that it will perform infinitely many transitions in a finite time; otherwise it is called *conservative*. In the special case of a Markov branching process, explosiveness occurs roughly speaking if family sizes are too large too often. In particular, it is well known that if N denotes a typical family size then the condition $EN=\infty$ is necessary for explosiveness. It is not, however, a sufficient condition, and it is a reasonable conjecture that there exists a function g with $g(n)\uparrow\infty$ as $n\to\infty$ such that $E[g(N)]=\infty$ is a sufficient condition for explosiveness. That this conjecture is false is a consequence of the following theorem, whose proof constitutes the remainder of this paper.

Theorem. Given a function g such that $g(n)\uparrow\infty$ as $n\to\infty$, there exists a conservative Markov branching process such that the typical family size N satisfies $E[g(N)]=\infty$.

2. Proof of the theorem

The 'classical' necessary and sufficient condition for the process to be explosive (see e.g. Harris (1963)) is that the integral

$$\int_{1-\varepsilon}^{1}\frac{ds}{s-h(s)}$$

converges for suitably small $\varepsilon>0$, where h is the probability generating function of the family size distribution. However, for our purposes, the following more recent result is more useful.

Theorem. (Doney (1984)). Let N be a typical family size and for each $n=1, 2, 3, \cdots$ let

$$l(n)=\sum_{r=0}^{n}P(N>r).$$

Then the Markov branching process is explosive if and only if

$$\sum_{n=1}^{\infty}\{nl(n)\}^{-1}<\infty.$$

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Before embarking on the proof, we note that if the tail of the distribution of N is regularly varying with

$$P(N>r)\sim r^{-\alpha}L(r) \quad \text{as} \quad r\to\infty$$

where $\alpha \geq 0$ and L is slowly varying, then Doney's theorem may easily be used to compute whether the Markov branching process is explosive or conservative; in particular it is always explosive if $\alpha<1$. To prove our theorem in cases such as $g(n)=n^{\beta}$ for $0<\beta<1$, therefore, we shall need a tail which is not regularly varying.

We consider distributions which have atoms of sizes $p_1, p_2, p_3, \cdots$ (>0) at points $n_1<n_2<n_3<\cdots$ and zero mass everywhere else. We show that the pairs (n_1, p_1), (n_2, p_2), $(n_3, p_3), \cdots$ may be successively chosen so that, for each $k=1, 2, 3, \cdots$, the following conditions are satisfied:

(1) $$\sum_{i=1}^{k} g(n_i)p_i \geqq k;$$

(2) $$\sum_{n=1}^{n_k-1} \{nl(n)\}^{-1} \geqq k;$$

and

(3) $$\sum_{n=n_k}^{\infty} \{n[l(n_k)+(n-n_k)q_k]\}^{-1} \geqq 2$$

where

$$q_k = 1-\sum_{i=1}^{k} p_i.$$

The manner of construction is similar to that used in Grey (1978). We prove by induction that the choice may be continued indefinitely. Therefore because (1) and (2) are true for all k we ultimately obtain a distribution which is proper (again because of (2)) and which satisfies the requirements of the theorem.

Choice of (n_1, p_1) poses no problem. Suppose, therefore, that we have chosen (n_i, p_i) for $i=1, 2, \cdots, k$. Now choose n_{k+1} sufficiently large that

(4) $$g(n_{k+1})q_k \geqq 2$$

and

(5) $$\sum_{n=n_k}^{n_{k+1}-1} \{n[l(n_k)+(n-n_k)q_k]\}^{-1} \geqq 1.$$

The latter choice is possible because of (3). Then choose p_{k+1} sufficiently close to, but smaller than, q_k that

(6) $$g(n_{k+1})p_{k+1} \geqq 1$$

and

(7) $$\sum_{n=n_{k+1}}^{\infty} \{n[l(n_{k+1})+(n-n_{k+1})q_{k+1}]\}^{-1} \geqq 2.$$

The former choice is possible because of (4); the latter is possible since as $q_{k+1}\downarrow 0$, the sum in (7) tends to infinity.

Now condition (6) ensures that (2) holds with k replaced by $k+1$; also, since

$$l(n)=l(n_k)+(n-n_k)q_k \quad \text{for} \quad n_k \leqq n < n_{k+1}$$

it follows from (5) that (2) is true with k replaced by $k+1$; and finally (7) is (3) with k replaced by $k+1$.

This completes the induction and the proof.

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