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OPTIMAL SMOOTH PORTFOLIO SELECTION FOR AN INSIDER

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Abstract

We study the optimal portfolio problem for an insider, in the case where the performance is measured in terms of the logarithm of the terminal wealth minus a term measuring the roughness and the growth of the portfolio. We give explicit solutions in some cases. Our method uses stochastic calculus of forward integrals.

Keywords: Insider trading; optimal portfolio; enlargement of filtration; log utility; information flow

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1. Introduction

There has been an increasing interest in the insider trading in recent years; see, for example, [1]–[6], [8]–[10], and the references therein. By an *insider* in a financial market we mean a certain investor who possesses more information than the information generated by the financial market itself. An insider may, for example, be an executive or simply an employee of a company. In probabilistic terminology, information is generally represented by a filtration. Usually an investor can only use the filtration generated by the market to make a decision. We call such investors *honest*. An insider has a larger filtration (more information) available to him and can use this larger filtration to make his decision; for example, to maximize his portfolio.

To simplify our presentation we assume that the market consists of the following two assets over the time period $[0, T]$. The first one is a bond whose price is determined by a stochastic process

$$\mathrm{d}S_0(t) = r(t)S_0(t)\,\mathrm{d}t, \qquad 0 \le t \le T.$$

Another asset is the stock whose price follows the following geometric Brownian motion:

$$\mathrm{d}S_1(t) = S_1(t)[\mu(t)\,\mathrm{d}t + \sigma(t)\,\mathrm{d}B(t)], \qquad 0 \le t \le T,$$

where $r(t)$, $\mu(t)$, and $\sigma(t)$ are deterministic functions, $B(t) = B_t(\omega)$, $0 \le t \le T$, $\omega \in \Omega$, is a Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge 0}, \mathrm{P})$, and $\mathrm{d}B(t)$ denotes the Itô-type stochastic differential. Denote the information generated by the market by $\mathcal{F}_t = \sigma(B_s,\ 0 \le s \le t)$. Assume, for example, that at the beginning $(t = 0)$ the insider knows, in addition to $\mathcal{F}_t$, the future value of the underlying Brownian motion at time T_0, where $T_0 > T$.

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Then his information filtration is given by $\mathcal{G}_t = \sigma(B_s, 0 \le s \le t) \vee \sigma(B_{T_0})$, the filtration generated by the Brownian motion up to time t and B_{T_0}. The insider may use this filtration (rather than as usual using only the filtration $\mathcal{F}_t$) to optimize his portfolio.

More explicitly, let us express the portfolio in terms of the fraction $\pi(t)$ of the total wealth invested in the stocks at time t. Let $X^{(\pi)}(t)$ denote the corresponding wealth at time t. In [9], Pikovsky and Karatzas considered the problem of maximizing the expectation of the logarithmic utility of terminal wealth,

$$\Phi_{\mathcal{G}} := \sup_{\pi} \{\mathrm{E}[\log(X^{(\pi)}(T))]\}, \tag{1.1}$$

where the supremum is taken over all $\mathcal{G}_t$-adapted portfolios $\pi(\cdot)$. They proved that in this case the optimal insider portfolio is

$$\pi^*(t) = \frac{\mu(t) - r(t)}{\sigma^2(t)} + \frac{B(T_0) - B(t)}{\sigma(t)(T_0 - t)}. \tag{1.2}$$

Moreover, the corresponding maximal expected utility $\Phi_{\mathcal{G}}$ is given by

$$\Phi_{\mathcal{G}} = \mathrm{E}\bigg[\int_0^T \bigg\{ r(s) + \frac{1}{2}\frac{(\mu(s) - r(s))^2}{\sigma^2(s)} + \frac{1}{2(T_0 - s)} \bigg\} \,\mathrm{d}s\bigg], \qquad T_0 \ge T.$$

In particular, if $T_0 = T$ we obtain

$$\Phi_{\mathcal{G}} = \infty.$$

This is clearly an unrealistic result. If $T_0 = T$ we see, by (1.2), that the optimal portfolio π^* needed to achieve $\Phi_{\mathcal{G}} = \infty$ will converge towards the derivative of $B(t)$ at $t = T_0^-$. Thus, $\pi^*(t)$ will consist of more and more wild fluctuations as $t \to T_0^-$. This is both practically impossible and also undesirable from the point of view of the insider; he does not want to expose a too conspicuous portfolio, compared to that of the honest trader, which in the optimal case is just

$$\pi^*_{\text{honest}}(t) = \frac{\mu(t) - r(t)}{\sigma^2(t)}.$$

To model this constraint we propose to modify (1.1) in the following way.

Problem 1.1. Find $\pi^* \in \mathcal{A}_{\mathcal{G}}$ and Φ such that

$$\begin{aligned}\Phi &= \sup_{\pi \in \mathcal{A}_{\mathcal{G}}} \mathrm{E}\bigg[\log(X^{(\pi)}(T)) - \int_0^T |\mathbb{Q}\pi(s)|^2 \,\mathrm{d}s\bigg] \\ &= \mathrm{E}\bigg[\log(X^{(\pi^*)}(T)) - \int_0^T |\mathbb{Q}\pi^*(s)|^2 \,\mathrm{d}s\bigg],\end{aligned}$$

where $\mathcal{A}_{\mathcal{G}}$ is a suitable family of admissible $\mathcal{G}_t$-adapted portfolios π. Here, $\mathbb{Q}\colon \mathcal{A}_{\mathcal{G}} \to \mathcal{A}_{\mathcal{G}}$ is some linear operator measuring the size and/or the fluctuations of the portfolio. For example, we could have

$$\mathbb{Q}\pi(s) = \lambda_1(s)\pi(s), \tag{1.3}$$

where $\lambda_1(s) \ge 0$ is some given weight function. This models the situation where the insider is penalized for large volumes of trade.

An alternative choice of $\mathbb{Q}$ would be

$$\mathbb{Q}\pi(s) = \lambda_2(s)\pi'(s) \tag{1.4}$$

for some weight function $\lambda_2(s) \geq 0$, where $\pi'(s) = (\mathrm{d}/\mathrm{d}s)\pi(s)$. In this case, the insider is penalized for large trade fluctuations. Other choices of $\mathbb{Q}$ are also possible, including combinations of (1.3) and (1.4).

We will return to Problem 1.1 in Section 3, after giving a brief introduction to the forward integral.

2. The forward integral

In general, $B(t)$ need not be a semimartingale with respect to a bigger filtration $\mathcal{G}_t \supset \mathcal{F}_t$. A simple example is

$$\mathcal{G}_t = \mathcal{F}_{t+\delta}, \qquad t \geq 0,$$

where $\delta > 0$ is a constant.

Therefore, to be able to deal with corresponding (anticipating) $\mathcal{G}_t$-adapted integrands $\phi(t, \omega)$, we must go beyond the semimartingale integral context. Following [3] we propose to use *the forward* integral to model such situations. This integral extends the semimartingale integral in the sense that the two integrals coincide if $B(t)$ is a semimartingale with respect to $\mathcal{G}_t$.

In this section we briefly review some basic concepts and results on forward integrals. We refer to [3] for the motivation for using forward integrals in insider trading, and to [11] and [12] for more information about forward integrals.

Definition 2.1. (*[11].*) Let $\phi(t, \omega)$ be a measurable process (not necessarily adapted). Then the forward stochastic integral of ϕ is defined as

$$\int_0^\infty \phi(t, \omega)\, \mathrm{d}^- B(t) = \lim_{\varepsilon \to 0} \int_0^\infty \phi(t, \omega) \frac{B(t+\varepsilon) - B(t)}{\varepsilon}\, \mathrm{d}t,$$

if the convergence is in probability.

Let $\pi : 0 = t_0 < t_1 < \cdots < t_n = t$ be a partition of $[0, T]$ and let $|\pi| = \max_{0 \leq j \leq n-1}(t_{j+1} - t_j)$. It is easy to see that if ϕ is càdlàg then

$$\int_0^T \phi(t, \omega)\, \mathrm{d}^- B(t) = \lim_{|\pi| \to 0} \sum_{j=0}^{n-1} \phi(t_j)(B(t_{j+1}) - B(t_j)); \tag{2.1}$$

see [3] for details. Here $\mathrm{d}^- B(t)$ indicates that the integral is interpreted in the forward integral sense.

Definition 2.2. By a (1-dimensional) forward process we mean a process $X(t) = X(t, \omega)$ of the form

$$X(t) = x + \int_0^t u(s, \omega)\, \mathrm{d}s + \int_0^t v(s, \omega)\, \mathrm{d}^- B(s), \qquad t > 0, \tag{2.2}$$

where $u(s, \omega)$ and $v(s, \omega)$ are measurable processes (not necessarily $\mathcal{F}_t$-adapted) such that

$$\int_0^t |u(s, \omega)|\, \mathrm{d}s < \infty, \quad \text{(almost surely (a.s.)) for all } t > 0,$$

and the Itô forward integral

$$\int_0^t v(s, \omega)\, \mathrm{d}^- B(s)$$

exists for all $t > 0$.

In accordance with the classical Itô process notation, we use the short-hand notation

$$\mathrm{d}^- X(t) = u(t)\,\mathrm{d}t + v(t)\,\mathrm{d}^- B(t)$$

for the integral equation (2.2).

Theorem 2.1. ([12]; an Itô formula for forward processes.) *Let*

$$\mathrm{d}^- X(t) = u(t)\,\mathrm{d}t + v(t)\,\mathrm{d}^- B(t)$$

be a forward process. Let $f \in C^2(\mathbb{R})$ and define

$$Y(t) = f(X(t)).$$

Then $Y(t)$ is also a forward process and

$$\mathrm{d}^- Y(t) = f'(X(t))\,\mathrm{d}^- X(t) + \tfrac{1}{2} f''(X(t)) v^2(t)\,\mathrm{d}t.$$

As an application of the Itô formula for forward integrals, we obtain the following result.

Corollary 2.1. ([3].) *Let $u(t)$ and $v(t)$ be measurable processes such that the following integrals exist for all $t > 0$:*

$$\int_0^t (|u(s)|^2 + |v(s)|^2)\,\mathrm{d}s, \qquad \int_0^t v(s)\,\mathrm{d}^- B(s).$$

Then the forward stochastic differential equation

$$\mathrm{d}X(t) = X(t)[u(t)\,\mathrm{d}t + v(t)\,\mathrm{d}^- B(t)], \qquad X(0) = x > 0,$$

has the following unique solution:

$$X(t) = x \exp\biggl(\int_0^t \Bigl(u(s) - \frac{1}{2} v^2(s)\Bigr)\,\mathrm{d}s + \int_0^t v(s)\,\mathrm{d}^- B(s)\biggr).$$

We also need the following result, which follows easily from Definition 2.1.

Lemma 2.1. *Suppose that $\phi(t)$ is forward integrable and that G is an $\mathcal{F}_T$-measurable random variable. Then we have*

$$\int_0^T G\phi(t)\,\mathrm{d}^- B(t) = G \int_0^T \phi(t)\,\mathrm{d}^- B(t).$$

3. Optimal smooth portfolio for an insider

We now return to Problem 1.1. We assume that the market consists of the following two investment possibilities:

(i) a bond, with price given by

$$\mathrm{d}S_0(t) = r(t) S_0(t)\,\mathrm{d}t, \qquad S_0(0) = 1,\ 0 \le t \le T,$$

(ii) a stock, with price given by

$$\mathrm{d}S_1(t) = S_1(t)[\mu(t)\,\mathrm{d}t + \sigma(t)\,\mathrm{d}B(t)], \qquad 0 \le t \le T,$$

where $T > 0$ is constant and $r(t)$, $\mu(t)$, and $\sigma(t)$ are given $\mathcal{F}_t$-adapted processes.

We assume that

$$\mathrm{E}\bigg[\int_0^T \{|\mu(t)| + |r(t)| + \sigma^2(t)\}\,\mathrm{d}t\bigg] < \infty,$$

$$\sigma(t) \neq 0 \quad \text{for almost all } (t,\omega) \in [0,T] \times \Omega.$$

Let $\mathcal{G}_t \supset \mathcal{F}_t$ be the information filtration available to the insider and let $\pi(t)$ be the portfolio chosen by the insider, measured in terms of the fraction of the total wealth $X(t) = X^{(\pi)}(t)$ invested in the stock at time $t \in [0,T]$. Then the corresponding wealth $X(t) = X^{(\pi)}(t)$ at time t is modeled by the forward differential equation

$$\begin{aligned} \mathrm{d}X(t) &= (1-\pi(t))X(t)r(t)\,\mathrm{d}t + \pi(t)X(t)[\mu(t)\,\mathrm{d}t + \sigma(t)\,\mathrm{d}^-B(t)] \\ &= X(t)[[r(t) + (\mu(t) - r(t))\pi(t)]\,\mathrm{d}t + \sigma(t)\pi(t)\,\mathrm{d}^-B(t)]. \end{aligned} \tag{3.1}$$

For simplicity, we assume that $X(0) = 1$. The motivation for using this forward integral model for the anticipating stochastic differential equation, (3.1), is the formula (2.1), which expresses the forward integral as a limit of Riemann sums of the Itô type, i.e. where the ith term has the form $\phi(t_i)(B(t_{i+1}) - B(t_i))$ with ϕ evaluated at the *left* end point t_i of the interval $[t_i, t_{i+1}]$. Moreover, if $B(t)$ happens to be a semimartingale with respect to $\mathcal{G}_t$ then the forward integral coincides with the semimartingale integral. See [3], [11], and [12] for more details on this.

We now specify the set $\mathcal{A} = \mathcal{A}_{\mathcal{G}}$ of the admissible portfolios π as follows.

Definition 3.1. In the following we let $\mathcal{A} = \mathcal{A}_{\mathcal{G}}$ denote a linear space of stochastic processes $\pi(t)$ such that (3.2)–(3.5) hold, where

$$\pi(t) \text{ is } \mathcal{G}_t\text{-adapted and the } \sigma\text{-algebra generated by } \{\pi(t);\ \pi \in \mathcal{A}\} \text{ is equal to } \mathcal{G}_t, \text{ for all } t \in [0,T], \tag{3.2}$$

$$\pi \text{ belongs to the domain of } \mathbb{Q}, \tag{3.3}$$

$$\sigma(t)\pi(t) \text{ is forward integrable}, \tag{3.4}$$

$$\mathrm{E}\bigg[\int_0^T |\mathbb{Q}\pi(t)|^2\,\mathrm{d}t\bigg] < \infty. \tag{3.5}$$

With these definitions we can now specify Problem 1.1 as follows.

Problem 3.1. Find Φ and $\pi^* \in \mathcal{A}$ such that

$$\Phi = \sup_{\pi \in \mathcal{A}} J(\pi) = J(\pi^*),$$

where

$$J(\pi) = \mathrm{E}\bigg[\log(X^{(\pi)}(T)) - \frac{1}{2}\int_0^T |\mathbb{Q}\pi(s)|^2\,\mathrm{d}s\bigg],$$

$\mathbb{Q}\colon \mathcal{A} \to \mathcal{A}$ is a given linear operator, and E denotes the expectation with respect to P. We call Φ the *value* of the insider and $\pi^* \in \mathcal{A}$ an optimal portfolio (if it exists).

We now proceed to solve Problem 3.2. Using Corollary 2.4 we find that the solution to (3.1) is

$$X(t) = \exp\bigg(\int_0^t \bigg\{r(s) + (\mu(s) - r(s))\pi(s) - \frac{1}{2}\sigma^2(s)\pi^2(s)\bigg\}\,\mathrm{d}s + \int_0^t \sigma(s)\pi(s)d^-B(s)\bigg).$$

Therefore, we obtain

$$J(\pi) = \mathrm{E}\bigg[\int_0^T \bigg\{r(t) + (\mu(t) - r(t))\pi(t) - \frac{1}{2}\sigma^2(t)\pi^2(t)\bigg\}\,\mathrm{d}t \\ + \int_0^T \sigma(t)\pi(t)\,\mathrm{d}^- B(t) - \frac{1}{2}\int_0^T |\mathbb{Q}\pi(t)|^2\,\mathrm{d}t\bigg]. \tag{3.6}$$

To maximize $J(\pi)$ we use a calculus of variation technique as follows. Suppose that an optimal insider portfolio $\pi = \pi^*$ exists (in the following we omit the '$*$'). Let $\theta \in \mathcal{A}$ be another portfolio. Then the function

$$f(y) := J(\pi + y\theta), \qquad y \in \mathbb{R},$$

is maximal for $y = 0$; and hence,

$$\begin{aligned} 0 &= f'(0) \\ &= \frac{\mathrm{d}}{\mathrm{d}y}[J(\pi + y\theta)]_{y=0} \\ &= \mathrm{E}\bigg[\int_0^T \{(\mu(t) - r(t))\theta(t) - \sigma^2(t)\pi(t)\theta(t)\}\,\mathrm{d}t \\ &\qquad + \int_0^T \sigma(t)\theta(t)\,\mathrm{d}^- B(t) - \int_0^T \mathbb{Q}\pi(t)\mathbb{Q}\theta(t)\,\mathrm{d}t\bigg]. \end{aligned} \tag{3.7}$$

Let $\mathbb{Q}^*$ denote the adjoint of $\mathbb{Q}$ in the Hilbert space $L^2([0, T] \times \Omega)$, i.e.

$$\mathrm{E}\bigg[\int_0^T \alpha(t)(\mathbb{Q}\beta)(t)\,\mathrm{d}t\bigg] = \mathrm{E}\bigg[\int_0^T (\mathbb{Q}^*\alpha)(t)\beta(t)\,\mathrm{d}t\bigg]$$

for all α and β in $\mathcal{A}$. Then we can rewrite (3.7) as

$$\mathrm{E}\bigg[\int_0^T \{\mu(t) - r(t) - \sigma^2(t)\pi(t) - \mathbb{Q}^*\mathbb{Q}\pi(t)\}\theta(t)\,\mathrm{d}t + \int_0^T \sigma(t)\theta(t)\,\mathrm{d}^- B(t)\bigg] = 0. \tag{3.8}$$

Now we apply this to a special choice of θ. Fix $t \in [0, T]$ and $h > 0$ such that $t + h < T$ and choose

$$\theta(s) = \theta_0(t)\,\mathbf{1}_{[t,t+h]}(s), \qquad s \in [0, T],$$

where $\theta_0(t)$ is $\mathcal{G}_t$-measurable. Then by Lemma 2.5 we have

$$\begin{aligned} \mathrm{E}\bigg[\int_0^T \sigma(s)\theta(s)\,\mathrm{d}^- B(s)\bigg] &= \mathrm{E}\bigg[\int_t^{t+h} \sigma(s)\theta_0(t)\,\mathrm{d}^- B(s)\bigg] \\ &= \mathrm{E}\bigg[\theta_0(t)\int_t^{t+h} \sigma(s)\,\mathrm{d}B(s)\bigg]. \end{aligned}$$

Combining this with (3.8) we obtain

$$\mathrm{E}\bigg[\bigg(\int_t^{t+h} \{\mu(s) - r(s) - \sigma^2(s)\pi(s) - \mathbb{Q}^*\mathbb{Q}\pi(s)\}\,\mathrm{d}s + \int_t^{t+h} \sigma(s)\,\mathrm{d}B(s)\bigg)\theta(t)\bigg] = 0.$$

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Since this holds for all such $\theta(t)$ we conclude that

$$\mathrm{E}[M(t+h) - M(t) \mid \mathcal{G}_t] = 0,$$

where

$$M(t) := \int_0^t \{\mu(s) - r(s) - \sigma^2(s)\pi(s) - \mathrm{E}[\mathbb{Q}^*\mathbb{Q}\pi(s) \mid \mathcal{G}_s]\}\,\mathrm{d}s + \int_0^t \sigma(s)\,\mathrm{d}B(s).$$

Since $\sigma \neq 0$ this proves the following result.

Theorem 3.1. *Suppose that an optimal insider portfolio $\pi \in \mathcal{A}$ for Problem 3.2 exists. Then*

$$\mathrm{d}B(t) = \mathrm{d}\hat{B}(t) - \frac{1}{\sigma(t)}\{\mu(t) - \rho(t) - \sigma^2(t)\pi(t) - \mathrm{E}[\mathbb{Q}^*\mathbb{Q}\pi(t) \mid \mathcal{G}_t]\}\,\mathrm{d}t, \tag{3.9}$$

where $\hat{B}(t) := \int_0^t \sigma^{-1}(s)\,\mathrm{d}M(s)$ is a $\mathcal{G}_t$-Brownian motion. In particular, $B(t)$ is a semimartingale with respect to $\mathcal{G}_t$.

We now use this to find an equation for an optimal portfolio π.

Theorem 3.2. *Assume that there exists a process $\gamma_t(s,\omega)$ such that $\gamma_t(s)$ is $\mathcal{G}_t$-measurable for all $s \le t$,*

$$t \to \int_0^t \gamma_t(s)\,\mathrm{d}s \quad \textit{is of finite variation a.s.}$$

and

$$N(t) := B(t) - \int_0^t \gamma_t(s)\,\mathrm{d}s \quad \textit{is a martingale with respect to } \mathcal{G}_t. \tag{3.10}$$

Assume that $\pi \in \mathcal{A}$ is optimal, then

$$\sigma^2(t)\pi(t) + \mathrm{E}[\mathbb{Q}^*\mathbb{Q}\pi(t) \mid \mathcal{G}_t] = \mu(t) - r(t) + \sigma(t)\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^t \gamma_t(s)\,\mathrm{d}s\right). \tag{3.11}$$

Proof. By comparing (3.9) and (3.10) we obtain

$$\sigma(t)\,\mathrm{d}N(t) = \mathrm{d}M(t),$$

i.e.

$$-\sigma(t)\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^t \gamma_t(s)\,\mathrm{d}s\right) = \mu(t) - r(t) - \sigma^2(t)\pi(t) - \mathrm{E}[\mathbb{Q}^*\mathbb{Q}\pi(t) \mid \mathcal{G}_t].$$

Next we turn to a partial converse of Theorem 3.2.

Theorem 3.3. *Suppose that (3.10) holds. Let $\pi(t)$ be a process solving (3.11). Suppose that $\pi \in \mathcal{A}$. Then π is optimal for Problem 3.2.*

Proof. Substituting

$$\mathrm{d}B(t) = \mathrm{d}N(t) + \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^t \gamma_t(s)\,\mathrm{d}s\right)\mathrm{d}t$$

and

$$\sigma(t)\pi(t)\,\mathrm{d}^-B(t) = \sigma(t)\pi(t)\,\mathrm{d}N(t) + \sigma(t)\pi(t)\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^t \gamma_t(s)\,\mathrm{d}s\right)\mathrm{d}t$$

into (3.6) we obtain

$$J(\pi) = \mathrm{E}\bigg[\int_0^T \bigg\{ r(t) + (\mu(t) - r(t))\pi(t) - \frac{1}{2}\sigma^2(t)\pi^2(t) + \sigma(t)\pi(t)\frac{\mathrm{d}}{\mathrm{d}t}\bigg(\int_0^t \gamma_t(s)\,\mathrm{d}s\bigg) - \frac{1}{2}|\mathbb{Q}\pi(t)|^2 \bigg\}\,\mathrm{d}t\bigg]. \tag{3.12}$$

This is a concave functional of π, so if we can find $\pi = \pi^* \in \mathcal{A}$ such that

$$\frac{\mathrm{d}}{\mathrm{d}y}[J(\pi^* + y\theta)]_{y=0} = 0 \quad \text{for all } \theta \in \mathcal{A},$$

then π^* is optimal. By a computation similar to the one leading to (3.8) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}y}[J(\pi^* + y\theta)]_{y=0} = \mathrm{E}\bigg[\int_0^T \bigg\{ \mu(t) - r(t) - \sigma^2(t)\pi^*(t) + \sigma(t)\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \gamma_t(s)\,\mathrm{d}s - \mathbb{Q}^*\mathbb{Q}\pi(t)\bigg\}\theta(t)\,\mathrm{d}t\bigg].$$

This is equal to 0 if $\pi = \pi^*$ solves (3.11).

We now apply this to some examples.

Example 3.1. Choose

$$\mathbb{Q}\pi(t) = \lambda_1(t)\sigma(t)\pi(t), \tag{3.13}$$

where $\lambda_1(t) \geq 0$ is deterministic.

Then (3.11) takes the form

$$\sigma^2(t)\pi(t) + \lambda_1^2(t)\sigma^2(t)\pi(t) = \mu(t) - r(t) + \sigma(t)\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \gamma_t(s)\,\mathrm{d}s$$

or

$$\pi(t) = \pi^*(t) = \frac{\mu(t) - r(t) + \sigma(t)(\mathrm{d}/\mathrm{d}t)\int_0^t \gamma_t(s)\,\mathrm{d}s}{\sigma^2(t)[1 + \lambda_1^2(t)]}. \tag{3.14}$$

Substituting this into (3.12) we obtain the following result.

Theorem 3.4. *Suppose that (3.10) and (3.13) hold. Let $\pi^*(t)$ be given by (3.14). If $\pi \in \mathcal{A}$ then π^* is optimal for Problem 3.2. Moreover, the insider value is*

$$\Phi = J(\pi^*) = \mathrm{E}\bigg[\int_0^T \bigg\{ r(t) + \frac{1}{2}(1 + \lambda_1^2(t))^{-1}\bigg(\frac{\mu(t) - r(t)}{\sigma(t)} + \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \gamma_t(s)\,\mathrm{d}s\bigg)^2\bigg\}\,\mathrm{d}t\bigg]. \tag{3.15}$$

In particular, if we consider the case mentioned in Section 1, where

$$\mathcal{G}_t = \mathcal{F}_t \vee \sigma(B(T_0)) \quad \text{for some } T_0 > T,$$

then, by a result of Itô [7], we have

$$\gamma_t(s) = \gamma(s) = \frac{B(T_0) - B(s)}{T_0 - s},$$

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and (3.14) becomes

$$\pi^*(t) = \sigma^{-2}(t)[1+\lambda_1^2(t)]^{-1}\bigg[\mu(t) - r(t) + \frac{\sigma(t)}{T_0 - t}(B(T_0) - B(t))\bigg].$$

The corresponding value is, by (3.15),

$$J(\pi^*) = \mathrm{E}\bigg[\int_0^T \bigg\{r(t) + \frac{1}{2}(1+\lambda_1^2(t))^{-1}\bigg(\frac{\mu(t)-r(t)}{\sigma(t)} + \frac{B(T_0)-B(t)}{T_0 - t}\bigg)^2\bigg\}\,\mathrm{d}t\bigg].$$

In particular, we see that if $\sigma(t) \geq \sigma_0 > 0$ and

$$\lambda_1(t) = (T_0 - t)^{-\beta} \quad \text{for some constant } \beta > 0, \tag{3.16}$$

then

$$J(\pi^*) \leq C_1 + C_2 \int_0^T (T_0 - t)^{-1+2\beta}\,\mathrm{d}t < \infty$$

for suitable constants C_1 and C_2, even if $T_0 = T$. Thus, if we penalize large investments near $t = T_0$ then, according to (3.16), the insider obtains a finite value even if $T_0 = T$.

Example 3.2. Next we put

$$\mathbb{Q}\pi(t) = \pi'(t) \qquad \bigg(= \frac{\mathrm{d}}{\mathrm{d}t}\pi(t)\bigg). \tag{3.17}$$

This means that the insider is being penalized for large portfolio fluctuations. Choose $\mathcal{A}$ to be the set of all continuously differentiable processes $\pi(t)$ satisfying (3.2)–(3.5) and, in addition,

$$\pi(0) = \pi(T) = 0 \quad \text{a.s.} \tag{3.18}$$

For simplicity, assume that

$$\sigma(t) \equiv 1.$$

Then (3.11) can be expressed in the form

$$\pi(t) - \pi''(t) = a(t),$$

where

$$a(t) = \mu(t) - r(t) + \frac{\mathrm{d}}{\mathrm{d}t}\bigg(\int_0^t \gamma_t(s)\,\mathrm{d}s\bigg).$$

Using the *variation of parameter* method we obtain the solution

$$\pi(t) = \int_0^t \sinh(t-s)a(s)\,\mathrm{d}s + K\sinh(t), \tag{3.19}$$

where, as usual, $\sinh(x) = \frac{1}{2}(\mathrm{e}^x - \mathrm{e}^{-x})$, $x \in \mathbb{R}$, is the hyperbolic sine function and the constant K is chosen such that $\pi(T) = 0$. In particular, if we again consider the case in which

$$\mathcal{G}_t = \mathcal{F}_t \vee \sigma(B(T_0)), \qquad T_0 > T,$$

so that

$$\gamma_t(s) = \gamma(s) = \frac{B(T_0) - B(s)}{T_0 - s}, \qquad 0 \leq s \leq T,$$

then, by (3.19), we obtain

$$\pi(t) = \int_0^t \sinh(t-s)\left[\mu(s) - r(s) + \frac{B(T_0) - B(s)}{T_0 - s}\right] \mathrm{d}s + K \sinh(t). \tag{3.20}$$

By (3.12), the corresponding value is

$$J(\pi) = \mathrm{E}\left[\int_0^T \left\{r(t) + (\mu(t) - r(t))\pi(t) - \frac{1}{2}\pi^2(t) + \pi(t)\frac{B(T_0) - B(t)}{T_0 - t} - \frac{1}{2}(\pi'(t))^2\right\} \mathrm{d}t\right].$$

Note that if $0 \le t \le T < T_0$ then

$$\mathrm{E}\left[\pi(t)\frac{B(T_0) - B(t)}{T_0 - t}\right] \le \mathrm{E}\left[\int_0^t \sinh(t-s)\frac{(B(T_0) - B(s))(B(T_0) - B(t))}{(T_0 - s)(T_0 - t)}\,\mathrm{d}s\right] = \int_0^t \frac{\sinh(t-s)}{T_0 - s}\mathrm{d}s.$$

Therefore,

$$J(\pi) \le \int_0^T \left(\int_0^t \frac{\sinh(t-s)}{T_0 - s}\mathrm{d}s\right) \mathrm{d}t \le \int_0^T \frac{\cosh(T-s) - 1}{T - s}\,\mathrm{d}s \quad \text{for all } T_0 > T.$$

We have proved the following result.

Theorem 3.5. *Suppose that* $\mathbb{Q}\pi(t) = \pi'(t)$ *and* $\mathcal{A}$ *is chosen as in (3.17) and (3.18), and assume that* $\sigma(t) = 1$. *Then the optimal insider portfolio is given by (3.19). In particular, if we choose*

$$\mathcal{G}_t = \mathcal{F}_t \vee \sigma(B(T_0)) \quad \text{with } T_0 > T,$$

then the optimal portfolio π *is given by (3.20) and the corresponding insider value* $J(\pi)$ *is uniformly bounded for* $T_0 > T$.

Remark 3.1. Both Examples 3.6 and 3.8 yield ways to penalize the insider investor so that he would not obtain infinite utility. In Example 3.6, $\lambda_1(t) = (T_0 - t)^{-\beta}$ for some $\beta > 0$. To use this penalization, we need to know T_0. In Example 3.8, T_0 is not required to be known.

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