HOMOLOGY IN THE EVOLUTION OF CLUSTER CORES

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Abstract. It is pointed out that in advanced phases of evolution the cores of clusters ought to evolve homologously and reasons are sought as to why this evolution should be at constant core energy. It is pointed out that continued evolution after infinite core densities have been achieved is an important area for future research.

The aim of this paper is to re-emphasize certain simplifying features of cluster evolution which have not been used in the most powerful modern methods of tackling the problem. Their reintroduction might lead to a simplified theory and a greater understanding. Research problems along these lines are re-emphasized.

At least for large clusters of equal mass stars in advanced stages of evolution the cluster cores have most of their mass at energies so well bound that Maxwell's distribution is a good approximation (Woolley, 1954; King, 1966; Spitzer *et al.*, 1972). The evolution proceeds through the changing temperature and density of this core. Now isothermal gas spheres have the same structure as one another in the sense that a scaling in radius and density suffices to bring their density profiles into the same standard shape. It is thus true that the central cores of clusters evolve almost homologously. This homology may extend even beyond the exactly isothermal energies, but it cannot extend to the whole cluster for reasons outlined below. Hénon's beautiful homological model of a whole cluster was only achieved at the cost of assuming an energy input at the centre (Hénon, 1961). It is important to discover in a neat form the homological structure of the evolving core and to predict the rate of core evolution. The discussion of the thermodynamics of isothermal spheres and their truncations given by Lynden-Bell and Wood (1968) shows that evolution of a well concentrated isothermal is not because of escape; rather it is because of the intrinsic gravothermal instability of the isothermal sphere. This gives one the hope that even the rate of the final dive to very great densities is not dependent on the outer parts of the cluster, but that the core evolves homologously and independently of the halo once it has developed sufficient central concentration.

The gravothermal instability occurs for a truncated isothermal sphere of equal mass stars when the dimensionless energy as measured from the centre is given by $u=\beta(\varepsilon+\psi_0)\sim 8.5$ where we have taken the Maxwellian at low energies to be

$$f \propto \exp(-\beta\varepsilon). \qquad (1)$$

Here $\varepsilon=v^2/2-\psi$ is the specific energy in the gravitational potential $\psi(r)$.

We shall find it useful in what follows to think of a star cluster as stratified in slices of different energy, rather as in stellar evolution a star is considered as stratified in shells of different mass (see Figure 1). For non-isothermal clusters it is useful to define

Hayli (ed.), Dynamics of Stellar Systems, 27–32.

$\beta(\varepsilon)$ an inverse temperature or coolness at each energy by $\beta(\varepsilon) = -\mathrm{d}\log f(\varepsilon)/\mathrm{d}\varepsilon$. We shall denote the least value $\beta(-\psi_0)$ by β_0. A useful dimensionless variable proportional to the excess energy above the lowest value is

$$u = \beta_0(\varepsilon + \psi_0). \tag{2}$$

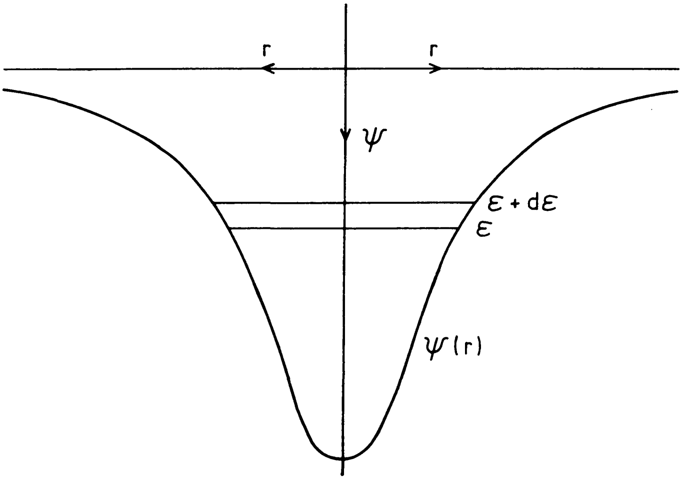

Fig. 1.

Since u is dimensionless it gives a scale invariant way of defining how far up the core a particular energy is. Since the core is evolving homologously the 'edge of the core' in energy will be defined by a particular value of u for all time. For definiteness we shall take this to be $u = 8.5$ by analogy with the place at which the gravothermal instability occurs.

Let the central core density be ϱ_0, the total mass of the core be M_c all of which mass has $u < 8.5$. Further let the internal energy of the core, that is the kinetic energy of these masses less their mutual potential energy, be E_c. Notice this definition has no contribution from the gravitational potential of the rest of the cluster. A good homological model would calculate $\beta_0(t)$, $\varrho(t)$, $M_c(t)$ and $E_c(t)$ from first principles as well as the distribution function $f(u)$. Let us first look at the dimensions of these quantities and the constant of gravity G.

$$\begin{aligned}
G &= [M^{-1}L^3T^{-2}] \\
\beta_0 &= [L^{-2}T^2] \\
\varrho_0 &= [ML^{-3}] \\
M_c &= [M] = [\beta_0^{-3/2}G^{-3/2}\varrho_0^{-1/2}] \\
E_c &= [ML^2T^{-2}] = [\beta_0^{-5/2}G^{-3/2}\varrho_0^{-1/2}] \\
u &= [1]
\end{aligned} \tag{3}$$

By the homology assumption the quantities $G^{3/2}\beta_0^{3/2}\varrho_0^{1/2}M_c$ and $G^{3/2}\beta_0^{5/2}\varrho_0 E_c$ are dimensionless and will thus be time independent. Now as Hénon has pointed out

(Hénon, 1969) there is no escape from an isolated cluster if its evolution is governed solely by a diffusion equation. Thus if we tried to make our whole cluster homologous we would have E and M constant and we could therefore deduce β_0 and ϱ_0 were constant. However there is always evolution in the presence of temperature gradients and there is no finite mass isothermal sphere. Hence our homology assumption applied to an isolated cluster is wrong. If alternatively we try to build a homologous evolution with a tidal mass loss which confines the mean density $M(\frac{4}{3}\pi r_e^3)^{-1}$ within the tidal radius to be constant, then by homology ϱ_0 must be constant. Further during escape by diffusion

$$\dot{E}=M\varepsilon_e=-\dot{M}\frac{GM}{r_e}=-\dot{M}GM^{2/3}\left(\frac{3\varrho_t}{4\pi}\right)^{1/3}. \tag{4}$$

Thus

$$E+\tfrac{3}{5}GM^{5/3}\left(\frac{3\varrho_t}{4\pi}\right)^{1/3}=E+\tfrac{3}{5}\frac{GM^2}{r_e}=constant=E_1. \tag{5}$$

The constant E_1 has the dimensions of energy, so $\beta^{5/2}G^{3/2}\varrho^{1/2}E_1$ must be constant by homology, which implies that β_0 is also constant unless E_1 is zero. If we define $\bar{r}$ by writing $E=(-GM^2/2\bar{r})$ the $E_1=0$ case gives $\bar{r}=\frac{5}{6}r_e$ which is so for a cluster of uniform density and can only be the case for a cluster with a weak central concentration. Homology therefore fails for complete clusters. However, let us return to homology as applied to the cores alone and consider escape from the core. We can write:

$$\dot{E}_c=\dot{M}_c\varepsilon_e-\dot{Q}, \tag{6}$$

where $\dot{Q}$ is the heat flux that comes out of the core and $\dot{M}\varepsilon_e$ is the energy change due to mass loss. We may write $\dot{Q}=\alpha\dot{M}_c\varepsilon_e$ where α is a constant by homology. Using definitions of r_e and $\bar{r}$ defined now for the core rather than the cluster we have

$$\dot{E}_c=\left(\frac{-GM_c}{r_e}\right)(1-\alpha)=\dot{M}_c\frac{E_c}{M_c}\frac{2\bar{r}}{r_e}(1-\alpha). \tag{7}$$

Thus

$$E_c\propto M_c^{\zeta}\quad\text{where}\quad\zeta=\frac{2\bar{r}}{r_e}(1-\alpha). \tag{8}$$

Homology alone will not give the value of ζ without further physical reasoning to give us the value of α, the ratio of the heat loss to the energy carried away by mass loss. There is a weak but not to my mind wholly convincing argument for taking $\alpha=1$ and $\zeta=0$. This argument demands that as each star leaves the core, the core must supply the wherewithal for its removal from core influence. Thus as each star leaves the core there must be just sufficient heat given out to ensure that the energy is available to free it completely from the gravity of the core. This implies.

$$\dot{M}_c\varepsilon_e=\dot{Q} \tag{9}$$

and therefore

$$\dot{E}_c=0. \tag{10}$$

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There is some evidence for the constancy of E_c from the detailed calculations of several investigators (Hénon, 1971, 1973; Spitzer *et al.*, 1972; Spitzer, 1973; Aarseth 1974; Larson, 1970) but I am not yet fully persuaded whether ζ is rather small as one would expect for $r_e \gg \bar{r}$ and $\alpha \sim \frac{5}{6}$ say, or whether ζ is really zero. If indeed E_c is accurately constant, surely there must be simple and fully convincing reason why it must be so. The problem of finding such an argument is an important challenge to theory.

It is perhaps worth recording the consequences of the E_c=constant assumption although equivalent results have been given many times before.

(1) Since E_c and $G^{3/2}\beta_0^{5/2}\varrho_0^{1/2}E_c$ are both constant hence $\beta_0^5\varrho_0$ is constant. Since $\beta_0^{3/2}\varrho_0^{1/2}M_cG^{3/2}$ is constant we have $M_c \propto \beta_0 \propto \varrho_0^{-1/5}$.

(2) Following a well-trodden path and introducing a relaxation time T we have by homology that

$$\frac{d\varrho_0}{dt} = c\frac{\varrho_0}{T}, \tag{11}$$

where c is a constant which depends on the precise definition of T. Now in homological evolution

$$T \propto \frac{v^3}{G^2 m\varrho \log N} \propto \frac{\beta_0^{-3/2}}{G^2 m\varrho_0 \log N}, \tag{12}$$

where m is the stellar mass and N is the number of stars in the cluster. Thus

$$\frac{1}{M_c}\frac{dM_c}{dt} = -\frac{1}{5}\frac{1}{\varrho_0}\frac{d\varrho_0}{dt} = -c_1\beta_0^{3/2}\varrho_0 \log N = -c_2 M_c^{-7/2}\log N \tag{13}$$

if we take N to be the constant number of stars in the whole system or ignore the variation in $\log N$ (see Appendix) we have

$$\begin{aligned} dt &\propto -M_c^{5/2}\, dM_c \\ t_0 - t &\propto M_c^{7/2}. \end{aligned} \tag{14}$$

Thus

$$\begin{aligned} M_c &\propto (t_0 - t)^{2/7} \propto \beta_0 \\ \varrho_0 &\propto (t_0 - t)^{-10/7} \\ r_c &\propto (t_0 - t)^{4/7} \end{aligned} \tag{15}$$

and of course by assumption E_c=constant. Notice that the temperature and density become infinite as $t_0 \to t$ but the mass of the core becomes zero leaving the energy constant.

Similar results can of course be written down for any particular value of ζ. In all cases the calculation of the distribution function and the value of the evolution is a harder task, but Hénon's vintage paper of 1961 shows us the way.

Since no mass is involved in the final dense core we should not stop here, but should be willing to attack the further evolution of clusters after infinite central density is formally achieved. Do they eventually create binaries at the centre and evolve into Hénon's homologous model with the otherwise mysterious energy source at the singularity, or do they continue with a non-homologous evolution? I should point out that Hénon's infinite density homologous model has a finite central temperature in contrast to the constant E_c assumption. At infinite density this can only be achieved with β_0 constant and $\zeta = 1$ and our formula for ζ makes this unreasonable.

What is certain is that the many body problem eventually gets replaced by the few body problem at its very centre. It is likely that in the end Aarseth's heavy central binary may form with a significant fraction of the cluster energy. It will be important to discuss the scenarios after that, bearing in mind that such heavy stars are short lived.

Appendix

Although the theory of the cut off in the calculation for the relaxation time in a cluster of variable density is not well established, it is probably more realistic not to take N constant but instead equal to the number of stars in the core, N_c. It is, I believe, an accident that this actually leads to a slow-down in cluster evolution just before infinite density is achieved because one gets

$$\frac{1}{N_c}\frac{\mathrm{d}N_c}{\mathrm{d}t} = -c_3 N_c^{-7/2} \log N_c$$

and so

$$t_0 - t \propto \int \frac{N_c^{5/2}\,\mathrm{d}N_c}{\log N_c}$$

which may be expressed in terms of the exponential integral; for $N_c \gg 1$ a rough approximation is $\propto N_c^{7/2}/\log N_c$ which is no doubt marginally better then the neglect of the $\log N$ variation, but it is a price not worth paying for the ugly formulae that result. As the real changes occur when the core is reduced to the few body problem the theory is no good at this level anyhow.

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DISCUSSION

Miller: It is always surprising that one thinks that thermodynamics might work for self-gravitating systems. Could you say something about why you think thermodynamics might provide a valid description for a star cluster?

Lynden-Bell: I would consider the only systems to which thermodynamics are not applicable in equilibrium are those which have divergences in phase space. Here the frozen equilibrium concept is important for eliminating binaries and consideration of the core alone eliminates the divergence at infinity.

Feix: Validity of thermodynamics for physical systems is not only connected to infinities (divergences) for large negative energies (or zero energy) but is a more general question which arises for all systems. In fact it is a question of how the 'total information' ($6N$ 'data') can be reduced to a few characteristics numbers (how and which are these numbers). Thermodynamics of equilibrium systems tells us that density and temperature are enough but the question is when can we tell that a system is in equilibrium. Plasma and self gravitating gas imply certainly more sophisticated information, probably non local, maybe non Markovian (i.e. implying the past history of the system). It is interesting to notice that this question is both fundamental in statistical physics and very practical in computational physics where obviously the amount of returned information must be finite.