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CONCENTRATION OF THE ERROR BETWEEN A FUNCTION AND ITS POLYNOMIAL OF BEST UNIFORM APPROXIMATION

by MAURICE HASSON

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Let f be a continuous real valued function defined on $[-1, 1]$ and let $E_n(f)$ denote the degree of best uniform approximation to f by algebraic polynomial of degree at most n. The supremum norm on $[a, b]$ is denoted by $\|.\|_{[a,b]}$ and the polynomial of degree n of best uniform approximation is denoted by P_n. We find a class of functions f such that there exists a fixed $a \in (-1, 1)$ with the following property

$$\|f - P_n\|_{[a-\frac{N}{n},a+\frac{N}{n}]} \geq CE_n(f), \quad n = 1, 2, 3, \dots$$

for some positive constants C and N independent of n. Moreover the sequence $\left(\frac{N}{n}\right)$ is optimal in the sense that if $\frac{N}{n}$ is replaced by $b_n = o\left(\frac{1}{n}\right)$ then the above inequality need not hold no matter how small $C > 0$ is chosen.

We also find another, more general class a functions f for which

$$\|f - P_n\|_{[a-\frac{N}{n},a+\frac{N}{n}]} \geq CE_n(f) \quad \text{infinitely often.}$$

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1. Introduction

Let f be a continuous real valued function defined on $[-1, 1]$ and let $E_n(f)$ denote the degree of best uniform approximation to f by algebraic polynomial of degree at most n. Thus

$$E_n(f) := \inf_{P_n} \|f - P_n\|$$

where $\|.\|$ denotes the supremum norm on $[-1, 1]$. If $a > -1$ and $b < 1$ the supremum norm on $[a, b]$ will be denoted by $\|.\|_{[a,b]}$. Let

$$e_n(x) := f(x) - P_n(x).$$

The classical alternation theorem [**5**] guarantees that $e_n(x)$ reaches the value $\pm E_n(f)$ alternatively at least $n + 2$ times. The asymptotic *distribution* of points x_i such that $|e_n(x_i)| = E_n(f)$ has been extensively studied. See, among others, [**3**, **6**] for the polynomial case and [**2**, **4**] for the setting where approximation by rational functions is considered, and the references therein. However, in general, there is no *fixed* point $a \in (-1, 1)$ such that $|e_n(a)| = E_n(f)$ or even $|e_n(a)| \geq CE_n(f)$ for some positive constant

C. It is then natural to ask whether there exist a *fixed* $a \in (-1, 1)$ and a sequence (a_n) of numbers tending to zero decreasingly with the following properties.

$$\|f - P_n\|_{[a-a_n, a+a_n]} \geq CE_n(f)$$

and (a_n) is optimal in the sense that if $b_n = o(a_n)$ then

$$\|f - P_n\|_{[a-b_n, a+b_n]} = o(E_n(f)).$$

Again we believe that, in general, a point a and sequence (a_n) satisfying the above properties do not exist. It is the purpose of this paper to describe a class of functions for which, precisely, such a point a and sequence (a_n) do exist. For such functions f we say that the error $e_n(x)$ is concentrated near a.

We then address the question of finding a more general class of functions f for which the weaker condition

$$\|f - P_n\|_{[a-a_n, a+a_n]} \geq CE_n(f) \quad \text{infinitely often.}$$

is satisfied, where (a_n) fulfils the same optimality condition as above.

Notice that we insist that a belongs to the *interior* $(-1, 1)$ of $[-1, 1]$, which, in final analysis, amounts to considering approximation of periodic functions by trigonometric polynomials. Because of the rapid oscillation of $e_n(x)$ near -1 and 1, we believe that the situation is completely different near the endpoints of the interval. See Section 4 where an open problem pertaining to that situation is stated.

Let $f \in C_{2\pi}^{k-1}$, the space of periodic (of period 2π) $k-1$-times differentiable functions. Then a well known theorem of Zygmund [12] states that $f^{(k-1)}$ belongs to the Zygmund class, i.e.

$$\omega_2\left(f^{(k-1)}, \frac{1}{n}\right) = O\left(\frac{1}{n}\right) \tag{1.1}$$

where $\omega_2(g, .)$ is the modulus of smoothness of the function g (see [11, p. 102]) if and only if

$$E_n^*(f) = O\left(\frac{1}{n^k}\right) \tag{1.2}$$

where $E_n^*(f)$ is the degree of approximation by trigonometric polynomials. This shows that the condition $E_n(f; [-1, 1]) = O\left(\frac{1}{n^k}\right)$ does not necessarily imply that f is in the Zygmund class of $[-1, 1]$. However f belongs to the Zygmund class of $[-1+\epsilon, 1-\epsilon]$, $\epsilon > 0$. We will prove our results in the $C^{k-1}[-1, 1]$ setting where it will be enough to assume that $E_n(f) = O\left(\frac{1}{n^k}\right)$. It will be clear from the proofs that the analogous results hold in the trigonometric case (where, as noted above, this condition on $E_n^*(f)$ amounts to assuming that f is in the Zygmund class.) The examination of the situation in the trigonometric case will yield, in turn, a slight improvement of the results in the algebraic case.

In [8] Steckin proved that, if in addition to (1.1), $\omega_2(f^{(k-1)})$ satisfies

$$\omega_2\left(f^{(k-1)}, \frac{1}{n}\right) \geq \frac{K}{n}, n = 1, 2, \ldots \tag{1.3}$$

then $E_n(f)$ is exactly of the order $\frac{1}{n^k}$, that is to say there exist positive constants K_1, K_2 such that for $n = 1, 2, \ldots$ one has

$$\frac{K_1}{n^k} \leq E_n(f) \leq \frac{K_2}{n^k}. \tag{1.4}$$

The Dini derivatives $D_+f(x)$ and $D^-f(x)$ of a function f at a point x are defined by

$$D_+f(x) := \liminf_{h\to 0^+} \frac{f(x+h)-f(x)}{h}$$

$$D^-f(x) := \limsup_{h\to 0^-} \frac{f(x+h)-f(x)}{h}.$$

$D^+f(x)$ and $D_-f(x)$ are similarly defined.

We record the following observation which will be used in the sequel: the relations $D^+f(0) = 1$, $f(0) = 0$ and $\epsilon > 0$ imply the existence of a sequence (x_l) of positive numbers with $\lim_{l\to\infty} x_l = 0$ such that

$$f(x_l) \geq (1-\epsilon)x_l, x_l \in (0, \delta]$$

for $\delta > 0$ small enough.

The following inequality, due to Bernstein, will play a fundamental role in the sequel (see [5]).

Theorem (Bernstein inequality). *Let $T_n(\theta)$ be a trigonometric polynomial of degree n of period 2π. Then*

$$\|T_n'\|_{\mathbf{R}} \leq n\|T_n\|_{\mathbf{R}}.$$

Here $\|.\|_{\mathbf{R}}$ denotes the supremum norm on the real line. It follows that if P_n is an algebraic polynomial of degree n then (with $\|.\|$ denoting now, and as before, the supremum norm on $[-1, 1]$)

$$\|P_n'\|_{[a,b]} \leq M_{a,b}n\|P_n\|, a > -1, b < 1 \tag{1.5}$$

where $M_{a,b} = \frac{1}{\sqrt{1-c^2}}$, $c = \max(|a|, |b|)$.

We are now ready to state our main results. It may be useful to remark that the Dini derivatives always exist, with the possible values of $\pm\infty$. In Theorem 1.1 below, as well as in Theorem 1.3, the inequalities (1.6), (1.7), (1.9) and (1.10) must be understood in the extended real number system.

Theorem 1.1. *Let k be a positive integer, $f \in C^{k-1}[-1,1]$. If there exists $a \in (-1,1)$ with one of the following properties:*

$$f_+^{(k)}(a) > f_-^{(k)}(a) \tag{1.6}$$

or

$$f_-^{(k)}(a) > f_+^{(k)}(a) \tag{1.7}$$

and if P_n is a sequence of polynomials of degree at most n such that

$$\|P_n - f\| = O(E_n(f)) = O\left(\frac{1}{n^k}\right)$$

then there exist positive constants N and K such that

$$\|P_n - f\|_{[a-\frac{N}{n}, a+\frac{N}{n}]} \geq \frac{K}{n^k}, \; n = 1, 2, \ldots \tag{1.8}$$

We remark that the relations $E_n(f) = O\left(\frac{1}{n^k}\right)$ and (1.8) imply the existence of a constant $C > 0$ such that

$$\|f - P_n\|_{[a-\frac{N}{n}, a+\frac{N}{n}]} \geq CE_n(f), \; n = 1, 2, 3, \ldots$$

That is to say, the error $e_n(x)$ is concentrated near a.

Corollary 1.2. *Let f be as in Theorem 1.1. Then $E_n(f)$ is exactly of the order $\frac{1}{n^k}$, i.e. satisfies* (1.4).

It is easy to see that $\omega_2\left(f^{(k-1)}, \frac{1}{n}\right) \geq \frac{K}{n}$, $n = 1, 2, \ldots$ if $f^{(k-1)}$ satisfies (1.6) or (1.7). It follows that Corollary 1.2 is also a consequence of Steckin's theorem. However it does not seem that the techniques used by Steckin are appropriate to derive Theorem 1.1. The papers [1] and [7] deal with the problem of finding functions f for which $n^k E_n(f)$ has a non zero finite limit as n goes to infinity. It would be of interest to see whether the condition $\omega_2\left(f^{(k-1)}, \frac{1}{n}\right) \geq \frac{K}{n}$, $n = 1, 2, \ldots$ implies (1.6) or (1.7). If this was the case then Theorem 1.1 would imply Steckin's theorem.

Theorem 1.3. *Let k be a positive integer,* $f \in C^{k-1}[-1,1]$. *If there exists* $a \in (-1,1)$ *with one of the following properties:*

$$D_+ f^{(k-1)}(a) \neq D^- f^{(k-1)}(a) \tag{1.9}$$

or

$$D_- f^{(k-1)}(a) \neq D^+ f^{(k-1)}(a) \tag{1.10}$$

and if P_n *is a sequence of polynomials of degree at most n such that*

$$\|P_n - f\| = O(E_n(f)) = O\left(\frac{1}{n^k}\right)$$

then there exist positive constants N and K such that

$$\|P_n - f\|_{[a-\frac{N}{n}, a+\frac{N}{n}]} \geq \frac{K}{n^k} \quad \text{infinitely often.} \tag{1.11}$$

Notice that relation $E_n(f) = O\left(\frac{1}{n^k}\right)$ together with (1.11) guarantee the existence of a constant $C > 0$ such that

$$\|f - P_n\|_{[a-\frac{N}{n}, a+\frac{N}{n}]} \geq CE_n(f) \quad \text{infinitely often.}$$

To use our previous terminology, the error $e_{n_l}(x)$ is concentrated near a for a sequence (n_l) of integers.

Corollary 1.4. *Let f be as in Theorem* 1.3. *Then* $E_{n_l}(f)$ *is exactly of the order* $\frac{1}{n_l^k}$ *for some sequence* (n_l), *that is to say*

$$\frac{K_1}{n^k} \leq E_n(f) \leq \frac{K_2}{n^k} \quad \text{infinitely often.} \tag{1.12}$$

As an example of application of Theorem 1.3 and its corollary, consider the function f defined by

$$\begin{cases} f(x) = x\sin(\log|x|), \ x \in [-1,1], \ x \neq 0 \\ f(0) = 0. \end{cases}$$

Because $|f'(x)| \leq \sqrt{2}$, $x \neq 0$, with equality reached, it is readily seen that f belongs to the Lipschitz class, with Lipschitz constant $\sqrt{2}$, of $[-1,0)$ and $(0,1]$ and it is easy to verify that it belongs to the Lipschitz class of $[-1,1]$, with the same Lipschitz constant. So f belongs to the Zygmund class of $[-1,1]$. Moreover, $D^+f(0) = D^-f(0) = 1$ and $D_+f(0) = D_-f(0) = -1$. It follows that for this function f relation (1.11), with $a = 0$

and $k=1$, and where P_n satisfy $\|P_n - f\| = O(E_n(f)) = O(\frac{1}{n})$, and relation (1.12), with $k=1$, hold true.

This paper is organized as follows. The next section is devoted to the proof of our main results, Theorems 1.1 and Theorem 1.3. Section 3 shows that Theorem 1.1 and Theorem 1.3 are sharp in the sense described above. The last section states an open problem pertaining to the situation where $a=-1$ or $a=1$.

2. Proofs of Theorem 1.1 and Theorem 1.3

Lemma 2.1. *Let $f \in C([-1,1])$ have the following property: there exists a sequence of polynomials of degree at most n, P_n, such that for some positive integer k one has*

$$\|P_n - f\| < \frac{C}{n^k}, n = 1, 2, \ldots$$

Then there exists a constant M such that, for $n = 1, 2, \ldots$

$$\|P_n^{(k+1)}\|_{[-\frac{1}{2},\frac{1}{2}]} \leq Mn.$$

Proof. Let r be the integer defined by $2^{r-1} \leq n < 2^r$. Then

$$P_n = P_n - P_{2^r} + \sum_{i=1}^{r}(P_{2^i} - P_{2^{i-1}}) + P_1.$$

The Bernstein inequality gives:

$$\|P_n^{(k+1)}\|_{[\frac{-1}{2},\frac{1}{2}]} \leq K\left(n^{k+1}\|P_n - P_{2^r}\| + \sum_{i=1}^{r} 2^{(k+1)i}\|P_{2^i} - P_{2^{i-1}}\|\right)$$

so that, in view of the inequality $\|P_n - f\| < \frac{C}{n^k}$,

$$\begin{aligned}
\|P_n^{(k+1)}\|_{[\frac{-1}{2},\frac{1}{2}]} &\leq K'\left(\frac{2n^{k+1}}{2^{rk}} + \sum_{i=1}^{r}\frac{2^{(k+1)i}}{2^{ik}}\right) \\
&\leq K'\left(\frac{2.2^{(k+1)r}}{2^{rk}} + \sum_{i=1}^{r} 2^i\right) \\
&\leq K''\left(n + \sum_{i=1}^{r} 2^i\right) \\
&\leq Mn.
\end{aligned}$$

It is of interest to note that the proof of Lemma 2.1 shows that

$$\|P_n^{(k)}\|_{[-\frac{1}{2},\frac{1}{2}]} \le M \log n.$$

Moreover it can be shown that this estimate cannot be improved.

The purpose of Lemma 2.3 below is to prove the following

Proposition 2.2. *Let k be a positive integer and let* $f \in C^{(k-1)}[0, d]$. *Assume that*

$$|f^{(k-1)}(x)| \ge x, \quad x \in [0, d].$$

Then there exists $c_k > 0$ *such that*

$$\|f\|_{[0,d]} \ge c_k d^k.$$

Proof. By the Hermite-Gennochi formula, we have

$$\int_0^h \cdots \int_0^h f^{(k-1)}(t_1 + \cdots + t_{k-1}) dt_1 \ldots dt_{k-1} = \Delta_h^{k-1} f(0),$$

where

$$\Delta_h^{k-1} f(0) = \sum_{j=0}^{k-1} (-1)^j \binom{k-1}{j} f(jh)$$

is the corresponding divided difference. Set $h = \frac{d}{k-1}$. Then

$$|\Delta_h^{k-1} f(0)| \le 2^{k-1} \|f\|_{[0,d]}.$$

Assume now, without loss of generality, that $f^{(k-1)}(x) \ge x$ on $[0, d]$. Then

$$\begin{aligned}\int_0^h \cdots \int_0^h f^{(k-1)}(t_1 + \cdots + t_{k-1}) dt_1 \ldots dt_{k-1} &\ge \int_0^h \cdots \int_0^h (t_1 + \cdots + t_{k-1}) dt_1 \ldots dt_{k-1} \\ &= \frac{k-1}{2} h^k \\ &= \frac{d^k}{2(k-1)^{k-1}}.\end{aligned}$$

Proposition 2.2 follows with $c_k = \frac{1}{2^k (k-1)^{k-1}}$.

We now have built the necessary tools to demonstrate our main results.

Proof of Theorem 1.1. We first make the following reductions. By replacing, if needed, f by $-f$ it is enough to make the proof only in the case (1.6). We suppose first that the two derivatives are finite. We now remark that $E_n(f + Q_k) = E_n(f)$, $n \ge k$, where Q_k is a polynomial of degree k. It follows that we may assume that

$$a = f^{(k-1)}(a) = 0$$

and that $f_+^{(k)}(a) = -f_-^{(k)}(a)$. Now $E_n(\alpha f) = |\alpha| E_n(f)$, $\alpha \in \mathbb{R}$. It follows, again without loss of generality, that we may assume that

$$f_+^{(k)}(a) = 1$$

and that

$$f_-^{(k)}(a) = -1.$$

Let P_n be as in Theorem 1.1 Lemma 2.1 gives

$$\|P_n^{(k+1)}\|_{[-\frac{1}{2},\frac{1}{2}]} \le Mn, n = 1, 2, \ldots \tag{2.1}$$

The mean value theorem and (2.1) yield

$$|P_n^{(k)}(x) - P_n^{(k)}(y)| \le \frac{1}{2}, x \in \left[-\frac{1}{4Mn}, \frac{1}{4Mn}\right] \tag{2.2}$$

The relations $f_+^{(k)}(0) = 1$, $f^{(k-1)}(0) = 0$ give

$$f^{(k-1)}(x) \ge \frac{3}{4}x, x \in [0, \delta] \tag{2.3}$$

if $\delta > 0$ is small enough. Suppose first that

$$P_n^{(k)}(x) \le \frac{1}{2}, x \in \left[0, \frac{1}{4Mn}\right]. \tag{2.4}$$

Then (2.3) and (2.4) yield that one of $(2.5)_1$ or $(2.5)_2$ hold:

$$|f^{(k-1)}(x) - P_n^{(k-1)}(x)| \ge \frac{1}{4}x, x \in \left[0, \frac{1}{16Mn}\right] \tag*{$(2.5)_1$}$$

or

$$\left|f^{(k-1)}\left(x + \frac{1}{8Mn}\right) - P_n^{(k-1)}\left(x + \frac{1}{8Mn}\right)\right| \ge \frac{1}{4}x, x \in \left[0, \frac{1}{16Mn}\right]. \tag*{$(2.5)_2$}$$

It follows from Proposition 2.2 and relation (2.5) that

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$$\|P_n - f\|_{[0,\frac{1}{4Mn}]} \geq \frac{K'}{n^k}, n = 1, 2, \ldots \tag{2.6}$$

Suppose now that there exists $x \in \left[0, \frac{1}{4Mn}\right]$ such that $P_n^{(k)}(x) > \frac{1}{2}$. It follows from (2.2) that

$$P_n^{(k)}(x) > 0, x \in \left[-\frac{1}{4Mn}, 0\right]. \tag{2.7}$$

Because $f_-^{(k)}(0) = -1$ and $f^{(k-1)}(0) = 0$ we have

$$f^{(k-1)}(x) \geq -\frac{3}{4}x, x \in [-\delta, 0] \tag{2.8}$$

for δ small enough. By (2.7), $P_n^{(k-1)}$ is increasing on $\left[-\frac{1}{4Mn}, 0\right]$. This fact and relation (2.8), yield that one of $(2.9)_1$ or $(2.9)_2$ hold:

$$|f^{(k-1)}(x) - P_n^{(k-1)}(x)| \geq \frac{3}{4}x, x \in \left[-\frac{1}{16Mn}, 0\right] \tag*{$(2.9)_1$}$$

or

$$\left|f^{(k-1)}\left(x - \frac{1}{8Mn}\right) - P_n^{(k-1)}\left(x - \frac{1}{8Mn}\right)\right| \geq \frac{3}{4}x, x \in \left[-\frac{1}{16Mn}, 0\right]. \tag*{$(2.9)_2$}$$

It follows by Proposition (2.2) and relation (2.9) that

$$\|P_n - f\|_{[\frac{-1}{4Mn},0]} \geq \frac{K_2}{n^k}, k = 1, 2, \ldots \tag{2.10}$$

We have proved, in view of (2.6) and (2.10), that

$$\|P_n - f\|_{[\frac{-1}{4Mn},\frac{1}{4Mn}]} \geq \frac{K}{n^k}, n = 1, 2, \ldots$$

This is relation (1.8) if we let $N := \frac{1}{4M}$. Theorem 1.1 is proved in the case when both derivatives are finite. If one derivative is infinite, then (2.3) and (2.8) still hold and the rest of the proof remains unchanged.

Theorem 1.1 is proved.

We now proceed to the proof of Theorem 1.3. In what follows $|I|$ designates the length of the interval I.

Lemma 2.3. *Let $f(x)$ belong to the Zygmund class of $[-1, 1]$, with constant 1, that is to say*

$$|f(x+h)+f(x-h)-2f(x)| \leq h, (h>0) \tag{2.11}$$

for $x,\ x+h,\ x-h \in [-1,1]$. *Assume that* $f(0) \geq 1$. *Then there exists an interval* I *contained in* $[-1,1]$, *with* $0 \in I$ *and* $|I| \geq \frac{1}{2}$ *such that* $f(x) \geq \frac{1}{2}$, $x \in I$. *It follows that* $\exists b \in \mathbb{R}$ *and* $J \subset [-1,1]$ *with* $|J| \geq \frac{1}{4}$, *such that* $f(x) \geq x+b$, $x \in J$. *In fact we can choose* $b \geq 0$.

Proof. The proof follows directly from relation (2.11).

Lemma 2.4. *Let* $\delta > 0$ *and let* (x_l) *be a sequence of positive numbers with* $\lim_{l\to\infty} x_l = 0$. *Then for each* l *there exists an integer* n_l *such that*

$$x_l \in \left[\frac{\delta}{2n_l}, \frac{\delta}{n_l}\right]. \tag{2.12}$$

Moreover n_l *can be chosen in such a way that*

$$x_l - \frac{\delta}{2n_l} > \frac{1}{6n_l} \qquad (2.13)_1$$

and

$$\frac{\delta}{n_l} - x_l > \frac{1}{6n_l}. \qquad (2.13)_2$$

Proof. That there exists a sequence (n_l) satisfying (2.12) follows from the fact that

$$\bigcup_{n=1}^{\infty}\left[\frac{\delta}{2n}, \frac{\delta}{n}\right) = (0, \delta).$$

Now (for $n \geq 5$) $\frac{\delta}{3n} \leq \frac{\delta}{n+1} - \frac{\delta}{2n}$ = length of the intersection of such two consecutive intervals. Relation (2.13) now follows by taking, if necessary, a subsequence, still denoted (n_l), of the sequence (n_l).

Proof of Theorem 1.3. Because of some similarities between the proofs of Theorems 1.1 and 1.3, we omit those details of the proof of Theorem 1.3 which are analogous to the corresponding arguments of the proof of Theorem 1.1. We first make, as before, the following reductions. Because $D_+g(a) = -D^+ - g(a)$ and $D^-g(a) = -D_- - g(a)$, we make the proof only in the case (1.10). We suppose first that the two Dini derivatives are finite. In that case we may suppose, without loss of generality, that

$$D^+f^{(k-1)}(a) = 1$$

and

$$D_-f^{(k-1)}(a) = -1.$$

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Again without loss of generality we may assume that

$$a = f^{(k-1)}(a) = 0.$$

Let P_n be as in Theorem 1.3. Lemma 2.1 gives

$$\|P_n^{(k+1)}\|_{[\frac{-1}{2},\frac{1}{2}]} \le Mn, n = 1, 2, \dots \tag{2.14}$$

The mean value theorem and (2.14) yield

$$|P_n^{(k)}(x) - P_n^{(k)}(y)| \le \frac{1}{2}, x \in \left[-\frac{1}{4Mn}, \frac{1}{4Mn}\right]. \tag{2.15}$$

The relations $D^+f^{(k-1)}(0) = 1$, $f^{(k-1)}(0) = 0$ give: there exists a sequence (x_l) of positive numbers with $\lim_{l\to\infty} x_l = 0$ such that

$$f^{(k-1)}(x_l) \ge \frac{3}{4}x_l, x_l \in (0, \delta] \tag{2.16}$$

for $\delta > 0$ small enough. Suppose first that

$$P_n^{(k)}(x) \le \frac{1}{2}, x \in \left[0, \frac{1}{4Mn}\right]. \tag{2.17}$$

Then (2.14), (2.17), Lemma 2.3 and Lemma 2.4 guarantee the existence of $b_l \ge 0$, $b_l' \ge 0$ such that

$$|f^{(k-1)}(x) - P_{n_l}^{(k-1)}(x)| \ge \frac{1}{4}x + b_l, x \in I_l \subset \left[0, \frac{1}{16Mn_l}\right] \qquad (2.18)_1$$

or

$$\left|f^{(k-1)}\left[(x + \frac{1}{8Mn_l}\right) - P_{n_l}^{(k-1)}\left(x + \frac{1}{8Mn_l}\right)\right| \ge \frac{1}{4}x + b_l', x \in J_l \subset \left[0, \frac{1}{16Mn_l}\right] \qquad (2.18)_2$$

where $|I_l|, |J_l| \ge \frac{K}{n_l}$. ($K$ depends on M and on the Zygmund constant.) It follows from Proposition 2.2 and relation (2.18) that

$$\|P_{n_l} - f\|_{\left[0,\frac{1}{4Mn_l}\right]} \ge \frac{K'}{n_l^k}, l = 1, 2, \dots \tag{2.19}$$

Suppose now that there exists $x \in \left[0, \frac{1}{4Mn}\right]$ such that $P_n^{(k)}(x) > \frac{1}{2}$. It follows from (2.15) that

$$P_n^{(k)}(x) > 0, x \in \left[-\frac{1}{4Mn}, 0\right]. \tag{2.20}$$

Because $D_- f^{(k-1)}(0) = -1$ and $f^{(k-1)}(0) = 0$ we have

$$f^{(k-1)}(x'_l) \geq -\frac{3}{4}x'_l, x'_l \in [-\delta, 0] \tag{2.21}$$

for a sequence x'_l of negative numbers with $\lim_{l\to\infty} x'_l = 0$ and for $\delta > 0$ small enough. By (2.20), $P_n^{(k-1)}$ is increasing on $\left[-\frac{1}{4Mn}, 0\right]$. This fact and relation (2.21), together with Lemma 2.3 and Lemma 2.4, yield that one of $(2.22)_1$ or $(2.22)_2$ hold for some b_l, $b'_l \geq 0$:

$$|f^{(k-1)}(x) - P_n^{(k-1)}(x)| \geq \frac{3}{4}x + b_l, x \in I_l \subset \left[-\frac{1}{16Mn_l}, 0\right] \tag*{$(2.22)_1$}$$

or

$$\left|f^{(k-1)}\left(x - \frac{1}{8Mn_l}\right) - P_{n_l}^{(k-1)}\left(x - \frac{1}{8Mn_l}\right)\right| \geq \frac{3}{4}x + b'_l, x \in J_l \subset \left[-\frac{1}{16Mn_l}, 0\right] \tag*{$(2.22)_2$}$$

where $|I_l|$, $|J_l| \geq \frac{K}{n_l}$. It follows by Proposition 2.2 and relation (2.22) that

$$\|P_{n_l} - f\|_{\left[\frac{-1}{4Mn_l}, 0\right]} \geq \frac{K_2}{n_l^k}, l = 1, 2, \ldots \tag{2.23}$$

We have proved, in view of (2.19) and (2.23), that

$$\|P_{n_l} - f\|_{\left[\frac{-1}{4Mn_l}, \frac{1}{4Mn_l}\right]} \geq \frac{K}{n_l^k}, l = 1, 2, \ldots$$

This is relation (1.11). Theorem 1.3 is proved in the case when both Dini derivatives are finite. If one Dini derivative is infinite, then (2.16) and (2.21) still hold and the rest of the proof remains unchanged.

Theorem 1.3 is proved.

We end this section with two remarks about the above proofs of Theorem 1.1 and 1.3.

(a) The crucial step in the proofs of Theorem 1.1 and 1.3 is relation (2.2) (and (2.15)) which could not have been obtained by estimating directly $\|P_n^{(k)}\|_{[-\frac{1}{2}, \frac{1}{2}]}$. Indeed, as noticed before, it can be shown that this sequence is bounded by $M \log n$ and that this estimate cannot be improved. This explains the role played by Lemma 2.1 in the proofs of Theorem 1.1 and 1.3.

(b) If the condition that $f^{(k-1)}$ belongs to the Zygmund class of $[-1, 1]$ is replaced by $f^{(k-1)}$ is in the Lipschitz class, then $\|P_n^{(k)}\|_{[-\frac{1}{2},\frac{1}{2}]} \leq C$. This observation will play an important role in the next section.

Finally we notice, as already pointed out previously, that Theorem 1.1 and 1.3 hold in the trigonometric setting. Going back to the algebraic case with the standard transformation $x = \cos(\theta)$, we see that these results can be slightly improved by showing higher concentration of the error towards the endpoints ± 1. This is addressed in Section 4 below.

3. The sharpness of Theorem 1.1 and of Theorem 1.3

The purpose of this section is to show that the conclusion of Theorem 1.1 and of Theorem 1.3 need not hold if the interval $\left[a-\frac{N}{n}, a+\frac{N}{n}\right]$ is replaced by $[a-b_n, a+b_n]$ where $b_n = o\left(\frac{1}{n}\right)$.

Lemma 3.1. *Let $f \in C^{k-1}[-1,1]$ with $f^{(k-1)} \in \text{Lip } 1$. Let P_n be a sequence of polynomials of degree n such that*

$$\|P_n - f\| = O(E_n(f)) = O\left(\frac{1}{n^k}\right). \tag{3.1}$$

Then

$$\|P_n^{(i)} - f^{(i)}\|_{[-\frac{1}{2},\frac{1}{2}]} \leq \frac{K}{n^{k-i}}, n = 1, 2, \ldots, 0 \leq i \leq k-1 \tag{3.2}$$

and

$$\|P_n^{(k)}\|_{[-\frac{1}{2},\frac{1}{2}]} \leq K, n = 1, 2, \ldots \tag{3.3}$$

As we observed before (3.3) need not hold if we assume only that $f^{(k-1)}$ is in the Zygmund class, even though in that case (3.2) still holds.

Proof. It follows from the Jackson (or Zygmund) theorem that there exists a sequence P_n satisfying (3.1). Relation (3.2) follows by an adaptation of the proof of Lemma 2.1. We now sketch the derivation of (3.3). We use the following (slight generalization of a) result proved in [**9**, p. 6]:

Let $g(\theta)$ be a periodic $(k-1)$ – times continuously differentiable function with $g^{(k-1)}(\theta)$ belonging to the Lipschitz class. Then for the (trigonometric) polynomials of best approximation to g, Q_n, one has:

$$\|Q_n^{(i)} - g^{(i)}\|_{\mathbf{R}} \leq \frac{K}{n^{k-i}}, n = 1, 2, \ldots, 0 \leq i \leq k-1 \tag{3.4}$$

$$\|Q_n^{(k)}\|_{\mathbf{R}} \leq K, n = 1, 2, \ldots \tag{3.5}$$

Here $\|.\|_{\mathbf{R}}$ denotes the supremum on the real line.

Let now $g(\theta) := f(\cos(\theta))$. Because $g(\theta)$ is even, $Q_n(\theta)$ is of the form $\sum_{k=0}^{n} a_k \cos(k\theta)$ so that $P_n(x) := \sum_{k=0}^{n} a_k T_k(x)$, where $T_k(x)$ are the Chebyshev polynomials, are the polynomials of best approximation to f on $[-1, 1]$. Now relations (3.4), (3.5) together with Bernstein inequality yields (3.3) (and (3.2).)

Consider now the function

$$f(x) = x^{k-1}|x|, \; x \in [-1, 1]. \tag{3.6}$$

$f^{(k-1)} \in \text{Lip } 1$ and $f_+^{(k)}(0) \neq f_-^{(k)}(0)$, so that f fulfils the conditions of Theorem 1.1. It follows that $E_n(f)$ is exactly of the order $\frac{1}{n^k}$, that is to say $\frac{K_1}{n^k} \leq E_n(f) \leq \frac{K_2}{n^k}$. (In fact a much more precise result is known for this particular function f : $n^k E_n(f)$ has a non zero limit as n goes to infinity. See [**11**, p. 417].) The function f also fulfils the conditions of Theorem 1.3.

Theorem 3.2. *Let k be a positive integer and let $f(x) = x^{k-1}|x|, x \in [-1, 1]$. There exists a sequence P_n of polynomials of degree at most n with the following properties:*

$$\|P_n - f\| \leq 2E_n(f) \leq \frac{C}{n^k}, \; n = 1, 2, \ldots \tag{3.7}$$

and if (b_n) is a sequence of positive numbers with

$$b_n = o\left(\frac{1}{n}\right) \tag{3.8}$$

then

$$\|f - P_n\|_{[b_n, b_n]} = o\left(\frac{1}{n^k}\right). \tag{3.9}$$

Proof. Let $P_n = Q_n - Q_n(0)$ where Q_n is the polynomial of degree n of best approximation to f. Clearly P_n satisfies (3.7). Now let $x > 0$. Then for some c^n with $0 < c^n < x$ we have:

$$P_n(x) = a_1^n x + a_2^n x^2 + \cdots + a_{k-1}^n x^{k-1} + P_n^{(k)}(c^n)\frac{x^k}{k!}, \tag{3.10}$$

where, in view of Lemma 3.1,

$$|a_i^n| = O\left(\frac{1}{n^{k-i}}\right), \; 1 \leq i \leq k-1 \tag{3.11}$$

and

$$|P_n^{(k)}(c^n)| \leq C. \tag{3.12}$$

It follows easily from (3.8), (3.10), (3.11) and (3.12) that

$$\|P_n(x) - x^{k-1}|x|\|_{[0,b_n]} = \|P_n(x) - x^k\|_{[0,b_n]} = o\left(\frac{1}{n^k}\right).$$

The proof is similar when $x<0$. (Or more directly by noticing that $P_n(x) - x^{k-1}|x|$ is even or odd according to whether k is odd or even.) Hence (3.9) is established.

Theorem 3.2 is proved.

4. Concluding remarks

As previously noted, Theorems 1.1 and 1.3 may be slightly improved by remarking first that they hold in the trigonometric case, and then by using the standard transformation $x = \cos(\theta)$.

Theorem 4.1. *Let k be a positive integer, $f \in C^{k-1}[-1,1]$. If there exists $a \in (-1,1)$ with one of the following properties:*

$$f_+^{(k)}(a) > f_-^{(k)}(a)$$

or

$$f_-^{(k)}(a) > f_+^{(k)}(a)$$

and if P_n is a sequence of polynomials of degree at most n such that

$$\|P_n - f\| = O(E_n(f)) = O\left(\frac{1}{n^k}\right)$$

then there exist positive constants N and K such that

$$\|P_n - f\|_{\left[a-\frac{N\sqrt{1-a^2}}{n},\, a+\frac{N\sqrt{1-a^2}}{n}\right]} \geq \frac{K}{n^k} \quad n = 1, 2, \ldots$$

Theorem 4.2. *Let k be a positive integer, $f \in C^{k-1}[-1,1]$. If there exists $a \in (-1,1)$ with one of the following properties:*

$$D_+ f^{(k-1)}(a) \neq D^- f^{(k-1)}(a)$$

or

$$D_- f^{(k-1)}(a) \neq D^+ f^{(k-1)}(a)$$

and if P_n is a sequence of polynomials of degree at most n such that

$$\|P_n - f\| = O(E_n(f)) = O\left(\frac{1}{n^k}\right)$$

then there exist positive constants N and K such that

$$\|P_n - f\|_{\left[a-\frac{N\sqrt{1-a^2}}{n},\, a+\frac{N\sqrt{1-a^2}}{n}\right]} \geq \frac{K}{n^k} \quad \textit{infinitely often.}$$

Because the error $e_n(x)$ oscillates much faster near the endpoints of the interval in the case of approximation by algebraic polynomials, one expects that the error is much more concentrated near these points. In that case we an interval of length $\frac{N}{n^2}$, instead of $\frac{N}{n}$, suffices to capture the error, at least in the analytic case and infinitely often.

Proposition 4.3. *Let f be analytic on (a neighbourhood of) $[-1, 1]$ and let P_n be the algebraic polynomial of degree n of best approximation to f on $[-1, 1]$. Then there exist positive constants C and N independent on n such that*

$$\|P_n - f\|_{\left[1-\frac{N}{n^2},1\right]} \geq CE_n(f) \quad \textit{infinitely often}$$

and

$$\|P_n - f\|_{\left[-1,-1+\frac{N}{n^2}\right]} \geq CE_n(f) \quad \textit{infinitely often.}$$

This follows from a result of Tashev **[10]**, see also **[5]**.

It would be also of interest to build a function g satisfying the hypotheses of Theorem 1.3 and for which, necessarily,

$$\|P_n - g\|_{\left[a-\frac{N}{n},a+\frac{N}{n}\right]} \geq \frac{K}{n^k} \quad \text{infinitely often} \tag{4.1}$$

but for which

$$\|P_{n_l} - g\|_{\left[a-\frac{N}{n_l},a+\frac{N}{n_l}\right]} = o\left(\frac{1}{n_l^k}\right) \tag{4.2}$$

for some sequence (n_l) of integers. It is of interest to note that for such a function $g_-^{(k)}(a)$ and $g_+^{(k)}(a)$ cannot both exist. Indeed if $g_-^{(k)}(a) = g_+^{(k)}(a)$ then conditions (1.9) and (1.10) of Theorem 1.3 are not fulfilled. If, on the other hand, $g_-^{(k)}(a) \neq g_+^{(k)}(a)$ then (4.2) cannot hold in view of Theorem 1.1. However we were not able to build such a function although we believe that the function $g(x) = x\sin(\log|x|)$ described page 4 following Corollary 1.4 satisfies (4.2) with $a = 0$ and $k = 1$ (in addition to satisfying (4.1) with $a = 0$ and $k = 1$ as already noticed.)

The existence of such a function g would show that, in general, the conclusion (1.11) of Theorem 1.3 cannot be replaced by the stronger one

$$\|P_n - f\|_{\left[a-\frac{N}{n}, a+\frac{N}{n}\right]} \geq \frac{K}{n^k} \quad \forall n.$$

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Department of Mathematics
Rutgers University
New Brunswick
New Jersey 08903
U.S.A.
E-mail: hasson@math.rutgers.edu