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TRANSFERRING RESULTS FROM RINGS OF CONTINUOUS FUNCTIONS TO RINGS OF ANALYTIC FUNCTIONS

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Introduction. Let $C(X)$ be the ring of all real-valued continuous functions on a completely regular topological space X, and let $A(Y)$ be the ring of all functions analytic on a connected non-compact Riemann surface Y. The ideal theories of these two function rings have been extensively studied since the fundamental papers of E. Hewitt on $C(X)$ [**12**] and of M. Henriksen on the ring of entire functions [**10**; **11**]. Despite the obvious differences between these two rings, it has turned out that there are striking similarities between their ideal theories. For instance, non-maximal prime ideals of $A(Y)$ [**2**; **11**] behave very much like prime ideals of $C(X)$ [**13**; **14**], and primary ideals of $A(Y)$ which are not powers of maximal ideals [**19**] resemble primary ideals of $C(X)$ [**15**]. In this paper we show that there are very good reasons for these similarities. It turns out that much of the ideal theory of $A(Y)$ is a special case of the ideal theory of rings of continuous functions. We develop machinery that enables one almost automatically to derive results about the ideal theory of $A(Y)$ from corresponding known results of ideal theory for rings of continuous functions.

In section 1 we present some facts about $A(Y)$ that are needed later. In section 2 we construct a special topological space X and analyze the structure of some of the important ideals of the ring $C^*(X)$ of bounded real-valued continuous functions on X. Section 3 contains the transfer machinery promised above. We define a map from the set of ideals of $A(Y)$ into the set of ideals of $C^*(X)$ and derive its basic properties in Theorems 3.5 and 3.6. Theorem 3.5 in particular shows that the differentially closed ideals of $A(Y)$ behave very much like the 'regular' ideals of $C^*(X)$. (An ideal is called regular if it is generated by a set of regular elements of the ring, i.e. elements that are not zero divisors.)

Finally in section 4 we show how the results of section 3 may be applied to transfer results of ideal theory from $C^*(X)$ to $A(Y)$. Many examples are provided in order to give a clear idea of what kinds of results can be transferred and how the transfer process works.

Corollary 3.7. together with some applications to local ideal theory was announced in [**1**]. The proof sketched there relies on valuation theory and the isomorphism of certain groups obtained from ultrapowers of the integers and

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the reals. In this paper we give a more direct proof which does not use valuation theory and we obtain in addition a great deal of information relating the global ideal theories of $A(Y)$ and $C^*(X)$.

1. The Ring $A(Y)$**.** The basic terminology and notation of this paper will be that of [**8**] with one exception: the whole ring will be considered an ideal. If F is an analytic function, $V_x(F)$ will denote the order of the zero of F at the point x.

Let $A(Y)$ be the ring of all functions analytic on a connected, non-compact Riemann surface Y. In this section we present facts about the ideal theory of $A(Y)$ that are needed later. For results mentioned but not proved in this section and for further information about $A(Y)$ see the papers of N. L. Alling [**2**; **3**] and M. Henriksen [**10**; **11**].

Suppose that M is a maximal ideal of $A(Y)$. Let F be a nonzero element of M, and set $D = Z(F)$. D is a nonempty, closed discrete subset of Y, either finite or countably infinite. Let $\mu = \{Z \cap D : Z \in Z[M]\}$. μ is an ultrafilter on the set D, and

(1.1) $M = \{F \in A(Y) : Z(F) \cap D \in \mu\}$.

If μ is a fixed (principal) ultrafilter on D, then M is called a *fixed maximal ideal.* If μ is a free ultrafilter on D, then M is called a *free maximal ideal.*

Now let I be a proper ideal of $A(Y)$. If M is a maximal ideal that contains I, we set

$$I_M{}^c = \{F \in A(Y) : GF \in I \text{ for some } G \in A(Y) - M\}.$$

$I_M{}^c$ is an ideal of $A(Y)$, and $I \subset I_M{}^c \subset M$. We will need two facts about the ideals $I_M{}^c$:

(1) A nonzero ideal I is contained in a unique maximal ideal M if, and only if, $I = I_M{}^c$.

(2) If I is any ideal and $v(I)$ is the set of maximal ideals containing I, then $I = \bigcap_{M \in v(I)} I_M{}^c$.

If M is a maximal ideal of $A(Y)$, then the ideal $P_M{}^* = \bigcap_{n \in N} M^n$ is the largest non-maximal prime ideal contained in M. If M is fixed, $P_M{}^* = (0)$. If M is free, however, $(0) \neq P_M{}^*$. In fact (using the notation above)

$$P_M{}^* = \{F \in A(Y) : \{x \in D : V_x(F) \geqq n\} \in \mu \text{ for all } n \in N\}.$$

1.2. *Definition.* Let M be a maximal ideal of $A(Y)$. We say that an ideal I of $A(Y)$ is $P_M{}^*$*-restricted* provided

$$I = \{F \in A(Y) : HF \in I \text{ for some } H \in A(Y) - P_M{}^*\}.$$

The notion of a $P_M{}^*$-restricted ideal is interesting only when M is free; for M fixed the only $P_M{}^*$-restricted ideals are (0) and $A(Y)$. If I is a proper $P_M{}^*$-restricted ideal, then clearly $I = I_M{}^c$; so if $I \neq (0)$, M is the only maximal ideal that contains I. The $P_M{}^*$-restricted ideals include all non-maximal prime

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ideals that are contained in M and all the primary ideals that are contained in M except for the powers M^n of M.

The result we record next, originally due to O. Helmer [9] for Y = the complex plane, has been a valuable tool in all studies of the ideal theory of $A(Y)$.

1.3. PROPOSITION. *Let* $\{F_1, \ldots, F_n\} \subset A(Y)$. *If* H *is any element of* $A(Y)$ *that satisfies* $V_x(H) = \min\{V_x(F_1), \ldots, V_x(F_n)\}$ *for all* $x \in Y$, *then* $(H) = (F_1, \ldots, F_n)$.

We will call an ideal I of $A(Y)$ *differentially closed* if $F' \in I$ whenever $F \in I$. We now give an elementary but useful description of the differentially closed ideals of $A(Y)$.

1.4. PROPOSITION. *The following are equivalent for any ideal* I *of* $A(Y)$.

(1) I *is differentially closed.*

(2) $G \in I$ *if, and only if, for some* $F \in I$ *and some integer* k, $V_x(G) \geqq V_x(F) - k$ *for all* $x \in Y$.

(3) *For all* $M \in v(I)$, $I_M{}^c$ *is* $P_M{}^*$*-restricted.*

Proof. (1) $\Rightarrow$ (2): Suppose $F \in I$ and $V_x(G) \geqq V_x(F) - k$ for all $x \in Y$. Let $F^{(k)}$ denote the kth derivative of F, and take $H \in A(Y)$ such that $V_x(H) = \min\{V_x(F), V_x(F^{(k)})\}$ for all $x \in Y$. (The existence of such an H follows from the generalized Weierstrass product theorem [6], which we use without comment throughout the remainder of the paper.) Note that $V_x(H) = V_x(F) - k$ for all $x \in Z(H)$. Since I is differentially closed, $(F, F^{(k)}) \subset I$, and so by 1.3 $H \in I$. Since $V_x(G) \geqq V_x(H)$ for all x, $G/H \in A(Y)$, and so $G \in I$.

(2) $\Rightarrow$ (3): Let $M \in v(I)$, and suppose that $FG \in I_M{}^c$, where $F \in A(Y) - P_M{}^*$. We want to show that $G \in I_M{}^c$. Now since $FG \in I_M{}^c$, $HFG \in I$ for some $H \in A(Y) - M$. Let D and μ be as in (1.1). Since $F \notin P_M{}^*$, there exists $k \in N$ such that $D_1 = \{x \in D : V_x(F) \leqq k\} \in \mu$. And since $H \notin M$, $D_2 = \{x \in D : V_x(H) = 0\} \in \mu$, so $D_1 \cap D_2 \in \mu$. Take $K \in A(Y)$ such that $Z(K) = Z(HF) - D_1 \cap D_2$ and $V_x(K) = V_x(HF)$ for all $x \in Z(K)$. Then $V_x(KG) \geqq V_x(HFG) - k$ for all x, so $KG \in I$ by (2). Since $K \notin M$, $G \in I_M{}^c$.

(3) $\Rightarrow$ (1): Since (1) holds for $I = A(Y)$, assume that $I \neq A(Y)$. Since $I = \bigcap_{M \in v(I)} I_M{}^c$, it is sufficient to show that any $P_M{}^*$-restricted ideal J is differentially closed. Let $F \in J$. Take $G \in A(Y)$ such that $Z(G) = Z(F) - Z(F')$ and $V_x(G) = 1$ for all $x \in Z(G)$. Then $GF'/F \in A(Y)$, so $GF' \in J$. Obviously $G \notin P_M{}^*$, so $F' \in J$.

1.5. COROLLARY. *An ideal of* $A(Y)$ *is differentially closed if, and only if, it is an intersection of* $P_{M_\alpha}{}^*$*-restricted ideals for some collection* $\{M_\alpha\}$ *of maximal ideals.*

The Corollary tells us that the class of differentially closed ideals, which we denote by $\mathscr{D}$, is quite large. $\mathscr{D}$ contains all non-maximal prime ideals, all primary ideals that are not powers of maximal ideals, and all intersections of such ideals.

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For an ideal I of $A(Y)$ let $\bar{I}$ be the differential closure of I (the smallest differentially closed ideal that contains I). Using 1.3 and 1.4 it is easy to see that $G \in \bar{I}$ if, and only if, for some $F \in I$ and some integer k, $V_x(G) \geqq V_x(F) - k$ for all $x \in Y$.

2. The ring $C^*(X)$. In this section we construct a topological space X and investigate certain ideals of the ring $C^*(X)$ of all bounded, continuous, real-valued functions on X. We use small letters $f, g, \ldots$ to denote elements of $C^*(X)$ to distinguish these functions from the elements $F, G, \ldots$ of $A(Y)$.

Let X be Y together with an additional point ω. The topology on X is defined as follows: Y has the discrete topology, and the neighborhoods of ω are complements of subsets of Y that are closed and discrete in the Riemann surface topology of Y. Note that the open-and-closed sets are precisely the closed discrete subsets of Y and their complements in X. (Whenever we use the phrase "closed discrete subset of Y" we mean a set that is closed and discrete in the Riemann surface topology of Y.) It is easy to verify that X is completely regular.

Now suppose that $f \in C^*(X)$. If $f(\omega) \neq 0$, then $f^{-1}(R - \{0\})$ is a neighborhood of ω, so $Z(f)$ is a closed discrete subset of Y. This observation enables us to describe the free z-ultrafilters (the ultrafilters of zero sets of elements of $C^*(X)$). If μ' is a free z-ultrafilter, then there exists $D \in \mu'$ such that $\omega \notin D$. D is therefore an infinite closed discrete subset of Y. Furthermore, the collection $\mu = \{Z \cap D : Z \in \mu'\}$ is a free ultrafilter on D, and $\mu' = \{Z \in Z(X) : Z \cap D \in \mu\}$. Hence the free z-ultrafilters are just the free ultrafilters on countably infinite closed discrete subsets of Y (with the obvious identification.)

Let βX be the Stone-Čech compactification of X. The points of $\beta X - X$ are, of course, in 1-1 correspondence with the free maximal ideals of $C^*(X)$. Let $p \in \beta X - X$, and denote the corresponding free maximal ideal by M^p. Suppose that μ' is the unique z-ultrafilter that converges to p, and let μ and D be as above. Then

$$M^p = \{f \in C^*(X) : \{x \in D : |f(x)| \leqq 1/n\} \in \mu \text{ for all } n \in N\}.$$

Denote by O^p the ideal

$$O^p = \{f \in C^*(X) : Z(f^\beta) \text{ is a } \beta X\text{-neighborhood of } p\},$$

where f^β is the continuous extension of f to βX. It is easy to see that

$$(2.1) \quad O^p = \{f \in C^*(X) : Z(f) \cap D \in \mu\}.$$

It follows that O^p is a prime ideal, and is therefore the unique minimal prime ideal of $C^*(X)$ contained in M^p [**8**, Theorem 7.15]. Hence the free minimal prime ideals of $C^*(X)$ are also in 1-1 correspondence with the points of $\beta X - X$. For a given free ultrafilter μ on some infinite closed discrete set D we will say that the free maximal ideal M of $A(Y)$ given by (1.1) is the maximal ideal of $A(Y)$ that *corresponds to* the minimal prime ideal O^p of $C^*(X)$ given by (2.1).

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Recall that an ideal I of $C^*(X)$ is *absolutely convex* if whenever $|f| \leqq |h|$ and $h \in I$, then $f \in I$.

2.2 PROPOSITION. *Every ideal I of $C^*(X)$ that contains a function that does not vanish at ω is absolutely convex.*

Proof. Suppose $g \in I$ and $g(\omega) \neq 0$. First we show that I is convex. Suppose that $0 \leqq f \leqq i$ for some $i \in I$. Let $j = i + g^2$. Then $j \in I$, and $0 \leqq f \leqq j$. $Z(j) \subset Z(g^2)$, so $Z(j)$ is an open-and-closed subset of X. Define h by $h(x) = f(x)/j(x)$ for $x \in X - Z(j)$; $h(x) = 1$ for $x \in Z(j)$. Then $h \in C^*(X)$, and $f = hj$, so $f \in I$.

Now by [**8**, Theorem 5.3] it is sufficient to show that if $f \in I$, then $|f| \in I$. Since $g(\omega) \neq 0$, there exists a closed discrete set D such that g is bounded away from 0 on $X - D$. Define functions $k, s, t \in C^*(X)$ as follows. $k(x) = 1/g(x)$ for $x \in X - D$, $k(x) = 1$ for $x \in D$. $s(x) = |f(x)|$ for $x \in X - D$, $s(x) = 0$ for $x \in D$. $t(x) = 0$ for $x \in X - D$; and for $x \in D$ we set $t(x) = 1$ if $f(x) \geqq 0$, $t(x) = -1$ if $f(x) < 0$. Then $|f| = skg + tf \in I$.

Recall that an element of a ring is called *regular* if it is not a zero divisor. In the ring $C^*(X)$ the regular elements are precisely the functions which are nowhere zero.

2.3. *Definition.* An ideal I of $C^*(X)$ is *regular* if there exists a set R of regular elements of $C^*(X)$ such that I is the ideal generated by R.

Note that (0) is a regular ideal of $C^*(X)$ according to this definition (with $R = \emptyset$). In the next section we will need a few facts about regular ideals.

2.4. PROPOSITION. *The following are equivalent for any nonzero ideal I of $C^*(X)$:*

(1) *Every element of I is a multiple of a regular element of I.*
(2) *I is regular.*
(3) *I contains a regular element.*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) is trivial. For (3) $\Rightarrow$ (1), suppose that I contains a regular element h. Let $f \in I$. By 2.2. I is absolutely convex, so $|f| + |h| \in I$. Now $(|f| + |h|)$ is also absolutely convex by 2.2. Therefore since $|f| \leqq |f| + |h|$, f is a multiple of the regular element $|f| + |h|$ of I.

2.5. PROPOSITION. *Any ideal of $C^*(X)$ that strictly contains a free minimal prime ideal is regular.*

Proof. Suppose that $O^p \subsetneq J$, where O^p is given by (2.1), and let $f \in J - O^p$. Since $f \notin O^p$, $S = \{x \in D : f(x) \neq 0\} \in \mu$. Define $g \in C^*(X)$ by $g(x) = 0$ for $x \in S$, $g(x) = 1$ for $x \in X - S$. Then $g \in O^p$, so $g + f^2 \in J$. Since $g + f^2$ is regular, J is regular by 2.4.

3. The transfer machinery. In this section we connect the rings $A(Y)$ and $C^*(X)$, first by defining a map from $A(Y)$ to $C^*(X)$, and then by using

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that map to define a map from the set of ideals of $A(Y)$ into the set of ideals of $C^*(X)$.

3.1. *Definition.* For $F \in A(Y) - \{0\}$ let $F^\# \in C^*(X)$ be defined by $F^\#(x) = \exp(-V_x(F))$ for $x \in Y$, $F^\#(\omega) = 1$. We set $0^\# = 0$.

3.2. PROPOSITION. *If $g \in C^*(X)$ is regular, then there exists a unit u of $C^*(X)$ and a function $F \in A(Y)$ such that $g = uF^\#$.*

Proof. Since $g(\omega) \neq 0$, there exists a closed discrete set D such that g is bounded away from 0 on $X - D$. Define $h_1 \in C^*(X)$ by $h_1(x) = 1/g(x)$ for $x \in X - D$, $h_1(x) = 1$ for $x \in D$. Define $h_2 \in C^*(X)$ by $h_2(x) = 1$ if $h_1g(x) > 0$, $h_2(x) = -1$ if $h_1g(x) < 0$. Let K be an upper bound for h_2h_1g on D. Define $h_3 \in C^*(X)$ by $h_3(x) = 1$ for $x \in X - D$, $h_3(x) = 1/eK$ for $x \in D$. $h_3h_2h_1g$ is equal to 1 outside D and is positive and $\leqq 1/e$ on D. Now for every $x \in D$ let $n(x)$ be the largest positive integer such that $\exp(-n(x)) \geqq h_3h_2h_1g(x)$. Define $h_4 \in C^*(X)$ by $h_4(x) = 1$ for $x \in X - D$, $h_4(x) = \exp(-n(x))/h_3h_2h_1g(x)$ for $x \in D$.

Let $u = 1/h_4h_3h_2h_1$. Then u is a unit of $C^*(X)$. $g(x)/u(x) = 1$ for $x \in X - D$, and $g(x)/u(x) = \exp(-n(x))$ for $x \in D$. Take $F \in A(Y)$ such that $Z(F) = D$ and $V_x(F) = n(x)$ for all $x \in D$. Clearly $g = uF^\#$.

It follows from this Proposition that a nonzero ideal of $C^*(X)$ is regular if, and only if, it contains an element $F^\#$ for some nonzero F. Now we define the map that will be our main tool throughout the remainder of the paper.

3.3 *Definition.* For an ideal I of $A(Y)$, we set

$$\tau(I) = \{gF^\# : g \in C^*(X),\ F \in I\}.$$

3.4. PROPOSITION. *$\tau(I)$ is the ideal of $C^*(X)$ generated by the set $\{F^\# : F \in I\}$.*

Proof. We need only to show that $\tau(I)$ is an ideal. It is sufficient to verify that for any $F, G \in I$ and $r, s \in C^*(X)$ there exists $H \in I$ and $t \in C^*(X)$ such that $rF^\# + sG^\# = tH^\#$. Take $H \in A(Y)$ such that $V_x(H) = \min(V_x(F), V_x(G))$ for all $x \in Y$. By 1.3, $H \in I$. Set $t = (rF^\# + sG^\#)/H^\#$. Since $|t| \leqq |r|F^\#/H^\# + |s|G^\#/H^\# \leqq |r| + |s|$, $t \in C^*(X)$.

Denote the set of ideals of $A(Y)$ by $\mathscr{I}(A(Y))$ and the set of regular ideals of $C^*(X)$ by $\mathscr{R}$.

3.5. THEOREM. *The map $\tau : \mathscr{I}(A(Y)) \rightarrow \mathscr{R}$ has the following properties:*

(1) *$\tau(IJ) = \tau(I)\tau(J)$ for all $I, J \in \mathscr{I}(A(Y))$.*

(2) *τ takes principal ideals to principal ideals.*

(3) *τ preserves sums.*

(4) *$\tau(I) = \tau(\bar{I})$ for all $I \in \mathscr{I}(A(Y))$.*

(5) *The restriction $\tau|\mathscr{D}$ of τ to $\mathscr{D}$ satisfies:*

  (a) *$\tau|\mathscr{D}$ is 1-1 and onto.*

  (b) *$\tau|\mathscr{D}$ preserves sums and nonzero intersections.*

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Proof. (1) For $\tau(I)\tau(J) \subset \tau(IJ)$ it is sufficient to show that if $F \in I$ and $G \in J$, then $F^\# G^\# \in \tau(IJ)$. Since $F^\# G^\# = (FG)^\#$, this is obvious. For the reverse inclusion we need to show that if $F \in IJ$, then $F^\# \in \tau(I)\tau(J)$. Since $F \in IJ$, $F = G_1H_1 + \ldots + G_nH_n$ for some $G_i \in I$, $H_i \in J$. Therefore

$$F^\# = (G_1H_1 + \ldots + G_nH_n)^\# \leqq G_1^\# H_1^\# + \ldots + G_n^\# H_n^\# \in \tau(I)\tau(J).$$

Since $\tau(I)\tau(J)$ is convex, $F^\# \in \tau(I)\tau(J)$.

(2) $\tau((F)) = (F^\#)$.

(3) Let $\{I_\alpha\} \subset \mathscr{I}(A(Y))$, and set $I = \sum I_\alpha$. Clearly $\sum \tau(I_\alpha) \subset \tau(I)$. We complete the proof by showing that if J is any ideal of $C^*(X)$ such that $\tau(I_\alpha) \subset J$ for all α, then $\tau(I) \subset J$. This is obvious if $I_\alpha = (0)$ for all α, so assume that at least one of the I_α is nonzero. Then J is absolutely convex. Let $G \in I$. Then $G = H_1 + \ldots + H_k$ for some $H_i \in I_{\alpha_i}$, so we have $0 \leqq G^\# \leqq H_1^\# + \ldots + H_k^\# \in J$. Hence $G^\# \in J$ by convexity.

(4) Since $I \subset \bar{I}$, $\tau(I) \subset \tau(\bar{I})$. Let $F \in \bar{I}$. Then there is a $G \in I$ and an integer k such that $V_x(F) \geqq V_x(G) - k$ for all $x \in Y$. Therefore $F^\# \leqq e^k G^\#$, so $F^\# \in \tau(I)$ by convexity.

(5) (a) To verify that $\tau|\mathscr{D}$ is 1-1 we show that for $I \in \mathscr{I}(A(Y))$ and $J \in \mathscr{D}$, if $\tau(I) \subset \tau(J)$, then $I \subset J$. Let $F \in I$. Then $F^\# \in \tau(J)$, so $F^\# = gH^\#$ for some $g \in C^*(X)$ and some $H \in J$. Hence $V_x(F) = V_x(H) - \log g(x) \geqq V_x(H) - k$ for all $x \in Y$ for some positive constant k. Since $J \in \mathscr{D}$, $F \in J$.

Next we show that $\tau|\mathscr{D}$ is onto. Let I be a nonzero regular ideal. We claim that for any $f \in I$ there exists $F \in A(Y)$ such that $f \in (F^\#)$ and $(F^\#) \subset I$. For, by 2.4 every $f \in I$ is a multiple of some regular $g \in I$, and by 3.2, $g = uF^\#$ for some $F \in A(Y)$ and some unit u of $C^*(X)$. Now for each $f \in I$ pick such an F, and let J be the ideal of $A(Y)$ generated by these F's. It is easy to verify that $\tau(J) = I$. Finally, $\bar{J} \in \mathscr{D}$, and $\tau(\bar{J}) = I$.

(5) (b) We have shown that τ preserves sums, and since the sum of any collection of differentially closed ideals is differentially closed, $\tau|\mathscr{D}$ preserves sums.

Suppose now that $\bigcap I_\alpha \neq (0)$ for some collection $\{I_\alpha\} \subset \mathscr{D}$. We want to show that $\tau(\bigcap I_\alpha) = \bigcap \tau(I_\alpha)$. Obviously $\tau(\bigcap I_\alpha) \subset \bigcap \tau(I_\alpha)$. Now since $\bigcap \tau(I_\alpha)$ is a nonzero regular ideal, for the reverse inclusion it is sufficient to deal with regular functions. Let $f \in \bigcap \tau(I_\alpha)$ be regular. $f = uF^\#$ for some $F \in A(Y)$ and some unit u of $C^*(X)$. For each α, $uF^\# \in \tau(I_\alpha)$, so $F^\# \in \tau(I_\alpha)$. It follows that $F \in I_\alpha$ since $I_\alpha \in \mathscr{D}$. Hence $F \in \bigcap I_\alpha$, and so $f \in \tau(\bigcap I_\alpha)$.

It should be noted that the map τ, unlike its restriction to $\mathscr{D}$, does not preserve nonzero intersections. For if it did, it would follow from Theorem 3.5 that $\overline{\bigcap I_\alpha} = \bigcap \bar{I}_\alpha$ for any collection $\{I_\alpha\}$ of ideals of $A(Y)$ such that $\bigcap I_\alpha \neq (0)$. This is not the case, as we now show. Let $D = \{x_1, x_2, \ldots\}$ be an infinite closed discrete subset of Y. Take $G \in A(Y)$ such that $Z(G) = D$ and $V_{x_n}(G) = n$, $n = 1, 2, \ldots$. For each positive integer k take $F_k \in A(Y)$ such that $Z(F_k) = D$ and $V_x(F_k) = k$ for all $x \in D$. Set $I_k = (F_k, G)$. Then for every k, $\bar{I}_k =$

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$A(Y)$, so $\bigcap \bar{I}_k = A(Y)$. But $\bigcap I_k = (G)$, so $\overline{\bigcap I_k} = \overline{(G)}$. It is true that τ preserves finite intersections since $\overline{I \cap J} = \bar{I} \cap \bar{J}$ for all $I, J \in \mathscr{I}(A(Y))$.

Now we give some information about the "local" behavior of τ.

3.6. THEOREM. *Let M be a free maximal ideal of $A(Y)$, and let O^p be the corresponding minimal prime ideal of $C^*(X)$.*

(1) *τ is a bijection of the nonzero P_M^*-restricted ideals of $A(Y)$ onto the ideals of $C^*(X)$ that strictly contain O^p.*

(2) *For every $f \in C^*(X)$ there exists $F \in A(Y)$ such that $\tau((F)) + O^p = (f) + O^p$.*

(3) *Let I be a nonzero P_M^*-restricted ideal. I is prime (respectively primary) if, and only if, $\tau(I)$ is prime (respectively primary).*

Proof. (1) By 1.5, P_M^*-restricted ideals are differentially closed. It follows from 3.5 that the map is 1-1. Now let I be a nonzero P_M^*-restricted ideal. We show first that $O^p \subset \tau(I)$. This is obvious if $I = A(Y)$, so assume $I \neq A(Y)$. Let $f \in O^p$. Let $G \in I - \{0\}$, and take $H \in A(Y)$ such that $Z(H) = Z(G) \cap Z(f) \cap D$ and $V_x(H) = V_x(G)$ for all $x \in Z(H)$. Since $Z(H) \subset Z(f)$, $f = f H^\#$, so we only need to show that $H \in I$. Take $K \in A(Y)$ such that $Z(K) = Z(G) - Z(H)$ and $V_x(K) = V_x(G)$ for all $x \in Z(K)$. Then $KH/G \in A(Y)$, so $KH \in I$. Since $Z(H) \in \mu$, $K \notin M$. Therefore since $I = I_{M^c}$, $H \in I$.

Now let J be an ideal of $C^*(X)$ which strictly contains O^p. We need to find a P_M^*-restricted ideal I such that $\tau(I) = J$. By 2.5, J is regular, so by 3.5 there exists $I \in \mathscr{D}$ such that $\tau(I) = J$. If $J = C^*(X)$, then $I = A(Y)$, which is P_M^*-restricted, and we are done. So assume $J \neq C^*(X)$. Then since $O^p \subset \tau(I)$, $\tau(I) \subset M^p$ and M^p is the only maximal ideal of $C^*(X)$ that contains $\tau(I)$ [**8**, Theorem 7.13]. Using the definition of τ, one can see that $I \subset M$ and M is the only maximal ideal of $A(Y)$ that contains I. Hence $I = I_{M^c}$, so by 1.4 I is P_M^*-restricted.

(2) Let O^p be given by (2.1), and let $f \in C^*(X)$. If $f \in O^p$, we can take $F = 0$; so assume $f \notin O^p$. Then $S = \{x \in D : f(x) \neq 0\} \in \mu$. Define $g \in C^*(X)$ by $g(x) = f(x)$ for $x \in S$, $g(x) = 1$ for $x \in X - S$. Since g is regular, there exists $F \in A(Y)$ and a unit u of $C^*(X)$ such that $g = uF^\#$. Then

$$\tau((F)) + O^p = (g) + O^p = (f) + O^p.$$

(3) Suppose that I is prime and that $fg \in \tau(I)$. Then $(fg) + O^p \subset \tau(I)$. Now $(fg) + O^p = ((f) + O^p)((g) + O^p)$ since $O^p = (O^p)^2$ [**8**, 2B.2], so $((f) + O^p)((g) + O^p) \subset \tau(I)$. Let $F, G \in A(Y)$ such that $\tau((F)) + O^p = (f) + O^p$, and $\tau((G) + O^p) = (g) + O^p$. Then $(\tau((F)) + O^p)(\tau((G)) + O^p) \subset \tau(I)$, and it follows that $\tau((FG)) \subset \tau(I)$. Since $I \in \mathscr{D}$, $FG \in I$. Therefore $F \in I$ or $G \in I$, say $F \in I$. Then we have $(f) + O^p = \tau((F)) + O^p \subset \tau(I)$, so $f \in \tau(I)$.

Conversely, suppose that $\tau(I)$ is prime and that $FG \in I$. Then $F^\# G^\# \in \tau(I)$, so either $F^\# \in \tau(I)$ or $G^\# \in \tau(I)$, say $F^\# \in \tau(I)$. Then $F \in I$ since $I \in \mathscr{D}$.

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The proof that I is primary if, and only if, $\tau(I)$ is primary is similar.

Let M and O^p be as in 3.6, and let $\mathscr{I}(C^*(X)/O^p)$ denote the set of ideals of $C^*(X)/O^p$. Consider $\mathscr{I}(C^*(X)/O^p)$ and the set of $P_M{}^*$-restricted ideals as totally ordered semigroups (under multiplication of ideals and inclusion). For $I \in \mathscr{I}(A(Y))$, set $\phi(I) = (\tau(I) + O^p)/O^p$.

The next result, which follows easily from 3.5 and 3.6, shows that the ideals of $C^*(X)/O^p$ behave very much like the $P_M{}^*$-restricted ideals.

3.7. COROLLARY. *The map* $\phi : \mathscr{I}(A(Y)) \to \mathscr{I}(C^*(X)/O^p)$ *has the following properties*:

(1) $\phi(IJ) = \phi(I)\phi(J)$ *for all* $I, J \in \mathscr{I}(A(Y))$.

(2) *The restriction of* ϕ *to the* $P_M{}^*$*-restricted ideals of* $A(Y)$ *is a surjective order preserving isomorphism.*

(3) ϕ *maps the set of principal ideals of* $A(Y)$ *onto the set of principal ideals of* $C^*(X)/O^p$.

(4) *If* $I \in \mathscr{I}(A(Y))$ *and* J *is a* $P_M{}^*$*-restricted ideal of* $A(Y)$, *then* $I \subset J$ *if and only if* $\phi(I) \subset \phi(J)$.

4. Applications. In this section we use the machinery developed in section 3 and known results of ideal theory of $C^*(X)$ to obtain results about the ideals of $A(Y)$. Examples 4.1, 4.2, 4.6, and 4.9 were proved directly in [**19**], and 4.4 is essentially contained in [**11**]. The remaining results appear to be new. We include a wide variety of examples to illustrate the power of the transfer machinery.

The first four examples deal with the local ideal theory of $A(Y)$. In [**1**] a result very much like Corollary 3.7 was used to derive a metamathematical Transfer Principle, and this device was used to transform results about the ideal theory of $C^*(X)/O^p$ into results about the $P_M{}^*$-restricted ideals of $A(Y)$. In this paper, however, we will use the map τ directly even in our local examples. In addition to the properties of τ which are listed in Theorems 3.5 and 3.6, we make free use of additional properties of τ which follow easily from these theorems. Among these latter properties are the following:

(1) Let $I \in \mathscr{D} - \{(0)\}$. I is prime (respectively primary) if and only if $\tau(I)$ is prime (respectively primary).

(2) Let $I \in \mathscr{D}$. Then $\tau(I^{1/2}) = [\tau(I)]^{1/2}$.

(3) Let $I, J \in \mathscr{D}$. Then $\tau(I : J) = \tau(I) : \tau(J)$.

4.1. *Example.* Let I be $P_M{}^*$-restricted. I is prime if and only if $I = I^2$.

Proof. We may take $I \neq (0)$. By [**18**, Corollary 2.2] an ideal J of $C^*(X)$ that contains a prime ideal is prime if and only if $J = J^2$. If I is prime, so is $\tau(I)$. But then $\tau(I) = [\tau(I)]^2 = \tau(I^2)$, so $I = I^2$.

If $I = I^2$, then $\tau(I) = [\tau(I)]^2$. Hence $\tau(I)$ is prime, and so I is prime.

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4.2. *Example.* Every non-prime primary ideal of $A(Y)$ is either an upper or a lower primary ideal.

Proof. The result is obvious for those primary ideals which are powers of maximal ideals. Any other primary ideal is P_M^*-restricted for some M. The chain of P_M^*-restricted primary ideals is order isomorphic to the chain of primary ideals of $C^*(X)$ that contain O^p. The result follows since every non-prime primary ideal of $C^*(X)$ is either an upper or a lower primary ideal [**15**, Theorem 1].

4.3. *Example.* No proper nonzero P_M^*-restricted ideal is countably generated. (In particular, no nonzero nonmaximal prime ideal of $A(Y)$ is countably generated.)

Proof. Note that for any $I \in \mathscr{I}(A(Y))$, if I is countably generated, then $\tau(I)$ is countably generated. By [**7**, Corollary 5.4] and [**16**, Theorem 2] no ideal J of $C^*(X)$ such that $O^p \subset J \subset M^p$ can be countably generated.

4.4. *Example.* The set of all upper prime ideals of $A(Y)$ properly between two given ones, $P \subsetneqq Q$, is an η_1-set if Q is nonmaximal.

Proof. The set of all upper prime ideals of $C^*(X)$ properly between $\tau(P)$ and $\tau(Q)$ is an η_1-set by [**8**, Theorem 14.19].

The remaining results are of a more global character. Since the restriction of τ to $\mathscr{D}$ is 1-1 and preserves nonzero intersections, it is easiest to derive results about the differentially closed ideals. The transfer process is somewhat less automatic than for local results. For the P_M^*-restricted ideals behave like *all* the ideals of $C^*(X)$ that contain O^p, while the differentially closed ideals correspond to the less well studied regular ideals of $C^*(X)$.

4.5. *Example.* Let $I, J \in \mathscr{D}$.
(1) If I and J are semiprime, then $I + J$ is semiprime.
(2) If $I \neq (0)$ is prime and J is semiprime, then $I + J$ is prime.

Proof. Recall that an ideal of a ring is semiprime if, and only if, it is an intersection of prime ideals. First we show that for an ideal $K \in \mathscr{D}$, K is semiprime if, and only if, $\tau(K)$ is semiprime. Since (0) is semiprime in both rings, we may assume $K \neq (0)$. If K is semiprime, then $K = \bigcap P_\alpha$, where the P_α are nonzero prime ideals. Hence $\tau(K) = \bigcap \tau(P_\alpha)$, so $\tau(K)$ is semiprime. If, conversely, $\tau(K)$ is semiprime, then $\tau(K) = \bigcap Q_\alpha$, where the Q_α are prime ideals of $C^*(X)$. Since $\tau(K)$ is regular, each Q_α is regular, so $Q_\alpha = \tau(P_\alpha)$, where P_α is a nonzero prime ideal of $A(Y)$. We have $\tau(K) = \tau(\bigcap P_\alpha)$, so $K = \bigcap P_\alpha$.

(1) If I and J are semiprime, then $\tau(I + J) = \tau(I) + \tau(J)$ is a semiprime ideal of $C^*(X)$ since the sum of two semiprime ideals of $C^*(X)$ is semiprime [**17**, Lemma 5.1]. Therefore $I + J$ is semiprime.

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(2) If $I \neq (0)$ is prime and J is semiprime, then $\tau(I + J) = \tau(I) + \tau(J)$ is a prime ideal of $C^*(X)$ since the sum of a prime and a semiprime ideal of $C^*(X)$ is prime [**17**, Theorem 5.3]. Therefore $I + J$ is prime.

4.6. *Example.* Let $I \in \mathscr{D}$.
(1) $I = I^2$ if and only if I is an intersection of prime ideals.
(2) If $I = I.I^{1/2}$ or $I = I : I^{1/2}$, then I is an intersection of primary ideals.

These results are true for the absolutely convex ideals of $C^*(X)$ by [**18**, Theorems 2.1 and 2.8]. They are therefore true for the differentially closed ideals of $A(Y)$ by arguments similar to those given in 4.5.

4.7. *Example.* Let $I \in \mathscr{D}$ and $\{I_\alpha\} \subset \mathscr{D}$.
(1) $I.I^{1/2}.(I.I^{1/2})^{1/2} = I.I^{1/2}$.
(2) $(\bigcap I_\alpha)^2 = \bigcap I_\alpha{}^2$.

Proof. (1) Note that if $J \in \mathscr{D}$, then $J^{1/2} \in \mathscr{D}$ since it is an intersection of nonmaximal prime ideals. Therefore each side of the equation we are considering is differentially closed. Now

$$\begin{aligned}\tau[I.I^{1/2}.(I.I^{1/2})^{1/2}] &= \tau(I).\tau(I)^{1/2}.(\tau(I).\tau(I)^{1/2})^{1/2}\\ &= \tau(I).\tau(I)^{1/2} \text{ by [18, Theorem 2.4]}\\ &= \tau(I.I^{1/2}),\end{aligned}$$

and the result follows since $\tau|\mathscr{D}$ is 1-1.

(2) The result is true for the absolutely convex ideals of $C^*(X)$ [**18**, Corollary 2.14]. It therefore holds for the differentially closed ideals of $A(Y)$ by an argument similar to (1).

4.8. *Example.* If $\bigcap P_{M_\alpha}{}^* \neq (0)$, then $\bigcap P_{M_\alpha}{}^*$ is not countably generated.

Proof. Set $J = \bigcap P_{M_\alpha}{}^*$. Then

$$\tau(J) = \bigcap \tau(P_{M_\alpha}{}^*) = \bigcap_{p \in A} M^p,$$

where A is some subset of $\beta X - X$, so $\tau(J)$ is a z-ideal of $C^*(X)$. In [**4**, Corollary, p. 575] the countably generated z-ideals of $C^*(X)$ are described. From the description it is easy to see that a countably generated z-ideal of $C^*(X)$ cannot contain a regular element.

4.9. *Example.* Let $I \in \mathscr{D}$. If I is an intersection of primary ideals, then $I^2 = I.(I : I^{1/2})$. Conversely, if the intersection of all the minimal primary ideals of I is irredundant, then the condition $I^2 = I.(I : I^{1/2})$ implies that I is an intersection of primary ideals.

This result is true for the absolutely convex ideals of $C^*(X)$ by [**18**, Theorems 2.15 and 2.17]. After one observes that the intersection of all the minimal primary ideals of I is irredundant if and only if the same is true for $\tau(I)$, the transfer argument is similar to arguments used above.

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4.10. *Example.* If $\bigcap P_{M_\alpha}{}^* \neq (0)$, then the Krull dimension of $\bigcap P_{M_\alpha}{}^*$ is infinite.

Proof. Set $L = \bigcap P_{M_\alpha}{}^*$. We want to show that for every positive integer n there exists an ascending chain of prime ideals of L of length n. $\tau(L) = \bigcap_{p \in A} M^p$ for some $A \subset \beta X - X$. Call this ideal I^A, and set $F^A = \bigcap_{p \in A} O^p$. Clearly $F^A \subsetneqq I^A$, so by [**5**, Corollary 3.6] there exists a chain $\{J_i\}$ of prime ideals of I^A such that

$$F^A \subsetneqq J_1 \subsetneqq J_2 \subsetneqq \ldots \subsetneqq J_n \subsetneqq I^A.$$

Now each J_i is $Q_i \cap I^A$, for some prime ideal Q_i of $C^*(X)$ by [**5**, Lemma 3.3]. No Q_i is a fixed ideal since F^A is contained in no fixed ideal. Furthermore it can be shown that no more than one of the Q_i can be a free minimal prime ideal. Therefore (by tossing the bad one out if necessary) we can assume that each Q_i strictly contains a free minimal prime ideal. Hence, by 2.5, each Q_i is $\tau(P_i)$ for some prime ideal P_i of $A(Y)$. We have

$$\tau(P_1) \cap \tau(L) \subsetneqq \tau(P_2) \cap \tau(L) \subsetneqq \ldots \subsetneqq \tau(P_n) \cap \tau(L),$$

so

$$P_1 \cap L \subsetneqq P_2 \cap L \subsetneqq \ldots \subsetneqq P_n \cap L;$$

and each $P_i \cap L$ is a prime ideal of L.

References

1. A. Adler and R. D. Williams, *Restricted ideals in rings of analytic functions* (to appear in Bull. Amer. Math. Soc.).

2. N. L. Alling, *The valuation theory of meromorphic function fields over open Riemann surfaces*, Acta Math. *110* (1962), 79–96.

3. ——— *The valuation theory of meromorphic function fields*, Proc. Sympos. Pure Math., vol. *11* (Amer. Math. Soc., Providence, R.I., 1968), 8–29.

4. G. De Marco, *On the countably generated z-ideals of* C(X), Proc. Amer. Math. Soc. *31* (1972), 574–576.

5. W. E. Dietrich, Jr., *On the ideal structure of* C(X), Trans. Amer. Math. Soc. *152* (1970), 61–77.

6. H. Florack, *Reguläre und meromorphe Funktionen auf nicht geschlossenen Riemannschen Flächen*, Schr. Math. Inst. Univ. Münster *1* (1948).

7. L. Gillman, *Countably generated ideals in rings of continuous functions*, Proc. Amer. Math. Soc. *11* (1960), 660–666.

8. L. Gillman and M. Jerison, *Rings of continuous functions*, The University Series in Higher Math. (Van Nostrand, Princeton, N.J., 1960).

9. O. Helmer, *Divisibility properties of integral functions*, Duke Math. J. *6* (1940), 345–356.

10. M. Henriksen, *On the ideal structure of the ring of entire functions*, Pacific J. Math. *2* (1952), 179–184.

11. ——— *On the prime ideals of the ring of entire functions*, Pacific J. Math. *3* (1953), 711–720.

12. E. Hewitt, *Rings of real valued continuous functions, I*, Trans. Amer. Math. Soc. *64* (1948), 54–99.

13. C. W. Kohls, *Prime ideals in rings of continuous functions*, Illinois J. Math. *2* (1958), 505–536.

14. ——— *Prime ideals in rings of continuous functions, II*, Duke Math. J. *25* (1958), 447–458.

https://doi.org/10.4153/CJM-1975-010-3 Published online by Cambridge University Press

15. ——— *Primary ideals in rings of continuous functions*, Amer. Math. Monthly *71* (1964), 980–984.

16. ——— *A note on countably generated ideals in rings of continuous functions*, Proc. Amer. Math. Soc. *12* (1961), 744–749.

17. D. Rudd, *On two sum theorems for ideals of* C(X), Michigan Math. J. *17* (1970), 139–141.

18. R. D. Williams, *Intersections of primary ideals in rings of continuous functions*, Can. J. Math. *24* (1972), 502–519.

19. ——— *Primary ideals in rings of analytic functions*, Trans. Amer. Math. Soc. *178* (1973), 37–49.

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