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# FRENET FORMULAE FOR HOLOMORPHIC CURVES IN THE TWO QUADRIC

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We give a complete description of holomorphic curves in the complex two quadric via the method of moving frames. For compact curves a Morse theory type integral formula is derived.

## 0. Introduction

Let $M$ be an orientable Riemannian two manifold. Also let $Q_2$ denote the hyperquadric in $CP^3$ which is identified with the real Grassmannian of oriented two planes in $R^4$. In this paper we study isometric holomorphic immersion of $M$ into $Q_2$, where $M$ is given the induced complex structure coming from its (two dimensional) Riemannian metric.

Viewing $Q_2$ as a homogeneous space $SO(4)/SO(2)\times SO(2)$ we apply the method of repère mobile. We succeed in finding a local normal form (10), and in doing so we stumble upon a global contact invariant which we call $\tau$. This invariant $\tau$ is our analogue of the torsion of real curves in Euclidean three space. Indeed the totality of holomorphic isometric immersions of $M$ into $Q_2$ is parametrized by solutions of a single differential equation (17) on $M$ involving the Gaussian curvature of $M$ and the invariant $\tau$. Moreover, given a solution $K$, $\tau$ of (17) an

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actual immersion of $M$ into $Q_2$ is constructed using integration involving ordinary differential equations only. Thus the equation (17) may be called the complete integrability condition of the exterior system describing holomorphic isometric immersions of $M$ into $Q_2$.

Assuming that $M$ is compact and connected now, integration of (17) yields an interesting integral formula (21). This formula relates the number of zeros of $\tau$, area of $M$, and the Euler-Poincaré characteristic of $M$ in a simple way. The significance of this Morse theory type formula is not apparent to the author.

Finally in section 3, as an application of our local normal form (10) the superminimality of the orthogonal maps of holomorphic curves in $Q_2$ is established.

## 1. Frenet frame construction

In this section we give a moving frame theoretic description of isometrically immersed holomorphic curves in the two-quadric. Besides finding a local normal form (10) for such curves we obtain a single global invariant, which we call $\tau$. The totality of holomorphic curves then is parametrized by the solutions of a *PDE* on $M$ (17) involving the Gaussian curvature and $\tau$.

Let $Q_2$ denote the complex hyperquadric in $CP^3$. $Q_2$ is also the real Grassmannian of oriented two-planes in $R^4$. As a homogeneous space $Q_2 = SO(4) / SO(2)\times SO(2)$.

The following index convention will be adhered to throughout this section: $1 \leq i,j,k, \ldots \leq 2$, $3 \leq a,b,c, \ldots \leq 4$, and $1 \leq \alpha,\beta,\gamma, \ldots \leq 4$.

If $A = (A_1,A_2,A_3,A_4) = (A_\alpha) \in SO(4)$ then the projection map $\pi: SO(4) \to Q_2$ is given by $\pi(A) = [A_1 \wedge A_2] = [A_1 + iA_2]$, where $[A_1 \wedge A_2]$ is the oriented two-plane in $R^4$ spanned by $A_1$ and $A_2$ and $[A_1 + iA_2]$ is the point in $CP^3$ represented by the homogeneour coordinate vector $A_1 + iA_2$.

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Let $\Omega = (\Omega^\alpha_\beta)$ denote the Maurer-Cartan form of $SO(4)$. Then a $SO(4)$-invariant hermitian metric on $Q_2$ is given by the pull-back of $\frac{1}{2} \sum_{a,i} (\Omega^a_i)^2$. (This is the induced metric of the inclusion $Q_2 \subset CP^3$, the latter with the standard Fubini-Study metric.)

We use $(M, ds^2)$ to denote a connected, orientable Riemannian two manifold. Let $\theta^1, \theta^2,$ be a local orthonormal coframe field so that $ds^2 = (\theta^1)^2 + (\theta^2)^2$. By decreeing that $\phi = \theta^1 + i\theta^2$ is of type (1,0) we introduce an almost complex structure on $M$ and by the Korn-Lichtenstein theorem this almost complex structure is actually complex, hence $M$ is now a Riemann surface.

Let $f : M \to Q_2$ be a holomorphic, isometric immersion. A local lifting $e = (e_\alpha) : U \subset M \to SO(4)$ (which exists) will be called a $SO(4)$-frame along $f$. Note that $\pi \circ e = f = [e_1 \wedge e_2]$.

Notation. $e^* \Omega^\alpha_\beta = \omega^\alpha_\beta$, $\phi^1 = \frac{1}{\sqrt{2}} (\omega^3_1 + i\omega^3_2)$, $\phi^2 = \frac{1}{\sqrt{2}} (\omega^4_1 + i\omega^4_2)$.

Then holomorphicity of $f$ is reflected by the fact that $\phi^1$ and $\phi^2$ are type (1,0) forms on $M$. Thus we can find locally defined complex valued functions $Z^1, Z^2$ such that

(1) $$\phi^1 = Z^1\phi , \quad \phi^2 = Z^2\phi .$$

Since $f$ is an isometric immersion we must have

(2) $$|Z^1|^2 + |Z^2|^2 = 1.$$

Given $e$, other local $SO(4)$-frames along $f$ are given by $\tilde{e} = ek$ where $k = (e^{it_1}, e^{it_2}) : U \to U(1)\times U(1) = SO(2)\times SO(2)$. (We use the identification $e^{it} \leftrightarrow \begin{pmatrix} \cos t, & -\sin t \\ \sin t, & \cos t \end{pmatrix}$.)

Notation. $\tilde{e}^*\Omega^\alpha_\beta = \tilde{\omega}^\alpha_\beta$, similary define $(\tilde{\phi}^i)$.

Let $\tilde{Z}^1, \tilde{Z}^2$ be locally defined complex valued functions so that

(3) $$\tilde{\phi}^1 = \tilde{Z}^1\phi , \quad \tilde{\phi}^2 = \tilde{Z}^2\phi .$$

Computations show that

$$(4) \qquad \begin{pmatrix} \tilde{Z}^1 \\ \tilde{Z}^2 \end{pmatrix} = \begin{pmatrix} \cos t_2, & \sin t_2 \\ -\sin t_2, & \cos t_2 \end{pmatrix} \begin{pmatrix} e^{it_1} Z^1 \\ e^{it_1} Z^2 \end{pmatrix}$$

Define $\tau = |(Z^1)^2 + (Z^2)^2|$, $\tilde{\tau} = |(\tilde{Z}^1)^2 + (\tilde{Z}^2)^2|$ . From (4) it follows immediately that $\tau = \tilde{\tau}$ , that is, $\tau$ is a global invariant. Note that $\tau : M \to [0,1]$.

PROPOSITION. i) *If* $\tau$ *is constant then either* $\tau \equiv 0$ *or* $\tau \equiv 1$*;* ii) *if* $\tau \equiv 0$ *then* $f(M)$ *is congruent to an open submanifold of (totally geodesic)* $CP^1 = U(2) / U(1)\times U(1) \subset Q_2$ *with the Gaussian curvature* $4$*;*
iii) *if* $\tau \equiv 1$ *then* $f(M)$ *is congruent to an open submanifold of* $Q_1 = S^2 = SO(3) / SO(2) \subset Q_2$ *with the Gaussian curvature 2.*

Proof. The proof of i) will be given at the end of this section.

Suppose that $\tau \equiv 0$. Put $Z = \begin{pmatrix} Z^1 \\ Z^2 \end{pmatrix}$ where $\phi^1 = Z^1\phi$, $\phi^2 = Z^2\phi$ as in (1). Then $\mathrm{Re}Z \perp \mathrm{Im}Z$ and $|\mathrm{Re}Z| = |\mathrm{Im}Z| = \frac{1}{\sqrt{2}}$ at every point of $M$.

Thus we can choose a $SO(4)$-frame $e$ about any point of $M$ so that, relative to $e$ ,

$$\phi^1 = \frac{1}{\sqrt{2}}\phi \ , \ \phi^2 = \frac{i}{\sqrt{2}}\phi \ , \text{ that is, } \ e^*(\Omega_1^3 + i\Omega_2^3) = \phi, \ e^*(\Omega_1^4 + i\Omega_2^4) = i\phi \ .$$

Consider the exterior system on $SO(4)$ given by $\{\Omega_1^3 = \Omega_2^4,\ \Omega_2^3 = -\Omega_1^4\}$ . This system defines a completely integrable left invariant distribution on $SO(4)$ whose analytic subgroup is $H = \left\{\begin{pmatrix} A & -B \\ B & A \end{pmatrix} \in SO(4)\right\} \cong U(2)$ .

It is now a fairly easy matter to check that $f(M)$ is congruent to an open submanifold of $H / G \cap \{SO(2)\times SO(2)\}$. The proof of iii) is omitted. (We just mention that the exterior system to consider for iii) is $\{\Omega_1^4 = 0 \ , \ \Omega_2^4 = 0\}$.) □

**Remark.** The above two cases $(\tau \equiv 0, \tau \equiv 1)$ give the only homogeneous holomorphic curves in $Q_2$; this now follows essentially from a theorem of E. Cartan ([2], p. 155, also [4], p. 41).

**LEMMA.** $\tau$ *is either identically zero or* $\tau^{-1}\{0\}$ *is an isolated set.*

**Proof.** Assume that $\tau$ is not identically zero. Recall that $\tau = (Z^1)^2 + (Z^2)^2$ where $Z^1, Z^2$ are defined by (1), which says $\phi^1 = Z^1\phi$, $\phi^2 = Z^2\phi$. Put $\tau = (Z^1)^2 + (Z^2)^2$. Then, though $\tau_C$ is defined only up to modulus, $\tau_C^{-1}\{0\}$ is a well-defined set and indeed $\tau_C^{-1}\{0\} = \tau^{-1}\{0\}$. Now using the structure equations of $SO(4)$ we obtain

$$(5)\qquad \begin{cases} d\phi^1 = i\omega_2^1 \wedge \phi^1 - \omega_4^3 \wedge \phi^2 = iZ^1\omega_2^1 \wedge \phi - Z^2\omega_4^3 \wedge \phi , \\ d\phi^2 = \omega_4^3 \wedge \phi^1 + i\omega_2^1 \wedge \phi^2 = Z^1\omega_4^3 \wedge \phi + iZ^2\omega_2^1 \wedge \phi. \end{cases}$$

We also have

$$(6)\qquad d\phi = i\omega \wedge \phi ,$$

where $\omega$ is the Levi-Civita connection form of $(M, ds^2)$.

It follows that (using $d\phi^i = d(Z^i\phi)$)

$$(7)\qquad \begin{aligned} dZ^1 &\equiv iZ^1(\omega_2^1 - \omega) - Z^2\omega_4^3 \pmod{\phi}, \\ dZ^2 &\equiv iZ^2(\omega_2^1 - \omega) + Z^1\omega_4^3 \pmod{\phi} . \end{aligned}$$

Thus

$$(8)\qquad d\tau_C \equiv 2i\tau_C(\omega_2^1 - \omega) \pmod{\phi} .$$

It now follows from a theorem of Chern ([3], section 4) that the zero set of $\tau_C$ (hence that of $\tau$) is isolated. Moreover, the theorem says that the zeros are all of finite multiplicities.

Assume that $\tau$ is not identically zero. Let $p \in M \setminus \tau^{-1}\{0\}$. Then, in a neighbourhood of $p$, $\tau$ is never zero, hence $\tau_C$, defined on a possibly smaller set, is never zero. Define real valued $\theta$ by $\tau_C = \tau e^{i\theta}$. Possibly restricting to a yet smaller neighbourhood of $p$ we can assume

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that $\theta$ is a smooth real valued function. (Take a smooth single-valued brance of $\theta$.) Let $t_1 = \frac{-\theta}{2}$, $k = (e^{it_1}, 1)$, and $\tilde{e} = ek$. Then using (4) we see that $(\tilde{Z}^1)^2 + (\tilde{Z}^2)^2$ is real valued. Putting $\tilde{Z} = \begin{pmatrix} \tilde{Z}^1 \\ \tilde{Z}^2 \end{pmatrix}$ we then must have $\text{Re}\tilde{Z} \perp \text{Im}\tilde{Z}$, and $|\text{Re}\tilde{Z}| > |\text{Im}\tilde{Z}|$. Now applying the rotation in the normal plane we can change $\tilde{Z}$ to $\begin{pmatrix} a \\ ib \end{pmatrix}$, where $a > |b| \geq 0$. Since $a^2 + b^2 = 1$ ((2)) we can find locally defined smooth $\alpha$ such that

$$(9) \qquad \begin{cases} a = \cos\alpha, \\ b = \sin\alpha, \\ -\frac{\pi}{4} < \alpha < \frac{\pi}{4}. \end{cases}$$

We summarize the preceding discussion as follows. In a neighbourhood of every point of $M \setminus \tau^{-1}\{0\}$ there exists a $SO(4)$-frame relative to which the following normal form holds;

$$(10) \qquad \begin{cases} \phi^1 = \cos\alpha\, \phi \\ \phi^2 = i\sin\alpha\, \phi, \text{ where } -\frac{\pi}{4} < \alpha < \frac{\pi}{4}. \end{cases}$$

So, on $M \setminus \tau^{-1}\{0\}$ we have

$$(11) \qquad \tau = \cos^2\alpha - \sin^2\alpha = \cos 2\alpha > 0.$$

Upon exterior differentiation we get

$$(12) \qquad d\tau = -2\sin 2\alpha\, d\alpha .$$

On the other hand (8) gives

$$(13) \qquad d\tau \equiv 2i\tau(\omega_2^1 - \omega) \pmod{\phi}.$$

Combining (12) and (13) and using the fact that $\tau$ is real we get

$$(14) \qquad [\sin 2\alpha\, d\alpha + i\cos 2\alpha\, (\omega_2^1 - \omega)] \wedge \phi = 0 .$$

It follows that

$$(15) \qquad 2\tau(\omega - \omega_2^1) = {*d\tau} ,$$

where $*$ is the Hodge operator of $(M, ds^2)$.

Rewriting,

$$(16) \qquad 2(\omega - \omega_2^1) = {*d\log\tau} .$$

Let $K$ denote the Gaussian curvature of $(M, ds^2)$ so that $d\omega = \frac{i}{2} K \phi \wedge \overline{\phi}$. Also let $\Delta$ denote the Laplace-Beltrami operator of $(M, ds^2)$ so that $d{*}d\log = \frac{i}{2} (\Delta\log\tau) \phi \wedge \overline{\phi}$.

Exterior differentiation of both sides of (16) now gives

$$2(K - 2) = \Delta\log \tau . \tag{17}$$

The equation (17) holds in $M\backslash\tau^{-1}\{0\}$.

**Proof of Proposition part i).** Assume that $\tau$ is constant and not zero. So the local normal form (10) is valid. Using (10) the equation in (7) becomes $d\cos\alpha \equiv -i\sin\alpha\ \omega_4^3 \pmod{\phi}$, $id\sin\alpha \equiv \cos\alpha\ \omega_4^3 \pmod{\phi}$. It follows that $\omega_4^3 = 0$. Hence $d\omega_4^3 = \omega_1^3 \wedge \omega_1^4 + \omega_2^3 \wedge \omega_2^4 = -i\cos\alpha\sin\alpha\ \phi \wedge \overline{\phi} = 0$. Since $-\frac{\pi}{4} < \alpha < \frac{\pi}{4}$ we must have $\sin\alpha = 0$, and $\alpha = 0$. This means that $\tau = \cos 2\alpha = 1$.

**Remark. i)** If $K \geq 2$ then (17) says that $\log\tau$ is subharmonic with singularities at $\tau^{-1}\{0\}$ where it goes to $-\infty$. Further if $M$ is compact then $\log\tau$ attains a maximum in $M$, hence is constant by the maximum principle for subharmonic functions. It follows that $K = 2$.
**ii)** Combining the Proposition part ii) with the preceding remark it follows that for compact $M, K = 4$ if and only if $\tau = 0$.

## 2. Integral formulae.

In this section we assume that $M$ is a compact, connected, orientable surface. Write $M = M_g$, $g$, the number of handles. We have $\chi(M) = 2 - 2g$, where $\chi(M)$ is the Euler-Poincaré characteristic.

Let $f : M \to Q_2$ be a holomorphic, isometric immersion as in section 1. Then in $M\backslash\tau^{-1}\{0\}$ the equation (17) holds and $\tau^{-1}\{0\}$ is a finite set (or $\tau$ is identically zero.) In the following we will give an integrated version of (17) relating $\chi(M)$, Area$(M)$, and the number of zeros of $\tau$. We have

(18) $$\text{Area}(M) = \frac{i}{2}\int_M \phi \wedge \overline{\phi}\ .$$

Note that though $\phi$ is defined only locally $\phi \wedge \overline{\phi}$ is a global $2$-form and $\frac{i}{2}\phi \wedge \overline{\phi}$ is the area form of $M$.

The Gauss-Bonnet-Chern theorem states

(19) $$\chi(M) = \frac{i}{4\pi}\int_M K\phi \wedge \overline{\phi}\ .$$

As an application of the elementary argument principle we get

(20) $$-\frac{i}{4\pi}\int_M \Delta\log\tau\ \phi \wedge \overline{\phi} = \#\,(\tau^{-1}\{0\})\ ,$$

where $\#\,(\tau^{-1}\{0\})$ is the number of zeros of $\tau$ each counted with multiplicity. Of course, here, we must assume that $\tau$ is not identically zero so that the formula makes sense in view of the Lemma in section 1.

We also know that $\frac{1}{\pi}\text{Area}(M)$ is a positive integer. This follows from the well-known equidistribution property of compact projective curves: Include $f(M) \subset Q_2 \subset CP^3$. Then for a non-planar $f(M)$ in $CP^3, \frac{1}{\pi}\text{Area}(M)$ is the intersection number (counted with multiplicity) between $f(M)$ and any $CP^2 \subset CP^3$.

Integrating (17) over $M$ now gives:

THEOREM A. *Let $M$ be a compact, connected, orientable surface with a Riemannian metric, equipped with the induced complex structure. Assume that the Gaussian curvature of $M$ is not identically equal to $4$. Also let $f : M \to Q_2$ be a holomorphic, isometric immersion. Then (with notation as above)*

(21) $$\#\,(\tau^{-1}\{0\}) = 2(\frac{1}{\pi}\text{Area}(M) - \chi(M))\ .$$

COROLLARY. i) *If $M = M_0 \cong S^2$ then $\text{Area}(M) \geq 2\pi$. Note the case $K \equiv 4$ has been excluded. Also, then, $\text{Area}(M) = 2\pi$ if and only if $f(M)$ is congruent to $Q_1$.*

**ii)** *if* $M = M_g$ *with* $g \geq 1$ *then* $\tau^{-1}\{0\}$ *is not empty, hence* # $(\tau^{-1}\{0\})$ *is a positive even integer.*

## 3. Normal superminimal surfaces

As an application of our frame construction in section 1, in particular the local normal form (10), we will show that the normal map of a holomorphic curve in $Q_2$ is superminimal. (See theorem B. below for a precise statement.) But first we review briefly the notion of superminimality.

Superminimal surfaces naturally arose as minimal spheres in Euclidean spheres or, more generally in spaces of constant curvature. (The nomenclature was first employed in [1].) Let $(M, ds^2)$ be a connected, orientable two-manifold endowed with the induced complex structure as in section 1. Also let $(N, ds_N^2)$ be a Riemannian manifold of dimension $n \geq 4$. We consider a smooth isometric immersion $g: M \to N$. We will use the following index convention for the rest of the paper: $1 \leq i, j, k, \ldots \leq 2$, $3 \leq a, b, c, \ldots \leq n$, $1 \leq \alpha, \beta, \gamma, \ldots \leq n$.

Let $(\theta^\alpha)$ be a (local, of course) orthonormal coframe on $N$. The Levi-Civita connection forms $(\theta^\alpha_\beta)$ are characterized by the structure equations $d\theta^\alpha = -\theta^\alpha_\beta \wedge \theta^\beta$. An orthonormal coframe $(\theta^\alpha)$ along the map $g$ is called a Darboux coframe if $(\theta^a) = 0$ on $M$. If $(\theta^a)$ is a Darboux coframe then $d\theta^a = -\theta^a_i \wedge \theta^i = 0$ for every $a$. Applying Cartan's lemma we get $\theta^a_i = h^a_{ij}\theta^j$ for some local functions $h^a_{ij} = h^a_{ji}$. Define $S^a = -h^a_{11} + ih^a_{12}$ and put $S = (S^a)$. Let $\phi = \theta^1 + i\theta^2$. Then $\phi$ is a type $(1,0)$ form on $M$ and $ds^2_M = \phi\bar{\phi}$. Then the quartic symmetric form of type $(4,0)$ $\Phi = {}^tSS\phi^4$ is a globally defined form on $M$.

DEFINITION. A smooth isometric immersion $g: M \to N$ is said to be superminimal if it is minimal and $\Phi$ vanishes identically.

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Now if $N$ is of constant curvature then one can show that $\Phi$ is holomorphic, hence by Riemann-Roch a minimal sphere in $N$ is superminimal with the induced metric. Of course, this is not the case for $N = Q_2$.

We now consider $f: M \to Q_2$ a holomorphic, isometric immersion as before. If $e = (e_1, e_2, e_3, e_4)$ is any $SO(4)$-frame along $f$ (that is, $f = [e_1 \wedge e_2]$) then the normal map $f^{\perp} : M \to Q_2$ is defined to be $[e_3 \wedge e_4]$ which is globally defined.

THEOREM B. *Let* $f: M \to Q_2$ *be a holomorphic, isometric immersion of a connected, orientable two-manifold* $(M, ds^2)$ *with the induced complex structure. Then the normal map* $f^{\perp} : M \to Q_2$ *is a superminimal (not* $\pm$ *holomorphic) immersion.*

Proof. If $e = (e_1, e_2, e_3, e_4)$ is any local $SO(4)$-frame along $f$ then $E = (E_1, E_2, E_3, E_4) = (e_3, e_4, e_1, e_2)$ is a local $SO(4)$-frame along $g = f^{\perp}$. Using the notation $\omega^{\alpha}_{\beta} = e^*\Omega^{\alpha}_{\beta}$ and $\hat{\omega}^{\alpha}_{\beta} = E^*\Omega^{\alpha}_{\beta}$ we get $\hat{\omega}^{\alpha}_{\beta} = \omega^{|\alpha+2|}_{|\beta+2|}$, where $|\alpha + 2|$ is defined to be $\alpha + 2$ (mod 4), and likewise for $|\beta + 2|$. Let $\hat{\phi}^1 = \frac{1}{\sqrt{2}}(\hat{\omega}^3_1 + i\hat{\omega}^3_2)$ and $\hat{\phi}^2 = \frac{1}{\sqrt{2}}(\hat{\omega}^4_1 + i\hat{\omega}^4_2)$. Recall that $\phi^1 = \frac{1}{\sqrt{2}}(\omega^3_1 + i\omega^3_2) = z^1\phi$, $\phi^2 = \frac{1}{\sqrt{2}}(\omega^4_1 + i\omega^4_2) = z^2\phi$, $|z^1|^2 + |z^2|^2 = 1$, $\phi$ a local unitary coframe on $M$. So the induced metric on $M$ by $g$ is $\hat{\phi}^1\bar{\hat{\phi}}^1 + \hat{\phi}^2\bar{\hat{\phi}}^2 = \frac{1}{2}((\omega^1_3)^2 + (\omega^1_4)^2 + (\omega^2_3)^2 + (\omega^2_4)^2) = \phi\bar{\phi}$. This shows that $g = f^{\perp}$ is an isometric immersion.

We first assume that $\tau$ is not identically zero so that the local normal form (10) is valid in $M\backslash\tau^{-1}\{0\}$ where $\tau^{-1}\{0\}$ is an isolated set. We will show that $g = f^{\perp}$ is superminimal in $M\backslash\tau^{-1}\{0\}$. Then $g$ has to be superminimal everywhere in $M$ by a simple continuity argument in view of the fact that $g$ is an isometric immersion. In a neighborhood of a point in $M\backslash\tau^{-1}\{0\}$ we can choose a $SO(4)$-frame along $f$ so that relative to it $\phi^1 = \cos\alpha\ \phi$, $\phi^2 = i\sin\alpha$ , where $-\frac{\pi}{4} < \alpha < \frac{\pi}{4}$. (This is (10).) This means that $\hat{\phi}^1 = -\cos\alpha\ \theta^1 + i\sin\alpha\ \theta^2$ , and $\hat{\phi}^2 = -\cos\alpha - i\sin\alpha\ \theta^1$ , where $\phi = \theta^1 + i\theta^2$. It is at once observed that $\hat{\phi}^1, \hat{\phi}^2$ are neither of type $(1,0)$ nor of type $(0,1)$, hence $f^{\perp}$ is neither holomorphic nor antiholomorphic.

Now $\frac{1}{\sqrt{2}}(\hat{\omega}_1^3, \hat{\omega}_1^4, \hat{\omega}_2^3, \hat{\omega}_2^4) = (\hat{\theta}^1, \hat{\theta}^2, \hat{\theta}^3, \hat{\theta}^4) = {}^t\hat{\theta}$ form an orthonormal coframe along the map $g = f^{\perp}$. Let $k$ be a $SO(4)$-valued (local) function on $M$ given by

$$k = \begin{pmatrix} -c & 0 & s & 0 \\ 0 & -c & 0 & s \\ 0 & s & 0 & c \\ -s & 0 & -c & 0 \end{pmatrix}, \text{ where } c = \cos\alpha,\ s = \sin\alpha.$$

Then $\theta = {}^t(\theta^1, \theta^2, \theta^3, \theta^4) = k^{-1}\hat{\theta}$ is a Darboux coframe along $g$, that is, $\theta^3 = \theta^4 = 0$ on $M$. Computations show that $\theta_1^3 = d\alpha = \theta_2^4$ and $\theta_2^3 = \tau\omega_3^4 = -\theta_1^4$. So $\theta_1^3 = h_{11}^3\theta^1 + h_{12}^3\theta^2 = \theta_2^4 = h_{21}^4\theta^1 + h_{22}^4\theta^2$ and $\theta_2^3 = h_{21}^3\theta^1 + h_{22}^3\theta^2 = -\theta_1^4 = -h_{11}^4\theta^1 - h_{12}^4\theta^2$. For $h_{12}^3 = h_{21}^3$ implies that $h_{11}^4 + h_{22}^4 = 0$ and $h_{12}^4 = h_{21}^4$ implies that $h_{11}^3 = h_{22}^3 = 0$. This proves that $f^{\perp}$ is minimal.

Now using the normal form (10), (7) becomes

$$\begin{cases} d\cos\alpha \equiv i\cos\alpha\ (\omega_2^1 - \omega) - i\sin\alpha\,\omega_4^3\ (\mathrm{mod}\,\phi), \\ id\sin\alpha \equiv -\sin\alpha\ (\omega_2^1 - \omega) + \cos\alpha\ \omega_4^3 (\mathrm{mod}\,\phi). \end{cases}$$

It follows that $d\alpha = -\sin\alpha\ d\cos\alpha + \cos\alpha\ d\sin\alpha \equiv i\tau\omega_3^4\ (\mathrm{mod}\,\phi)$. Using the fact that $d\alpha$ is real it follows that $\tau\omega_3^4 = {*d\alpha}$. But this means that $\theta_1^4 = {*\theta_1^3}$. So $\theta_1^4 = h_{11}^4\theta^1 + h_{12}^4\theta^2 = {*\theta_1^3} = -h_{12}^3\theta^1 + h_{11}^3\theta^2$ ,and

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$t_{SS} = (-h^3_{11} + ih^3_{12})^2 + (-h^4_{11} + ih^4_{12})^2 = 0$. Hence the quartic form $\Phi$ vanishes proving that $f^{\perp}$ is superminimal.

We now consider the case $\tau$ identically zero. Then in a neighbourhood about any point of $M$ we can choose a $SO(4)$-frame along $f$ so that relative to it $\phi^1 = \frac{1}{\sqrt{2}}\phi$, $\phi^2 = \frac{i}{\sqrt{2}}\phi$. (See the proof of the Proposition in section 1.) Using this local normal form an argument completely analogous to the one given for the case $\tau$ not identically zero finishes the proof of the Theorem. □

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