TURBULENCE : DETERMINISM AND CHAOS

Y. POMEAU

CEA, DPhT, Gif sur Yvette, FRANCE

After the article of Ruelle and Takens[1], there has been recently much interest in the problem of the "onset of turbulence". That is, instead of trying to understand the structure of a well established turbulence flow, one studies the way in which a flow "jumps" from a quiet stable laminar state to a turbulent state when its Reynolds (or Rayleigh) number increases.

The idea of turbulence is connected with the one of "chaos". The ergodic theory[2] allows one to give a precise content to this last notion (Take care that this differs from the one given in the review paper by May[3], see also[4]). One first considers a time dependent quantity, say u(t), as, for instance, the fluid velocity at a given point in a turbulent flow of fluid under constant (or periodic, or eventually stationnary "in average") external conditions, in such a way that one may define a gliding average as

$$<\phi[u(t)]> \;=\; \lim_{T \to \infty} \frac{1}{T} \int_t^{t+T} dt' \; \phi(u(t')),$$

where ϕ is any smooth function.

We assume that these averaged quantities are independent of the initial conditions, at least for "almost" any choice of them and that they do not depend on t. The "signal" u(.) has the property of *mixing*, that we shall consider as defining the *chaos* if :

$$<\psi[u(t)]\,\Phi[u(t+t')] \;-\; <\psi><\Phi>> \;\underset{t'\to\infty}{\to}\; 0$$

for any smooth ψ and Φ . This property expresses the idea that, after a sufficiently long interval of time, say t', the system "forgets" the detail of the initial conditions (= the fluctuations of u at two very distant times are uncorrelated).

A good example of such a "chaotic" signal, with astrophysical implications, is provided by the time dependence of the magnetic field of earth[5]. The geological data show that the earth's magnetic dipole has reversed a large number of times. It is of interest to know whether these reversals occur "regularly" or at random. For that purpose, let us consider the autocorrelation function which is built up from the data[6] as follows : Φ and ψ represent the same function. This function is equal to +1 when the polarity is the same as now, and to (-1) in the reversed case. The autocorrelation function of this random signal is given in Fig. 1. It shows rather clearly that, from this point of view, the reversals are at random, and the reversals follow approximately a Poisson law.

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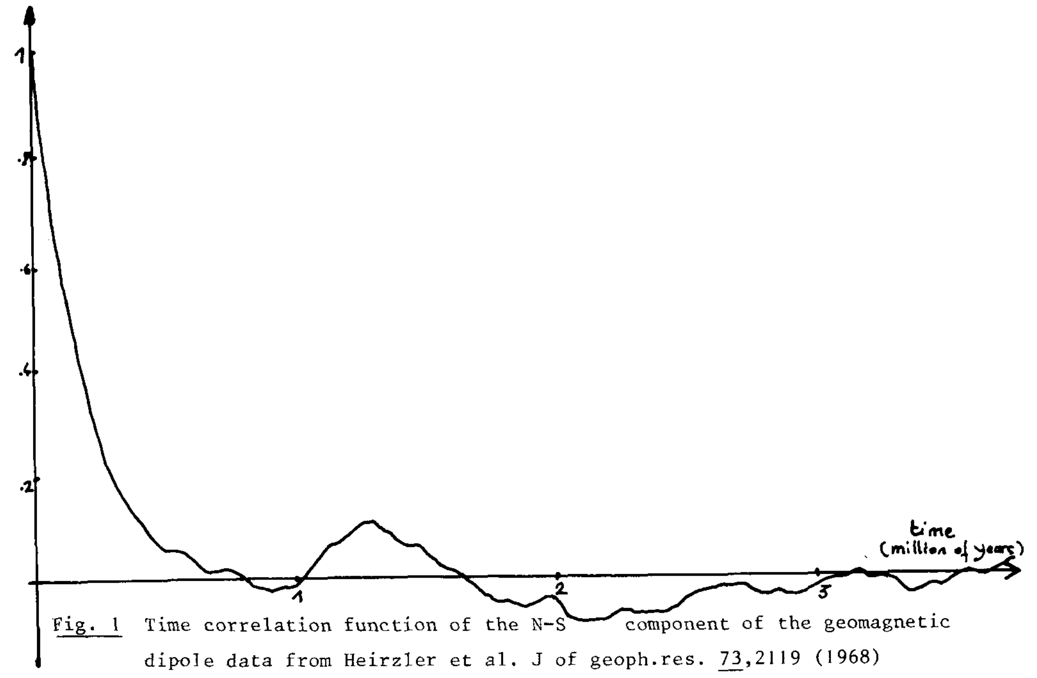

Fig. 1 Time correlation function of the N-S component of the geomagnetic dipole data from Heirzler et al. J of geoph.res. 73,2119 (1968)

There is often a misunderstanding about this idea of chaos, which is implicitly connected with the one of "noise". At least, at the level of the equations of the motion, it is often thought that chaos *must* be introduced by some noise source and that chaos may exist in non-deterministic systems only (as, say, a damped harmonic oscillator in contact with a heat bath). In order to understand how chaos may arise very simply from a *deterministic* process, let us consider a "discrete" dynamical system. This dynamical system mimics a system depending continuously on time, wherein measurements are made at discrete instants, say t_1, $t_1+\tau$, $t_1+2\tau$, ..., $t_1+n\tau$,.... Thus we shall define for this dynamical system a "variable" and a time translation operator (that is an operator which allows one to jump from the value of the variable at any time t to its value at time $t+\tau$). This is a *dynamical system* if the transformation acts continuously and is inversible, in mathematical terms it is a *homeomorphism*) (= time can be reversed to get the initial data from the final state). To define the variable of our dynamical system, we consider a set K with a finite number, say k, of elements and the doubly infinite sequences of elements of K :

$$u_t = \left\{\ldots i_{-n}, \ldots\; i_{-1}, i_o, i_1, \ldots, i_n, i_{n+1}, \ldots\right\} \quad ,$$

where i_j $(j = 1,2,\ldots,k) \in K$. Thus, giving the initial data $u(t_1)$, means that a particular sequence is known. The transformation that allows one to find u_{t+T} , once u_t is given, is just the *shift* of the sequence ; by definition :

$$i_n(u_{t+T}) = i_{n-1}(u_t) \quad .$$

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If we consider now the class of functions of u defined by

$$\psi(u) = \sum_{m=-\infty}^{+\infty} \psi_m(i_m)$$

with

$$\sum_{m=-\infty}^{+\infty} \left| \max_{i_m \in K} \psi_m(i_m) \right|^2 < \infty \quad ,$$

it is easy to see that, if the i_m's are taken at random in K with the probabilities $p_1,\ldots,p_k$ (k = cardinality of K) such as $\sum_{j=1}^{k} p_j = 1$, then

$$\langle\psi(u)\rangle = \sum_{m=-\infty}^{+\infty} \sum_{n=1}^{K} p_n \psi_m(y_n) \quad ,$$

where y_n is the n^{th} element of K , and

$$\langle[\psi(u_t) - \langle\psi\rangle] [\Phi(u_{t+N}) - \langle\Phi\rangle]\rangle \xrightarrow[N\to\infty]{} 0 \quad ,$$

which proves the property of mixing for our discrete (and *deterministic*) system. Actually this last property expresses a very simple fact : ψ and Φ depend on terms of the infinite sequence $\{i_m\}$ which are located in a fixed part of this sequence. Shifting at each step this sequence on the left, one "loses " part of the knowledge of the values of the $\{i_m\}$ in this region, as new i_m's come from the right which are uncorrelated with the already known i_m's . Obviously this double infinite sequence looks very much as the k-ary expansion of a real number (except that it is doubly - instead of singly - infinite). This helps to understand that, in a mixing dynamical system, the noise source might just be the infinite (say decimal) expansion of the real numbers defining the initial datas.

Of course this notion of *temporal* chaos is not sufficient to define turbulence, as another typical feature of turbulence in unbounded flows is the absence of *spatial* correlation with an infinite range. There is an obvious extension of the mixing property to the *spatial* case, that is a position dependent function $u(\vec{r})$ has this mixing property iff

$$\langle\psi[u(\vec{r})]\ \phi[u(\vec{r}+\vec{R})] - \langle\psi\rangle\ \langle\phi\rangle\rangle \xrightarrow[|\vec{R}|\to\infty]{} 0 \quad ,$$

where the averages are now to be understood as gliding space averages. This definition implies, of course, that the flow is unbounded in some direction and that the turbulent state is invariant under the translations along this direction. Although there is clear evidence[7] that the turbulent flows have the mixing property both in time *and* space, we shall only consider the time dependent properties, as the connection between spatial chaos and the original non linear Navier-Stokes equation is rather unclear at the present time.

On the contrary, if one neglects completely this question of the spatial structure, one is led to consider the fluid motion as described by the solution of a system of *ordinary* differential equations (O.D.E.). Of course, one is mainly interested in the *qualitative* properties of these O.D.E., as the fluid equations cannot be replaced by a fully equivalent system of a finite number of O.D.E. with an explicit form. This qualitative theory of the O.D.E. has been the subject of detailed investigations[8,9], in particular one may understand quite well how the solutions of such a system may have the property of mixing.

Recently Ruelle and Takens[1] have drawn attention to the possible connection between the onset of turbulence in flows and some bifurcation properties of O.D.E. . Without going into too many details, I will just explain what is presently known about the onset of turbulence. Following Martin and McLaughlin[10], one may consider three different ways for the occurrence of turbulence .

1. *The onset of turbulence in the Lorenz system*

By a drastic reduction of the Oberbeck-Boussinesq equations for a flow convecting in a horizontal layer, Lorenz[11] has obtained the following system of O.D.E. :

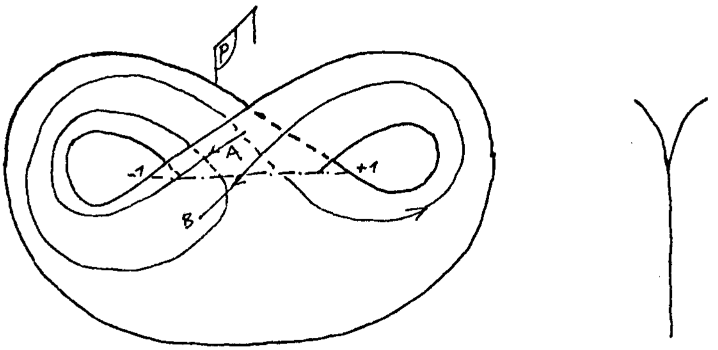
$$\text{1.a} \quad \frac{dx}{dt} = \sigma(y-x)$$

$$\text{1.b} \quad \frac{dy}{dt} = -xz + rx - y$$

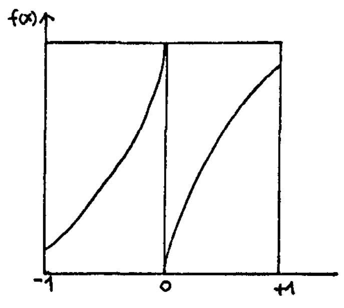
$$\text{1.c} \quad \frac{dz}{dt} = xy - bz \quad ,$$

where σ , r and b are numerical parameters. By numerical computations he has shown that in some range of values of these parameters the motion described by these equations is chaotic and that the trajectory, after some transient, reaches very rapidly an "attractor", which does not depend on the initial conditions. This attractor is very interesting, as it is presumably structurally stable, that is it exists (and remains attracting) for values of the parameters in open intervals. The idea of structural stability is actually much more general[12], for O.D.E., like the system (1), it just means[13] that one may add a small perturbation depending on x , y , z on the right hand side, without changing the topology of the velocity field defined by this system .This means that by a homeomorphic mapping of space (i.e. a change of variables $\vec{x}' = f(\vec{x})$ that is both continuous and inversible) one may change the trajectories of the perturbed system into the ones of the unperturbed system. The structural stability is essential, as it means that the properties of the system under consideration do not depend on the details of the equations, and that they remain essentially the same under any kind of (small enough, but *finite*) perturbation (actually one thinks of perturbations arising from a lack of knowledge of the exact form of the equations).

The structure of the Lorenz attractor has been recently studied by a number of authors(14), and I will just give a brief (and I hope clear enough) account of these works. A simple way to describe it is the one of Williams(14). He first considers a "semi-flow", that is a flow on a surface where two sheets may collapse. This cannot really represent the solution of O.D.E., as the motion cannot be traced back unambiguously, as it should be allowed for O.D.E. But this helps to understand the structure of the Lorenz attractor. I have tried to draw as clearly as possible this surface on Fig.2 . It has two holes and the line along which the two sheets collapse is the dashed line. On the right part of the figure is the section of the surface by the mid-vertical plane.



The trajectories run on this surface approximately as follows :
They revolve around each hole by diverging slowly and if at one of these revolutions the trajectory cut the dashed line beyond the middle point , it is inserted at the next turn close to the other hole and revolves by diverging slowly around this "new" hole. Finally it jumps at random from a hole to the other. To understand why this motion is non periodic, one considers the so-called Poincaré transform on the shaded segment where the two sheets merge. Let us define on this segment a coordinate, say x which varies between -1 and $+1$. The Poincaré transform defines a function $f(x)$, $-1 \leq f(x) \leq +1$ if $-1 \leq x \leq +1$: if the trajectory crosses the shaded line at x , its next crossing will be at $f(x)$. The function $f(x)$ has a discontinuity at $x = 0$, and looks approximately as represented in Fig.3 .



If one assumes that its derivative (when it exists) is everywhere larger than 1, then it is clear that the application $x \to f(x)$ cannot have any stable fixed point or even any stable period*. It is important to understand that this is possible only because $f(x)$ has a discontinuity, otherwise if f and its derivative were continuous $|f'(x)|$ could be not everywhere larger than 1, if $f[-1,1] \subset [-1,1]$.

Now the question of the mixing character of the motion is turned into the question of the mixing character of the application $x \to f(x)$. This mixing property is rather obvious (if one does not want to get a rigorous proof) : consider two points $\{x_1, x_2\}$ very close to each other, and their successive transform :$\{f(x_1), f(x_2)\}$:

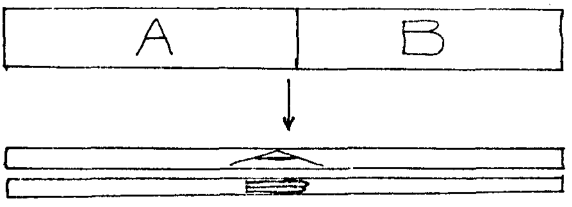
$$\left\{f[f(x_1)],\ f[f(x_2)]\right\},\ldots;\ \left\{f^{(n)}(x_1),\ f^{(n)}(x_2)\right\},\ldots$$

(By definition $f^{(n)}(x) = f[f^{(n-1)}(x)]$ and $f^{(1)} \equiv f$) . The distance between the tansforms of x_1 and x_2 is multiplied after each application of f by a quantity larger than min $|\frac{df}{dx}|$ that is larger than 1, thus it increases at least exponentially. This means that, at some time, the two image points will be separated by the discontinuity of f, and the subsequent trajectories starting from x_1 or x_2 will be completely different from each other. This is a version of the mixing property : a small fluctuation in the initial conditions yields, after some time, a huge difference in the arrival points ; in other terms unless one knows the initial conditions with an infinite accuracy, the motion is unpredictible after some *finite* time.

Let us come back now to a more realistic description of the Lorenz attractor I have already noted that it cannot be considered as a surface in the usual sense, since two sheets cannot merge owing to the deterministic character of the equations of the motion. To understand what really happens, it is only necessary to replace the shaded line where the two sheets collapse by a small surface parallel to this line (as a thin stick).Now the Poincaré transform is no longer given by a function of one variable, but by a plane transform: that is, given a starting point inside the stick, one wonders what is the next crossing point of the trajectory inside this stick.Essentially (although things are a little bit more complicated), the Poincaré transform looks very much like a Baker's transform [15] : the stick is first cut in two pieces, (this cutting remembers of course very much the discontinuity in f), each piece is stretched and the two resultant pieces are put together inside the stick (see fig. 4). The transform of the coordinate parallel to the stick is very similar to the one dimensional transform represented in fig. 3, but now the Poincaré transform has been made invertible as the coordinate perpendicular to x allows one to

*Actually Lasota and Yorke (Trans. of the A.M.S. 186, 481 (1973)) show that there is an absolutely continuous invariant measure for such f, and f is ergodic for this measure.

distinguish between the arrival points with the same x. By a repeated action of the transform pictured on Fig. 4, one obtains a section of the attractor, which is the "object" stable under an infinite number of applications of Poincaré transform. If one cuts this section by a line perpendicular to the axis of the stick, one easily sees that, after each application of the transform, the central part of the stick is deleted, then the central part of the remaining segments is deleted, and so on. This is precisely the way in which one generates Cantor sets. Accordingly the Lorenz attractor has the structure of a Cantor set perpendicular to its "surface". It is a surface with a number of sheets which has the power of a continuum.



Let us notice that these properties of the Lorenz attractor have actually not been *proved* from a rigorous study of the system (1), although it is a very reasonable extrapolation from the computer studies.

Another important feature of the Lorenz attractor is that it appears by an inverted bifurcation from a pair of stable fixed points : in a domain of values of the parameters (r,σ,b), one reaches, from some initial conditions, one of the stable fixed points, or/and some other Lorenz attractor. This manner of occurrence of turbulence is well known, for instance in Poiseuille flow[16]; in a range of values of the Reynolds number the laminar flow is linearly stable, but unstable against perturbations with a finite amplitude, and at the upper limit of this domain the laminar flow becomes linearly unstable. It must be stressed that the stability of convection flows is much less well known than the one of the Poiseuille flow, so that it is not clear if the Lorenz system describes, even in a rough way, the bifurcation toward turbulence of convective flows. From this point of view the existence (or the non existence) of two sorts of flows for the same value of the Rayleigh number would be an important test.

2. *The theories of Landau-Hopf and of Ruelle-Takens*

Landau[17] and Hopf have given a quite convincing picture of the onset of turbulence. In order to understand their idea, it is necessary to introduce the notion of *quasiperiodic function*. Let us consider a dynamical system with a periodic limit cycle. The existence of such an oscillatory behaviour is known to occur[18] in convective flows.

The theory[19] shows that, by a certain type of bifurcation (called the Hopf bifurcation), this limit cycle may give rise to a motion with two incommensurate frequencies, say ω_1 and ω_2 (these frequencies are incommensurate if no non zero integers p and q exist such that $p\omega_1 = q\omega_2$). Then any function of time in this flow should be *quasiperiodic*. To define such a function, let us consider a function of two variables, say t_1 and t_2, which is periodic of period 2π with respect to each of the variables:

$$f(t_1 + 2\pi n,\ t_2 + 2\pi m) \quad = \quad f(t_1, t_2)$$

whatever the integers n and m are.

From this function we may build the quasiperiodic function $\phi(t) = f(\omega_1 t, \omega_2 t)$. If ω_1 and ω_2 are incommensurate, this function will *appear* (at least at first sight) as completely choatic, although it is not *chaotic* in the sense of the mixing property. Its frequency spectrum is concentrated at the frequencies ω_1, ω_2, and more generally at any linear combination $p\omega_1 + q\omega_2$ with integer coefficients.

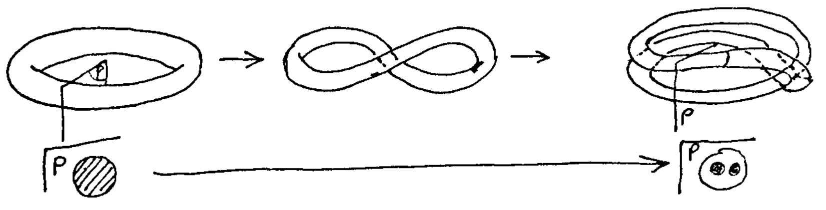
The extension of this construction to a function that depends periodically with the period 2π on any set of variables, say $t_1, t_2, \ldots, t_n$, allows one to define the most general quasiperiodic function which has *not* the mixing property.

One must take care that such a quasiperiodic behaviour is *not* structurally stable (except for the case of a single period). If a parameter as, say, the Rayleigh number varies in the domain of quasiperiodic behaviour, then a *periodic* (and *structurally stable*) limit cycle should be reached every time when ω_1 and ω_2 (which both depend on the Rayleigh number) are commensurate. The non periodic behaviour is reached for *isolated* values of the parameter R_a only[20]. The idea of Landau and Hopf is the following: when the Rayleigh number increases many new bifurcations occur, which always correspond to frequencies incommensurate with the already existing ones. Then, one can show that, if the quasiperiodic function depends actually on all these frequencies, then it tends toward a chaotic signal (in the sense of the mixing property) after an infinite number of frequencies have appeared.

One may wonder first about the validity of this theory, as it is unclear whether an infinite or finite Rayleigh (or Reynolds) number is required for the occurrence of an infinite number of bifurcations.

Landau and Lifshitz[21] relate the number of "degrees of freedom" of a turbulent motion with its Reynolds number R_e. Asymptotically this number should increase as $(R_e)^{9/4}$, so that the existence of an infinite number of degrees of freedom requires an infinite Reynolds number. They assume that each "degree of freedom" is actually connected with the freedom in the choice of the phase at a bifurcation where a new frequency appears. Although this notion of "degree of freedom" is widely used in theoretical physics, the way in which the $(R_e)^{9/4}$ formula of Landau Lifshitz counts the number of bifurcations between $R_e = 0$ and a large value of R_e is rather unclear.

On the other hand following Ruelle and Takens[1], the onset of turbulence, as described by Landau and Lifshitz, cannot be a "generic" phenomenon. They show that, after the occurrence of a few non commensurate frequencies, the next bifurcation is toward a non periodic attractor. This non-periodic attractor is built as follows : let us consider the case of a quasiperiodic motion with four incommensurate frequencies, if one represents the trajectory by the motion of a point in a four dimensional space, it intersects a 3-dimensional hyperplane (i.e. a usual 3-d space) following a full torus. Again one considers that the motion describes a one to one application of this torus into itself (that is, each point inside the torus has an image which is the next crossing of the trajectory with the hyperplane). It is possible[22] to find a transformation of the torus into itself that is continuous, invertible , and which transforms the torus into a strange attractor after an infinite number of applications (see fig. 5). This attractor is also structurally stable.



There have been attempts[23] to find if this picture of the onset of turbulence is valid for Taylor instabilities, it is not completely clear whether the experimental findings are or not in agreement with this theory.

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3. *Approach of chaos by succesive bifurcations*

For some values of the parameters, the Lorenz system (1) has one (or two) periodic limit cycle[24]. When a parameter such as r varies in this domain, this limit cycle becomes very rapidly much more complicated by a mechanism of "cascading bifurcations". These bifurcations are rather striking as they describe a smooth transition from a periodic limit cycle with a single period toward a strange (i.e. non-periodic) attractor.

For $r \geq r_1$, the period of the limit cycle is, say T ; for r just below r_1(bifurcation point) the period is 2T ; but as the motion is anharmonic, the amplitude of the Fourier component of frequency 1/2T start from a zero value at r_1, then increases continuously as r becomes smaller than r_1. In order to make clear possibility of this mechanism of frequency division, it is enough to draw a closed trajectory (= the limit cycle) in the space of the variables x,y,z (Fig. 6), when r becomes just a little smaller than r_1, this closed curve with *single orb* becomes a closed curve with *two orbs*, as drawn (approximately) in fig. 7 .

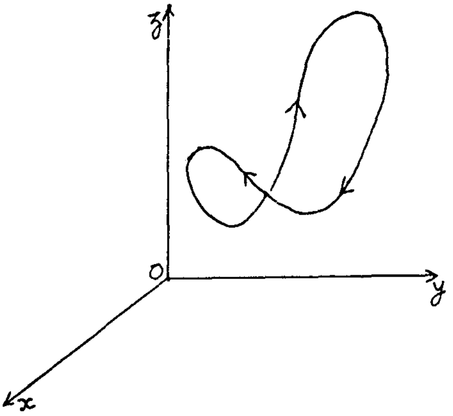

Fig. 6

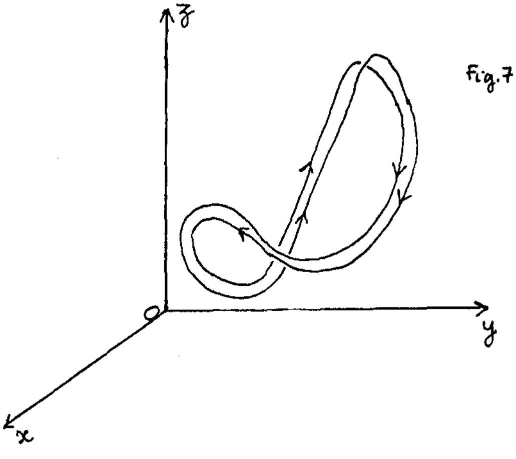

Fig. 7

There is a series of such splitting of the limit cycle when r decreases, so that the period of the limit cycle becomes 2T, 4T, ... 2^nT. There is an infinite number of such bifurcations when r decreases from r_1 to, say r_∞ . The limit r_∞ is approached in a geometric fashion : let r_{2^n} be the value of r which the limit cycle of period 2^nT becomes unstable giving birth to a limit cycle of period 2^{n+1}T, then the quantity $(r_{2^n} - r_\infty)/(r_{2^{n+1}} - r_\infty)$ tends to a limit as n increases, which is an universal number (= independent of the detail of the equations) approximately equal to 0.2141 693. At the end of these bifurcations (i.e. at r_∞ , and for lower values of r) the period of the motion is infinite,which means that this motion is chaotic. Actually, the recurrence time for a given point on the limit cycle is the period. It is easy to see that the autocorrelation function for such a periodic system must take the same finite value at separation time of one period, two periods,... N periods, which forbids it to tend to zero at infinite separation times (so that a periodic system is obviously not *chaotic*). On the contrary, if the period of the motion is infinite, the autocorrelation function may tend to zero for large separation times, and the motion may be chaotic.

At the present time,these are only theoretical examples for this occurrence of chaos by an infinite number of bifurcations in a finite domain of variation of the parameters. However this is probably[25] the best understood case.

CONCLUSION

Even if the proofs of theorems are quite remote, there is some hope at the present time for understanding the way in which turbulence may occur in flows (convective or not). However let us emphasize again that this approach leaves aside the question of spatial chaos. Further studies in this domain are needed in order to know whether the spatial chaos occurs or not in the infinite Rayleigh (or Reynolds) number limit,as required by the Landau theory of turbulence.

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