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Some inequalities in trigonometric approximation

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For a nonconstant $L^2(-\pi, \pi)$ function f, we prove that $\frac{1}{\pi}\omega_2\left(\frac{\pi}{n+1}; f\right) < \|\sigma_n(f)-f\|_2 < \frac{1}{\sqrt{2}}\omega_2\left(\frac{\pi}{n+1}; f\right)$ and that the inequalities are sharp.

Let $s_n(f)$ be the n-th partial sum and $\sigma_n(f)$ the n-th Cesàro means of the Fourier series of an $L^2 = L^2(-\pi, \pi)$ function f. Extend f periodically to the real line and let $\omega_2(\delta; f)$ denote the L^2 integral modulus of continuity of f. For nonconstant f, Černyh [1] proved that

$$\|s_n(f)-f\|_2 < \frac{1}{\sqrt{2}}\omega_2\left(\frac{\pi}{n+1}; f\right) \tag{1}$$

and that the constant $1/\sqrt{2}$ cannot be made smaller. In this note, we show that Černyh's proof can be improved to give

$$\frac{1}{\pi}\omega_2\left(\frac{\pi}{n+1}; f\right) < \|\sigma_n(f)-f\|_2 < \frac{1}{\sqrt{2}}\omega_2\left(\frac{\pi}{n+1}; f\right) \tag{2}$$

for all nonconstant $f \in L^2$ and all n. We also note that the constant $1/\pi$ cannot be made larger, and hence, the inequalities in (2) are best possible. In general, it is well-known that $\|\sigma_n(f)-f\|_p < C_p\omega_p\left(\frac{\pi}{n+1}; f\right)$. However, the best constants C_p, $p \neq 2$, do not seem to be known to our knowledge.

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To prove the inequalities in (2), we write

$$\|\sigma_n(f)-f\|_2^2 = \sum_{|k|\le n} \left(\frac{k}{n+1}\right)^2 |a_k|^2 + \sum_{|k|>n} |a_k|^2$$

and

$$\omega_2^2\left(\frac{\pi}{n+1};\ f\right) = \sup_{0\le t\le \frac{\pi}{n+1}} \sum_{k=-\infty}^{\infty} 4|a_k|^2 \sin^2\left(\frac{kt}{2}\right) ,$$

where the a_k are the Fourier coefficients of $f \in L^2$. It can be shown that

$$\sum_{|k|\le n+1} \left(\frac{k}{n+1}\right)^2 |a_k|^2 \le \sup_{0\le t\le \frac{\pi}{n+1}} \sum_{|k|\le n+1} |a_k|^2 \sin^2\left(\frac{kt}{2}\right) ;$$

and following the proof of the theorem in [1], we have

$$\sum_{|k|>n+1} |a_k|^2 < \sup_{0\le t\le \frac{\pi}{n+1}} \sum_{|k|>n+1} 2|a_k|^2 \sin^2\left(\frac{kt}{2}\right)$$

for nonconstant f. This gives the second inequality in (2). The other inequality in (2) also follows, since if f is not constant, then

$$\begin{aligned}\omega_2^2\left(\frac{\pi}{n+1};\ f\right) &\le \sup_{0\le t\le \frac{\pi}{n+1}} \sum_{|k|\le n} 4|a_k|^2 \sin^2\left(\frac{kt}{2}\right) + 4 \sum_{|k|>n} |a_k|^2 \\ &< \pi^2 \sum_{|k|\le n} \left(\frac{k}{n+1}\right)^2 |a_k|^2 + 4 \sum_{|k|>n} |a_k|^2 \\ &\le \pi^2 \|\sigma_n(f)-f\|_2^2 .\end{aligned}$$

That the constant $1/\pi$ cannot be made larger follows simply from the example $f\left(e^{it}\right) = e^{it}$.

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Reference

[1] Н.И. Черных [N.I. Černyh], "О неравенстве Джексона в L_2 ", [On Jackson's inequality in L_2], *Trudy Mat. Inst. Steklov.* 88 (1967), 71-74; quoted from *Proc. Steklov Inst. Math. (Amer. Math. Soc.)* 88 (1969), 75-78.

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