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JAKUBOWSKI STARLIKE INTEGRAL OPERATORS

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Abstract

Let $S(m, M)$ be the set of functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ regular and satisfying $|zf'(z)/f(z) - m| < M$ in $|z| < 1$, where $|m - 1| < M \leqslant m$; and let $S^*(\rho)$ be the set of starlike functions of order ρ, $0 \leqslant \rho < 1$. In this paper we obtain integral operators which map $S(m, M)$ into $S(m, M)$ and $S^*(\rho) \times S(m, M)$ into $S^*(\rho)$. Our results improve and generalize many recent results.

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1. Introduction

Let S denote the set of functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ that are regular in the open unit disc $U = \{z: |z| < 1\}$. A function f of S is said to belong to $S^*(\rho)$, the set of starlike univalent functions of order ρ, if $\operatorname{Re}\{zf'(z)/f(z)\} > \rho$ for $z \in U$, $0 \leqslant \rho < 1$. The set S^* of starlike univalent functions is identified by $S^*(0) \equiv S^*$. A function f of S is said to belong to the set $S(m, M)$ if $|zf'(z)/f(z) - m| < M$ for $z \in U$, where $|m - 1| < M \leqslant m$. The set $S(m, M)$ was introduced by Jakubowski [6]. It is clear that $m > 1/2$ and $S(m, M) \subset S^*(m - M) \subset S^*$.

Many authors (see [1], [2], [3], [4]) have studied, for $\eta = 0$, the integral operators of the form

$$(1.1) \qquad I(f) \equiv \left[\frac{\gamma + \alpha + \eta}{z^{\gamma}} \int_0^z u^{\gamma+\eta-1} f^{\alpha}(u)\, du\right]^{1/(\alpha+\eta)}$$

and

$$I(f, g) \equiv \left[\frac{\gamma + 2\alpha}{z^{\gamma+\alpha}} \int_0^z u^{\gamma-1} f^{\alpha}(u) g^{\alpha}(u)\, du \right]^{1/\alpha}, \tag{1.2}$$

where α and γ are suitably chosen real constants and f and g belong to some favoured classes of univalent functions. Recently, Miller *et al.* [9, Theorem 4] have shown that, for $\eta \geqslant 0$, $I(f)$ maps S^* into itself. Their result provides the expressions of curious forms for the starlike functions. In the present paper the authors obtain a result from which it follows that, for $\eta \geqslant 0$, $I(f)$ maps $S(m, M)$ into itself. An integral operator has been also obtained which is more general as well as applicable than $I(f, g)$ and maps $S^*(\rho) \times S(m, M)$ into $S^*(\rho)$.

Our results improve some recent results of Goel and Mehrok [3] and Gupta and Jain [4], and generalize some recent results of Bhargava and Shukla [1] and Miller *et al.* [9].

2. Preliminary lemmas

The following lemma may be found in [6], [8].

LEMMA 2.1. *The function f belongs to $S(m, M)$ if and only if there exists a function w regular in U and satisfying $w(0) = 0$, $|w(z)| < 1$ for $z \in U$ such that*

$$z\frac{f'(z)}{f(z)} = \frac{1 + aw(z)}{1 - bw(z)}, \qquad z \in U, \tag{2.1}$$

where $a = (M^2 - m^2 + m)/M$ and $b = (m - 1)/M$.

Next we have the well known Jack's lemma [5].

LEMMA 2.2. *If the function w is regular for $|z| \leqslant r < 1$, $w(0) = 0$ and $|w(z_0)| = \max_{|z|=r} |w(z)|$, then $z_0 w'(z_0) = kw(z_0)$, where k is a real number such that $k \geqslant 1$.*

Lastly we prove a lemma which plays an important role in establishing one of our main results.

LEMMA 2.3. *Let α, β, m and M be real numbers such that $0 < \alpha \leqslant \beta$, $|m - 1| < M \leqslant m$. If $t = (m\alpha + \beta - \alpha)/\beta$ and $T = M\alpha/\beta$ then $S(t, T) \subset S(m, M)$.*

PROOF. We need only to consider the case when $\alpha < \beta$. In order to establish the lemma it suffices to show that

$$m - M < t - T \quad \text{and} \quad t + T < m + M. \tag{2.2}$$

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Let $m - M \geqslant t - T$. Then $((m - M)\alpha + \beta - \alpha)/\beta \leqslant m - M$, which implies that $m - M \geqslant 1$. But this is contrary to the assumption $|m - 1| < M$. Next, suppose that $t + T \geqslant m + M$. Then $m + M \leqslant ((m + M)\alpha + \beta - \alpha)/\beta$, which implies that $m + M \leqslant 1$. But this is also contrary to $|m - 1| < M$. Therefore the inequalities in (2.2) hold and hence the required result follows.

Hereafter in this paper t and T are the same as in the above lemma.

3. Integral operators that map $S(m, M)$ into $S(m, M)$

An integral operator which is defined on $S(m, M)$ and maps $S(m, M)$ into (or onto) itself is called Jakubowski starlike integral operator. We now prove the following:

THEOREM 3.1. *Let α, β, γ and δ be real constants such that $0 < \alpha \leqslant \beta$ and $\gamma + \beta = \delta + \alpha$. If $f \in S(m, M)$ then the function F defined by*

$$F(z) = \left[\frac{\gamma + \beta}{z^{\gamma}} \int_0^z u^{\delta - 1} f^{\alpha}(u)\, du\right]^{1/\beta} \tag{3.1}$$

also belongs to $S(m, M)$, provided $\gamma \geqslant -[\beta + \alpha(m - M - 1)]$.

In (3.1) all powers are principal ones.

PROOF. First of all we show that $F \in S(t, T)$. Let us choose a function w such that

$$z\frac{F'(z)}{F(z)} = \frac{1 + cw(z)}{1 - ew(z)} \tag{3.2}$$

where $w(0) = 0$ and w is either regular or meromorphic in U; here $c = (T^2 - t^2 + t)/T$ and $e = (t - 1)/T$. From (3.1) and (3.2) we have

$$(\gamma + \beta) z^{\delta - \gamma} \frac{f^{\alpha}(z)}{F^{\beta}(z)} = \frac{(\gamma + \beta) + (c\beta - e\gamma)w(z)}{1 - ew(z)}. \tag{3.3}$$

Logarithmic differentiation of (3.3) yields

$$\frac{\alpha}{\beta}\left[z\frac{f'(z)}{f(z)} - m\right] = \frac{(1 - t) + (c + et)w(z)}{1 - ew(z)} + \frac{(c + e)zw'(z)}{[1 - ew(z)][(\gamma + \beta) + (c\beta - e\gamma)w(z)]}. \tag{3.4}$$

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$|w(z)| \neq 1$ in $|z| < r_0$, w cannot have a pole at $|z| = r_0$. Therefore w is regular and satisfies $|w(z)| < 1$, for z in U.

Thus from (3.2) and Lemma 2.1, $F \in S(t, T)$. But Lemma 2.3 ensures that $S(t, T) \subset S(m, M)$. Hence $F \in S(m, M)$ and the proof is completed now.

COROLLARY 3.1. *If* $0 \leqslant \alpha \leqslant 1/(1 - m + M)$, $\alpha \leqslant \beta$, *and if* $f \in S(m, M)$, *then the function* F *defined by*

$$F(z) = \left[z^{\beta - 1} \int_0^z \left(\frac{f(u)}{u} \right)^\alpha du \right]^{1/\beta}$$

also belongs to $S(m, M)$.

The above corollary follows by taking $\gamma = 1 - \beta$ and $\delta = 1 - \alpha$ in Theorem 3.1.

REMARK. When $m = M$ and $m \to \infty$, a result of Miller *et al.* [9, Theorem 3] follows from this corollary.

Another direct but important consequence of Theorem 3.1 is the following result which enables us to obtain the expressions for the functions in $S(m, M)$ of curious forms.

COROLLARY 3.2. *Let* α *and* η *be real constants such that* $\alpha > 0$, $\eta \geqslant 0$. *If* $f \in S(m, M)$ *then the function* F *defined by*

$$F(z) = \left[\frac{\gamma + \alpha + \eta}{z^\gamma} \int_0^z u^{\gamma + \eta - 1} f^\alpha(u)\, du \right]^{1/(\alpha + \eta)} \tag{3.9}$$

also belongs to $S(m, M)$, *provided* $\gamma + \eta \geqslant -\alpha(m - M)$.

The above result is obtained if, in Theorem 3.1, we take $\beta = \alpha + \eta$ and $\delta = \gamma + \eta$.

For $\gamma + \eta = 1$, $\alpha = 1$, $\eta = 0, 1, 2, \ldots$, we obtain the sequence

$$\left[2 z^{n-1} \int_0^z f(u)\, du \right]^{1/(n+1)} = z + \cdots, \qquad n = 0, 1, 2, \ldots,$$

and for $\gamma = 0$, $\alpha = 1$, $\eta = 0, 1, 2, \ldots$, we obtain the sequence

$$\left[(n + 1) \int_0^z u^{n-1} f(u)\, du \right]^{1/(n+1)} = z + \cdots, \qquad n = 0, 1, 2, \ldots,$$

of elements of $S(m, M)$.

REMARK. A recent result of Bhargava and Shukla [1] follows by taking $\eta = 0$ in Corollary 3.2.

Let us choose $m = N - \rho(N-1)$ and $M = N(1-\rho)$, where $N \geqslant 1$ and $0 \leqslant \rho < 1$. Then $|m-1| < M \leqslant m$, $a = \rho/N + (1-2\rho)$ and $b = 1 - 1/N$. Now as $N \to \infty$, $a \to (1-2\rho)$ and $b \to 1$. In this case the relation (2.1) reduces to

$$z\frac{f'(z)}{f(z)} = \frac{1+(1-2\rho)w(z)}{1-w(z)}, \qquad z \in U,$$

which is a necessary and sufficient condition for f to be in $S^*(\rho)$. The undermentioned corollary follows now from Corollary 3.2.

COROLLARY 3.3. *Let α and η be real constants such that $\alpha > 0$, $\eta \geqslant 0$. If $f \in S^*(\rho)$ then the function F defined by* (3.9) *also belongs to $S^*(\rho)$ provided $\gamma + \eta \geqslant -\alpha\rho$.*

REMARKS. (i) A result of Miller *et al.* [9, Theorem 4] follows by taking $\rho = 0$ in the above corollary.

(ii) A result of Gupta and Jain [4, Theorem 1] follows by taking $\eta = 0$ in Corollary 3.3. However the transform

$$I(f) = \left[\frac{11}{4z^2}\int_0^z u f^{3/4}(u)\,du\right]^{4/3} = z + \cdots$$

can be studied by Corollary 3.3 and not by the result of Gupta and Jain, since, the technique followed by them fails when α and γ are not positive integers.

Let $-1 < B < A \leqslant 1$. If we set $m = (1-AB)/(1-B^2)$ and $M = (A-B)/(1-B^2)$, then (2.1) becomes

$$z\frac{f'(z)}{f(z)} = \frac{1+Aw(z)}{1+Bw(z)}, \qquad z \in U. \tag{3.10}$$

Let us denote by $S^*[A, B]$, $-1 \leqslant B < A \leqslant 1$, the class of functions f satisfying (3.10). Then, including the limiting case $B \to -1$ (Corollary 3.3), Theorem 3.1 provides:

COROLLARY 3.4. *Let γ be any real number such that $\gamma \geqslant -(1-A)/(1-B)$. If $f \in S^*[A, B]$, then the function F defined by*

$$F(z) = \frac{\gamma+1}{z^\gamma}\int_0^z u^{\gamma-1} f(u)\,du \tag{3.11}$$

also belongs to $S^*[A, B]$.

REMARK. The above corollary improves a recent result of Goel and Mehrok [3] who proved it when γ is a positive integer. Here it is worth noting that, however, Goel and Mehrok also proved it with the help of Jack's lemma, but the classical technique used by them fails when γ is other than a positive integer.

Let $C^*[A, B]$ be the class of functions f of S for which there exists a function g in $S^*[A, B]$ such that $\operatorname{Re}\{zf'(z)/g(z)\} > 0$, $z \in U$. Clearly the functions in $C^*[A, B]$ are close-to-convex in U. Goel and Mehrok [3] have shown that, if $f \in C^*[A, B]$, then the function F defined by (3.11) also belongs to $C^*[A, B]$ when γ is a positive integer. In continuation we now improve that result for real γ.

THEOREM 3.2. *Let γ be a real number such that $\gamma \geqslant -(1-A)/(1-B)$. If $f \in C^*[A, B]$, then the function F defined by* (3.11) *also belongs to* $C^*[A, B]$.

PROOF. Since $f \in C^*[A, B]$, there exists a function g in $S^*[A, B]$ such that $\operatorname{Re}\{zf'(z)/g(z)\} > 0$, $z \in U$. By Corollary 3.4, for $g \in S^*[A, B]$, the function G defined by

$$G(z) = \frac{\gamma+1}{z^\gamma}\int_0^z u^{\gamma-1}g(u)\,du \tag{3.12}$$

also belongs to $S^*[A, B]$. It is easy to obtain from (3.11) and (3.12) that

$$\frac{zf'(z)}{g(z)} = \left(\frac{zF'(z)}{G(z)}\right)\left[\frac{\gamma+1+zF''(z)/F'(z)}{\gamma+zG'(z)/G(z)}\right].$$

Substituting $p(z) = zF'(z)/G(z)$ and $q(z) = zG'(z)/G(z)$, the above relation reduces to

$$\frac{zf'(z)}{g(z)} = p(z) + \frac{zp'(z)}{\gamma+q(z)}.$$

Our theorem is proved if we can show that

$$\operatorname{Re}\left\{p(z) + \frac{zp'(z)}{\gamma+q(z)}\right\} > 0 \quad \text{implies} \quad \operatorname{Re}\{p(z)\} > 0, z \in U.$$

Let a function w regular in U such that $w(0) = 0$ and $w(z) \neq 1$ be defined by

$$p(z) = \frac{1+w(z)}{1-w(z)}. \tag{3.13}$$

We claim that $|w(z)| < 1$ for $z \in U$. For, otherwise by Lemma 2.2, there exists a z_0, $|z_0| < 1$ such that

$$z_0 w'(z_0) = kw(z_0) \tag{3.14}$$

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with $|w(z_0)| = 1$ and $k \geqslant 1$. From (3.13) and (3.14) we get

$$\begin{aligned}\operatorname{Re}\left\{\frac{z_0 f'(z_0)}{g(z_0)}\right\} &= \operatorname{Re}\left\{p(z_0) + \frac{z_0 p'(z_0)}{\gamma + q(z_0)}\right\}\\ &= 2k \operatorname{Re}\left\{\frac{w(z_0)}{(1 - w(z_0))^2} \cdot \frac{1}{\gamma + q(z_0)}\right\}\\ &= -2k\mu \operatorname{Re}\{\gamma + q(z_0)\}^{-1}\\ &\leqslant 0,\end{aligned}$$

here $\mu = -w(z_0)/(1 - w(z_0))^2 \geqslant 1/4$ and $\operatorname{Re}\{\gamma + q(z_0)\}^{-1} \geqslant 0$ for $\gamma \geqslant -(1 - A)/(1 - B)$. But this is contrary to the fact that $f \in C^*[A, B]$. Hence $|w(z)| < 1$ for $z \in U$ and by (3.13) $F \in C^*[A, B]$.

We now consider the integral operator defined in (3.1) in a limiting case. When $\alpha = \beta$, the relation (3.1) can be written as

$$f(z) = \left[\{\gamma + \beta z F'(z)/F(z)\}/(\gamma + \beta)\right]^{1/\beta}.$$

When $\beta \to 0$, the above relation reduces to

$$f(z) = F(z) \exp\left[\{zF'(z)/F(z) - 1\}/\gamma\right] \tag{3.15}$$

where $\gamma > 0$. It follows from (3.15) that

$$F(z) = f(z) \exp\left[-z^{-\gamma} \int_0^z u^{\gamma - 1}\left\{u\frac{f'(u)}{f(u)} - 1\right\} du\right]. \tag{3.16}$$

We now prove the following:

THEOREM 3.3. *If $f \in S(m, M)$ and $\gamma > 0$, then the function F defined by* (3.16) *also belongs to $S(m, M)$.*

PROOF. Let us choose a function w such that

$$z\frac{F'(z)}{F(z)} = \frac{1 + aw(z)}{1 - bw(z)} \tag{3.17}$$

where $w(0) = 0$ and w is either regular or meromorphic in U; here $a = (M^2 - m^2 + m)/M$ and $b = (m - 1)/M$. Differentiating (3.15) logarithmically and using (3.17) we get

$$z\frac{f'(z)}{f(z)} - m = \frac{(1 - m) + (a + bm)w(z)}{1 - bw(z)} + \frac{(a + b)zw'(z)}{\gamma[1 - bw(z)]^2}.$$

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The required result can be proved now on the lines of the proof of Theorem 3.1.

4. Integral operators that map $S^*(\rho) \times S(m, M)$ into $S^*(\rho)$

THEOREM 4.1. *Let α, β, γ, δ and σ be real constants such that $\alpha > 0$, $\beta \geqslant \alpha$, $\sigma \geqslant 0$, $\alpha + \delta = \beta + \gamma$ and $\gamma > -(\sigma + \beta\rho)$. If $f \in S^*(\rho)$ and $g \in S(m, M)$, $(m, M) \in E = \{(m, M)\colon |m - 1| < M \leqslant m^*\}$, where $m^* = \min\{m, (m-1) + \beta(1-\rho)/(2\sigma(\gamma + \sigma + \beta\rho))\}$, then the function F defined by*

$$F(z) = \left[\frac{\gamma + \beta + \sigma}{z^{\gamma+\sigma}} \int_0^z u^{\delta-1} f^{\alpha}(u) g^{\sigma}(u)\, du\right]^{1/\beta} \tag{4.1}$$

also belongs to $S^*(\rho)$.

In (4.1) all powers are principal ones.

PROOF. Let us choose a function w such that

$$z\frac{F'(z)}{F(z)} = \frac{1 + (2\rho - 1)w(z)}{1 + w(z)} \tag{4.2}$$

where $w(0) = 0$ and w is either regular or meromorphic in U.

From (4.1) and (4.2) we have

$$\eta z^{\delta-\gamma-\sigma}\left\{\frac{f^{\alpha}(z)g^{\sigma}(z)}{F^{\beta}(z)}\right\} = \frac{\eta + \xi w(z)}{1 + w(z)} \tag{4.3}$$

where $\eta = \gamma + \beta + \sigma$ and $\xi = \gamma + (\sigma - \beta) + 2\beta\rho$.

Logarithmic differentiation of (4.3) yields

$$\begin{aligned} z\frac{f'(z)}{f(z)} = \frac{\sigma}{\alpha}(1 - m) - \frac{\sigma}{\alpha}\left\{z\frac{g'(z)}{g(z)} - m\right\} + \left(\frac{\beta}{\alpha}\right)\left[\frac{1 + (2\rho - 1)w(z)}{1 + w(z)}\right] \\ - \left(\frac{\beta - \alpha}{\alpha}\right) - \frac{2\beta(1-\rho)zw'(z)}{\alpha\{1 + w(z)\}\{\eta + \xi w(z)\}}. \end{aligned} \tag{4.4}$$

Let r^* be the distance from the origin of the pole of w nearest the origin. Then w is regular in $|z| < r_0 = \min\{r^*, 1\}$. By Lemma 2.2, for $|z| \leqslant r$ $(r < r_0)$, there is a point z_0 such that

$$z_0 w'(z_0) = k w(z_0), \qquad k \geqslant 1. \tag{4.5}$$

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From (4.4) and (4.5) we get

(4.6)

$$\begin{aligned}
\operatorname{Re}\left\{z_0\frac{f'(z_0)}{f(z_0)}\right\} &\leqslant \frac{\sigma}{\alpha}(1-m)+\frac{\sigma}{\alpha}\left|z_0\frac{g'(z_0)}{g(z_0)}-m\right| \\
&\quad +\left(\frac{\beta}{\alpha}\right)\left[\frac{\operatorname{Re}\left[\{1+(2\rho-1)w(z_0)\}\{1+\overline{w(z_0)}\}\right]}{|1+w(z_0)|^2}\right]-\left(\frac{\beta-\alpha}{\alpha}\right) \\
&\quad -\frac{2\beta k(1-\rho)\operatorname{Re}\left[w(z_0)\{1+\overline{w(z_0)}\}\{\eta+\xi\overline{w(z_0)}\}\right]}{\alpha|1+w(z_0)|^2|\eta+\xi w(z_0)|^2} \\
&< \frac{\sigma}{\alpha}\{M-(m-1)\}-\left(\frac{\beta-\alpha}{\alpha}\right) \\
&\quad +\left(\frac{\beta}{\alpha}\right)\left[\frac{1+2\rho\operatorname{Re}w(z_0)+(2\rho-1)|w(z_0)|^2}{1+2\operatorname{Re}w(z_0)+|w(z_0)|^2}\right] \\
&\quad -\frac{2\beta k(1-\rho)\operatorname{Re}\left[\eta w(z_0)+2(\gamma+\sigma+\beta\rho)|w(z_0)|^2+\xi|w(z_0)|^2\overline{w(z_0)}\right]}{\alpha\{1+2\operatorname{Re}w(z_0)+|w(z_0)|^2\}\{\eta^2+2\eta\xi\operatorname{Re}w(z_0)+\xi^2|w(z_0)|^2\}}.
\end{aligned}$$

Now suppose that it were possible to have $\mathfrak{M}(r,w)=\max_{|z|=r}|w(z)|=1$ for some r ($r<r_0\leqslant 1$). At the point z_0 where this occurred, we would have $|w(z_0)|=1$. Then, in view of $k\geqslant 1$, we have from (4.5) and (4.6) that

$$\begin{aligned}
\operatorname{Re}\left\{z_0\frac{f'(z_0)}{f(z_0)}\right\} &\leqslant \frac{\sigma}{\alpha}\{M-(m-1)\}-(\beta-\alpha)/\alpha+\left(\frac{\beta}{\alpha}\right)\rho \\
&\quad -\frac{2\beta(1-\rho)(\gamma+\sigma+\beta\rho)}{\alpha\{\eta^2+2\eta\xi\operatorname{Re}w(z_0)+\xi^2\}} \\
&\leqslant \rho+\frac{2(\gamma+\sigma+\beta\rho)[2\sigma(\gamma+\sigma+\beta\rho)\{M-(m-1)\}-\beta(1-\rho)]}{\alpha\{\eta^2+2\eta\xi\operatorname{Re}w(z_0)+\xi^2\}} \\
&\leqslant \rho,\quad \text{provided } M\leqslant(m-1)+\frac{\beta(1-\rho)}{2\sigma(\gamma+\sigma+\beta\rho)}.
\end{aligned}$$

But this is contrary to the fact that $f\in S^*(\rho)$. So we cannot have $\mathfrak{M}(r,w)=1$. Thus $|w(z)|\neq 1$ in $|z|<r_0$. Since $w(0)=0$, $|w(z)|$ is continuous in $|z|<r_0$ and $|w(z)|\neq 1$ there, w cannot have a pole at $|z|=r_0$. Therefore $|w(z)|<1$ and w is regular in U.

Hence, from (4.2), $F\in S^*(\rho)$.

REMARK. In the particular case when $\alpha = \beta = \sigma$ and $\alpha > 0$, a recent result of Bhargava and Shukla [1] follows from Theorem 4.1. In the sequel it is worth noting that the transforms of the form

$$F(z) = \left[\frac{\gamma + \beta}{z^{\gamma}} \int_0^z u^{\gamma - 1} f^{\beta}(u)\, du \right]^{1/\beta},$$

that map $S^*(\rho)$ into itself, can be studied from Theorem 4.1 and not from that result of Bhargava and Shukla.

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