INTEGRATION OF E-FUNCTIONS WITH RESPECT TO THEIR PARAMETERS

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(Received 15th January, 1959)

1. Introductory. A number of integrals of E-functions with respect to their parameters have been given by Ragab [**1, 5**]. Two further integrals of this type are given in § 2 and § 3. In § 4 it is shown that these can be employed to sum series of products of E-functions.

The four following formulae will be made use of in the proofs [**2**].

If $R(\rho_{q+1}) > R(\alpha_{p+1}) > 0$,

$$\int_0^1 \lambda^{\alpha_{p+1}-1}(1-\lambda)^{\rho_{q+1}-\alpha_{p+1}-1} E(p\,;\ \alpha_r:\ q\,;\ \rho_s:\ z/\lambda)\,d\lambda$$

$$= \Gamma(\rho_{q+1}-\alpha_{p+1})E(p+1\,;\ \alpha_r:\ q+1\,;\ \rho_s:\ z). \quad\text{.........(1)}$$

If $R(\alpha_{p+1}) > 0$,

$$\int_0^\infty e^{-\lambda}\lambda^{\alpha_{p+1}-1} E(p\,;\ \alpha_r:\ q\,;\ \rho_s:\ z/\lambda)\,d\lambda = E(p+1\,;\ \alpha_r:\ q\,;\ \rho_s:\ z). \quad\text{.........(2)}$$

If $|\operatorname{amp} z| < \pi$,

$$E(\alpha::z) = \Gamma(\alpha)(1+1/z)^{-\alpha} = z^\alpha E(\alpha::1/z). \quad\text{.........................(3)}$$

If $|\operatorname{amp} z| < \pi$,

$$E(p\,;\ \alpha_r:\ q\,;\ \rho_s:\ z) = \frac{1}{2\pi i}\int \frac{\Gamma(\zeta)\Pi\Gamma(\alpha_r-\zeta)}{\Pi\Gamma(\rho_s-\zeta)} z^\zeta\,d\zeta, \quad\text{.......................(4)}$$

where the integral is taken up the η-axis, with loops, if necessary, to ensure that the pole at the origin lies to the left and the poles at $\alpha_1, \alpha_2, \ldots, \alpha_p$ to the right of the contour. Zero and negative integral values of the parameters are excluded and the α's must not differ by integral values. If $p < q+1$, the contour is bent to the left at both ends. If $p > q+1$, the formula is valid for $|\operatorname{amp} z| < \frac{1}{2}(p-q+1)\pi$.

2. The first integral. If $p \geqslant q$,

$$|\operatorname{amp} z| < \tfrac{1}{2}(p-q+2)\pi,\quad R(\delta-\alpha-\beta) > 0,\quad R(\rho_n) > R(\alpha_n) > 0\quad (n = 1, 2, \ldots, q),$$
$$R(\alpha_n) > 0\quad (n = q+1, \ldots, p),$$

$$\frac{1}{2\pi i}\int \frac{\Gamma(\zeta)\Gamma(\alpha-\zeta)\Gamma(\beta-\zeta)}{\Gamma(\delta-\zeta)} E\left(\begin{matrix}\delta-\alpha-\beta,\ \alpha_1-\zeta,\ \ldots,\ \alpha_p-\zeta:\ z\\ q\,;\ \rho_s-\zeta\end{matrix}\right) z^\zeta\,d\zeta$$

$$= \frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\delta-\alpha-\beta)}{\Gamma(\delta-\alpha)\Gamma(\delta-\beta)} E\left(\begin{matrix}\delta-\alpha,\ \delta-\beta,\ \alpha_1,\ \ldots,\ \alpha_p:\ z\\ \delta,\ \rho_1,\ \ldots,\ \rho_q\end{matrix}\right), \quad\text{......(5)}$$

the path of integration being as in (4), with loops, if necessary, to ensure that α and β are to the right of the contour.

In proving (5) note that, in virtue of (1) and (2), the integral can be put in the form

$$\frac{1}{2\pi i}\int \frac{\Gamma(\zeta)\Gamma(\alpha-\zeta)\Gamma(\beta-\zeta)}{\Gamma(\delta-\zeta)} z^{\zeta}\, d\zeta \left[\prod_{n=1}^{q}\Gamma(\rho_n-\alpha_n)\right]^{-1} \prod_{n=1}^{q}\int_0^1 \lambda_n^{\alpha_n-\zeta-1}(1-\lambda_n)^{\rho_n-\alpha_n-1}\, d\lambda_n$$
$$\times \prod_{n=q+1}^{p}\int_0^{\infty} e^{-\lambda_n}\lambda_n^{\alpha_n-\zeta-1}\, d\lambda_n\, E(\delta-\alpha-\beta : : z/(\lambda_1 \ldots \lambda_p)).$$

Here change the order of integration, putting the first integral last, and get

$$\left[\prod_{n=1}^{q}\Gamma(\rho_n-\alpha_n)\right]^{-1} \prod_{n=1}^{q}\int_0^1 \lambda_n^{\alpha_n-1}(1-\lambda_n)^{\rho_n-\alpha_n-1}\, d\lambda_n$$
$$\times \prod_{n=q+1}^{p}\int_0^{\infty} e^{-\lambda_n}\lambda_n^{\alpha_n-1}\, d\lambda_n\, \Gamma(\delta-\alpha-\beta)\left(1+\frac{\lambda_1\ldots\lambda_p}{z}\right)^{\alpha+\beta-\delta}$$
$$\times \frac{1}{2\pi i}\int \frac{\Gamma(\zeta)\Gamma(\alpha-\zeta)\Gamma(\beta-\zeta)}{\Gamma(\delta-\zeta)}\left(\frac{z}{\lambda_1\ldots\lambda_p}\right)^{\zeta} d\zeta.$$

Now, from (4), the last line is equal to

$$E\begin{pmatrix}\alpha,\beta: & z/(\lambda_1\ldots\lambda_p)\\ \delta & \end{pmatrix} = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\delta)} F\begin{pmatrix}\alpha,\beta; & -(\lambda_1\ldots\lambda_p)/z\\ \delta & \end{pmatrix}$$
$$= \left(1+\frac{\lambda_1\ldots\lambda_p}{z}\right)^{\delta-\alpha-\beta}\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\delta-\alpha)\Gamma(\delta-\beta)} E\begin{pmatrix}\delta-\alpha,\delta-\beta: & z/(\lambda_1\ldots\lambda_p)\\ \delta & \end{pmatrix};$$

and, on applying (1) and (2), the result is obtained.

Note. The restrictions on the ρ's can be omitted, as the paths of integration from 0 to 1 can be replaced by contours starting from 0, passing round 1, and returning to 0.

3. The second integral. If $p \geqslant q$, $|\operatorname{amp} z| < \frac{1}{2}(p-q+2)\pi$, $R(\alpha_n) > 0$ $(n = 1, 2, \ldots, p)$,

$$\frac{1}{2\pi i}\int \Gamma(\zeta)\Gamma(\beta-\zeta)E(\gamma, \alpha_1-\zeta, \ldots, \alpha_p-\zeta : q ; \rho_s-\zeta : z)z^{\zeta}\, d\zeta$$
$$= B(\beta,\gamma)E(\beta+\gamma, \alpha_1, \ldots, \alpha_p : q ; \rho_s : z), \qquad (6)$$

where the path of integration is that of (4), with a loop, if necessary, to ensure that β lies to the right of the contour.

The integral is equal to

$$\frac{1}{2\pi i}\int \Gamma(\zeta)\Gamma(\beta-\zeta)z^{\zeta}\, d\zeta \left[\prod_{n=1}^{q}\Gamma(\rho_n-\alpha_n)\right]^{-1} \prod_{n=1}^{q}\int_0^1 \lambda_n^{\alpha_n-\zeta-1}(1-\lambda_n)^{\rho_n-\alpha_n-1}\, d\lambda_n$$
$$\times \prod_{n=q+1}^{p}\int_0^{\infty} e^{-\lambda_n}\lambda_n^{\alpha_n-\zeta-1}\, d\lambda_n\, E(\gamma : : z/(\lambda_1\ldots\lambda_p))$$
$$= \left[\prod_{n=1}^{q}\Gamma(\rho_n-\alpha_n)\right]^{-1} \prod_{n=1}^{q}\int_0^1 \lambda_n^{\alpha_n-1}(1-\lambda_n)^{\rho_n-\alpha_n-1}\, d\lambda_n$$
$$\times \prod_{n=q+1}^{p}\int_0^{\infty} e^{-\lambda_n}\lambda_n^{\alpha_n-1}\, d\lambda_n\, E(\gamma : : z/(\lambda_1\ldots\lambda_p))\frac{1}{2\pi i}\int \Gamma(\zeta)\Gamma(\beta-\zeta)\left(\frac{z}{\lambda_1\ldots\lambda_p}\right)^{\zeta} d\zeta.$$

But the last integral is equal to

$$2\pi i\, E(\beta : : z/(\lambda_1\ldots\lambda_p)),$$

and, from (3),

$$E(\beta:\,:z/(\lambda_1\ldots\lambda_p))E(\gamma:\,:z/(\lambda_1\ldots\lambda_p)) = B(\beta,\gamma)E(\beta+\gamma:\,:z/(\lambda_1\ldots\lambda_p)).$$

Hence, on applying (1) and (2), the result is obtained.

4. Summation of series. Special cases of (5) and (6) are

$$\frac{1}{2\pi i}\int\frac{\Gamma(\zeta)\Gamma(\alpha-\zeta)\Gamma(\beta-\zeta)}{\Gamma(\delta-\zeta)}E(\delta-\alpha-\beta,\gamma-\zeta:\,:z)\,z^{\zeta}\,d\zeta$$
$$=\frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\delta-\alpha-\beta)}{\Gamma(\delta-\alpha)\Gamma(\delta-\beta)}E(\delta-\alpha,\,\delta-\beta,\gamma:\;\delta:\;z),\quad\ldots\ldots\ldots(7)$$

where $|\operatorname{amp} z| < \frac{3}{2}\pi$, $R(\delta-\alpha-\beta) > 0$, $R(\gamma) > 0$; and

$$\frac{1}{2\pi i}\int\Gamma(\zeta)\Gamma(\beta-\zeta)E(\gamma,\alpha-\zeta:\,:z)\,z^{\zeta}\,d\zeta = B(\beta,\gamma)E(\alpha,\,\beta+\gamma:\,:z),\quad\ldots\ldots\ldots(8)$$

where $|\operatorname{amp} z| < \frac{3}{2}\pi$, $R(\alpha) > 0$.

These may be employed to sum two series given by Ragab.
The first [3] is

$$\sum_{r=0}^{\infty}\frac{z^{-2r}}{r!\,\Gamma(\gamma+r)}E(\gamma+r,\,\alpha+\beta-\delta+r:\,:z)\,E(\gamma+r,\,\delta-\alpha+r,\,\delta-\beta+r:\;\delta+r:\;z)$$
$$=\frac{\Gamma(\delta-\alpha)\Gamma(\delta-\beta)\Gamma(\alpha+\beta-\delta)}{\Gamma(\alpha)\Gamma(\beta)}E(\alpha,\,\beta,\gamma:\;\delta:\;z),\quad\ldots\ldots\ldots(9)$$

where $|\operatorname{amp} z| < \frac{3}{2}\pi$, $R(\alpha+\beta-\delta) > 0$, $R(\gamma) > 0$.

To prove this, substitute on the left from (4), so getting

$$\sum_{r=0}^{\infty}\frac{z^{-2r}}{r!\,\Gamma(\gamma+r)}\frac{1}{2\pi i}\int\Gamma(\zeta)\Gamma(\gamma+r-\zeta)\Gamma(\alpha+\beta-\delta+r-\zeta)\,z^{\zeta}\,d\zeta$$
$$\times\frac{1}{2\pi i}\int\Gamma(w)\frac{\Gamma(\gamma+r-w)\Gamma(\delta-\alpha+r-w)\Gamma(\delta-\beta+r-w)}{\Gamma(\delta+r-w)}z^{w}\,dw.$$

Here replace ζ and w by $\zeta+r$ and $w+r$, and change the order of integration and summation. Then the expression becomes

$$\frac{1}{\Gamma(\gamma)}\frac{1}{2\pi i}\int\Gamma(w)\frac{\Gamma(\gamma-w)\Gamma(\delta-\alpha-w)\Gamma(\delta-\beta-w)}{\Gamma(\delta-w)}z^{w}\,dw$$
$$\times\frac{1}{2\pi i}\int\Gamma(\zeta)\Gamma(\gamma-\zeta)\Gamma(\alpha+\beta-\delta-\zeta)F(w,\zeta;\;\gamma;\;1)\,z^{\zeta}\,d\zeta.$$

Now apply Gauss's Theorem and get

$$\frac{1}{2\pi i}\int\Gamma(w)\frac{\Gamma(\delta-\alpha-w)\Gamma(\delta-\beta-w)}{\Gamma(\delta-w)}E(\alpha+\beta-\delta,\,\gamma-w:\,:z)\,z^{w}\,dw,$$

and the result follows from (7).

The second series [4] is

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$$\sum_{r=0}^{\infty} \frac{z^{-2r}}{r!\,\Gamma(\alpha+r)} E(\alpha+r,\ \beta+r : : z) E(\alpha+r,\ \gamma+r : : z) = B(\beta, \gamma) E(\alpha,\ \beta+\gamma : : z), \quad \ldots\ldots(10)$$

where $|\operatorname{amp} z| < \frac{3}{2}\pi$, $R(\alpha) > 0$.

Proceeding as before it is seen that the series is equal to

$$\frac{1}{\Gamma(\alpha)} \frac{1}{2\pi i} \int \Gamma(\zeta)\Gamma(\alpha-\zeta)\Gamma(\beta-\zeta) z^{\zeta}\, d\zeta \frac{1}{2\pi i} \int \Gamma(w)\Gamma(\alpha-w)\Gamma(\gamma-w) F(\zeta,\ w\ ;\ \alpha\ ;\ 1)\, z^{w}\, dw$$

$$= \frac{1}{2\pi i} \int \Gamma(\zeta)\Gamma(\beta-\zeta) E(\gamma,\ \alpha-\zeta : : z)\, z^{\zeta}\, d\zeta,$$

and from (8) the result follows.

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