H. Fujimoto
Nagoya Math. J.
Vol. 185 (2007), 111–122

FINITENESS OF ENTIRE FUNCTIONS SHARING A FINITE SET

HIROTAKA FUJIMOTO

Abstract. For a finite set $S = \{a_1, \dots, a_q\}$, consider the polynomial $P_S(w) = (w-a_1)(w-a_2)\cdots(w-a_q)$ and assume that $P'_S(w)$ has distinct k zeros. Suppose that $P_S(w)$ is a uniqueness polynomial for entire functions, namely that, for any nonconstant entire functions ϕ and ψ, the equality $P_S(\phi) = cP_S(\psi)$ implies $\phi = \psi$, where c is a nonzero constant which possibly depends on ϕ and ψ. Then, under the condition $q > k + 2$, we prove that, for any given nonconstant entire function g, there exist at most $(2q-2)/(q-k-2)$ nonconstant entire functions f with $f^*(S) = g^*(S)$, where $f^*(S)$ denotes the pull-back of S considered as a divisor. Moreover, we give some sufficient conditions of uniqueness polynomials for entire functions.

§1. Introduction

A finite subset S of $\mathbf{C}$ is called a uniqueness range set for meromorphic functions (or entire functions) if $f^*(S) = g^*(S)$ implies $f = g$ for arbitrary nonconstant meromorphic functions (or entire functions) f and g on $\mathbf{C}$, where $f^*(S)$ and $g^*(S)$ denote the pull-backs of S considered as a divisor, namely, the inverse images of S counted with multiplicities by f and g respectively. For $S := \{a_1, a_2, \dots, a_q\}$, we consider the polynomial

$$P_S(w) := (w-a_1)(w-a_2)\cdots(w-a_q). \tag{1}$$

We call a nonconstant monic polynomial $P(w)$ a uniqueness polynomial for meromorphic functions (or entire functions) if, for any nonconstant meromorphic functions (or entire functions) ϕ and ψ on $\mathbf{C}$, the equation $P(\phi) = cP(\psi)$ implies $\phi = \psi$, where c is a nonzero constant which possibly depends on ϕ and ψ. Obviously, if S is a uniqueness range set for meromorphic functions (or entire functions), then $P_S(w)$ is a uniqueness polynomial for meromorphic functions (or entire functions).

Received December 17, 2004.
Revised November 8, 2005.
2000 Mathematics Subject Classification: 30D35.

Assume that $P'_S(w)$ has k distinct zeros e_ℓ with multiplicities q_ℓ $(1 \leq \ell \leq k)$. In [1], the author gave some sufficient conditions for uniqueness range set under the condition

(H) $P_S(e_\ell) \neq P_S(e_m)$ *for* $1 \leq \ell < m \leq k$.

Main results in [1] are stated as follows.

THEOREM 1.1. *Let S be a finite subset of $\mathbf{C}$ such that $P_S(w)$ is a uniqueness polynomial for meromorphic functions* (*or entire functions*) *which satisfies the above condition* (H). *Assume that $k \geq 3$, or $k = 2$ and* $\min\{q_1, q_2\} \geq 2$. *If $q > 2k+6$* (*or $q > 2k+2$*)*, then S is a uniqueness range set for meromorphic functions* (*or entire functions*).

We now introduce the following definition.

DEFINITION 1.2. A finite subset S of $\mathbf{C}$ is called a *finiteness range set for entire functions* if, for any given nonconstant entire function g, there exist only finitely many nonconstant entire functions f such that $f^*(S) = g^*(S)$.

The purpose of this paper is to give some sufficient conditions for a finiteness range set for entire functions. The main result is stated as follows.

THEOREM 1.3. *Take a finite set $S = \{a_1, a_2, \ldots, a_q\}$ and assume that, for the polynomial $P_S(w)$ defined by* (1), *$P'_S(w)$ has distinct k zeros. If $P_S(w)$ is a uniqueness polynomial for entire functions and $q > k+2$, then S is a finiteness range set for entire functions. More precisely, for an arbitrarily given nonconstant entire function g, there exist at most* $(2q-2)/(q-k-2)$ *entire functions f such that $f^*(S) = g^*(S)$.*

The poof of Theorem 1.3 is given in the next section.

We give some sufficient conditions for uniqueness polynomials for entire functions in the last section. For example, the polynomial

$$P(w) = w^5 + \frac{5}{2}w^4 + \frac{5}{3}w^3 + c \quad \left(c \neq 0, \frac{1}{6}, \frac{1}{12}\right)$$

is a uniqueness polynomial for entire functions (cf. Theorem 3.4) which satisfies the condition $q = 5 > k + 2 = 4$, and so the set of zeros of $P(w)$ gives a finiteness range set for entire functions consisting of 5 values. In fact, $P(w)$ has no multiple zero by the condition $c \neq 0, 1/6$, and $P'(w)$ has two distinct zeros satisfying the conditions of Theorem 3.4. It is a very interesting problem to ask if there are smaller finiteness range sets for entire functions.

The author thanks Professor K. Yamanoi for many valuable comments on this paper.

§2. Proof of Main Theorem

We first introduce some notations. By a divisor we mean a map $\nu : \mathbf{C} \to \mathbf{Z}$ such that the set $\{z; \nu(z) \neq 0\}$ has no accumulation point. The counting function $N(r, \nu)$ of a divisor ν is defined by

$$N(r,\nu) = \int_0^r \left(\sum_{0<|z|\le t} \nu(z) \right) \frac{dt}{t} + \nu(0)\log r,$$

and set $\bar{N}(r,\nu) := N(r, \min\{\nu, 1\})$.

In the following, a meromorphic function means a meromorphic function defined on $\mathbf{C}$. For a nonconstant meromorphic function f and another meromorphic function (possibly, a constant) α, we define the divisor ν_f^α by

$$\nu_f^\alpha(z) := \begin{cases} 0 & \text{if } f - \alpha \text{ does not vanish at } z \\ m & \text{if } f - \alpha \text{ has a zero of multiplicity } m \text{ at } z, \end{cases}$$

and $\nu_f^\infty := \nu_{1/f}^0$. As usual, by $T(r,f)$ and $m(r,f)$ we denote the order function and proximity function of f respectively, and $S(r,f)$ means a function of r satisfying the condition

$$S(r,f) = o(T(r,f)) \parallel,$$

where the notation $\parallel$ means that the inequality holds for every positive number r excluding a measurable set E with $\int_E dr < +\infty$.

The main tool for the proof of Theorem 1.3 is the truncated second main theorem for moving targets, which was proved by K. Yamanoi. A particular case of his result [3, Theorem 1] is stated as follows.

THEOREM 2.1. *Let f be a nonconstant meromorphic function and let $\alpha_1, \ldots, \alpha_q$ be mutually distinct meromorphic functions with $f \not\equiv \alpha_i$ $(1 \le i \le q)$. Then, for every $\varepsilon > 0$, there exists a positive constant $C(\varepsilon)$ such that*

$$(q - 2 - \varepsilon)T(r,f) \le \sum_{i=1}^q \bar{N}(r, \nu_f^{\alpha_i}) + C(\varepsilon)\left(\sum_{i=1}^q T(r,\alpha_i)\right) + O(1)$$

for any positive number r *excluding some set* $E \subset (1, +\infty)$ *with* $\int_E d\log\log r < +\infty$.

We now give the following;

DEFINITION 2.2. Let f be a nonconstant meromorphic function on $\mathbf{C}$. A meromorphic function $\alpha(\neq f)$ is called *a small function with respect to* f if $T(r, \alpha) = S(r, f)$.

As an immediate consequence of Theorem 2.1, we have the following.

THEOREM 2.3. *Let* f *be a nonconstant meromorphic function and let* $\alpha_1, \alpha_2, \ldots, \alpha_q$ *be mutually distinct small functions with respect to* f. *Then, for every* $\varepsilon > 0$,

$$(q - 2 - \varepsilon)T(r, f) \leq \sum_{j=1}^{q} \bar{N}(r, \nu_f^{\alpha_j}) + O(1)$$

for any positive number r *excluding some set* $E \subset (1, +\infty)$ *with* $\int_E d\log\log r < +\infty$.

Now, we start the proof of Theorem 1.3. Assume that, for some N with $N > (2q-2)/(q-k-2)$, there exists a nonconstant entire function g such that $f_j^*(S) = g^*(S)$ for mutually distinct N nonconstant entire functions f_j $(1 \leq j \leq N)$, where we set $g = f_1$. As in §1, for $S = \{a_1, \ldots, a_q\}$, we consider the polynomial $P_S(w)$ defined by (1). By assumption, we can find entire functions α_j such that

$$P_S(g) = e^{\alpha_j} P_S(f_j) \quad (1 \leq j \leq N). \tag{2}$$

In this situation, we can show the following.

(2.4) *There are some positive numbers* K_1, K_2 *such that*

$$K_1 T(r, g) \leq T(r, f_j) \leq K_2 T(r, g)\|.$$

In fact, by the second main theorem and $f_j^{-1}(S) = g^{-1}(S)$,

$$\begin{aligned}(q-1)T(r, g) &\leq \sum_{i=1}^{q} \bar{N}(r, \nu_g^{a_i}) + S(r, g) \\ &= \sum_{i=1}^{q} \bar{N}(r, \nu_{f_j}^{a_i}) + S(r, g) \leq qT(r, f_j) + o(T(r, g))\|,\end{aligned}$$

https://doi.org/10.1017/S0027763000025769 Published online by Cambridge University Press

whence $T(r,g)=O(T(r,f_j))\ \|$ and, similarly, $T(r,f_j)=O(T(r,g))\ \|$.

By (2.4), a small function with respect to g is also a small function with respect to any f_j.

We take the logarithmic derivatives of the identities (2) and get

$$\frac{P'_S(g)g'}{P_S(g)}=\alpha'_j+\frac{P'_S(f_j)f'_j}{P_S(f_j)}. \tag{3}$$

Set $\varphi_j := P'_S(f_j)f'_j/P_S(f_j)$ and $\varphi=\varphi_1$. Then, we have the following assertion.

(2.5) *There exist some positive numbers K_1, K_2 such that*

$$K_1T(r,g)\le T(r,\varphi_j)\le K_2T(r,g)\| \quad (1\le j\le N).$$

In fact, we get $T(r,\varphi_j)=O(T(r,g))\|$ by using the logarithmic derivative lemma. On the other hand, the second main theorem gives

$$\begin{aligned}(q-1)T(r,g)&\le\sum_{i=1}^{q}\bar{N}(r,\nu_g^{a_i})+S(r,g)\\&\le N(r,\nu_{\varphi_j}^{\infty})+S(r,g)\le T(r,\varphi_j)+S(r,g).\end{aligned} \tag{4}$$

(2.6) *Each function α'_j is a small function with respect to φ.*

In fact, by the logarithmic derivative lemma, we have

$$m(r,\varphi_j)=S(r,P_S(f_j))=S(r,f_j)=S(r,\varphi),$$

and so the identity (3) gives

$$T(r,\alpha'_j)=m(r,\alpha'_j)\le m(r,\varphi)+m(r,\varphi_j)+O(1)=S(r,\varphi).$$

(2.7) *The functions α'_j are mutually distinct.*

To see this, we assume that $\alpha'_i=\alpha'_j$ for some distinct i and j. Then, there is a constant c_0 with $\alpha_i=\alpha_j+c_0$ and hence

$$e^{c_0}P_S(f_i)=e^{\alpha_i-\alpha_j}P_S(f_i)=P_S(f_j).$$

This contradicts the assumption that $P_S(w)$ is a uniqueness polynomial for entire functions.

https://doi.org/10.1017/S0027763000025769 Published online by Cambridge University Press

We now apply Theorem 2.3 to the function φ and small functions α'_j with respect to φ to show that, for any ε with $0 < \varepsilon < N - 2$,

$$(N-2-\varepsilon)T(r,\varphi) \leq \sum_{j=1}^{N} \bar{N}(r, \nu_{\varphi}^{\alpha'_j}) + O(1)$$

for any positive number r excluding a set $E \subset (1, +\infty)$ with $\int_E d \log \log r < +\infty$.

By (3) we have

$$\bar{N}(r, \nu_{\varphi}^{\alpha'_j}) = \bar{N}(r, \nu_{\varphi_j}^{0}) \leq \bar{N}(r, \nu_{f'_j}^{0}) + \sum_{\ell=1}^{k} \bar{N}(r, \nu_{f_j}^{e_\ell}),$$

where $e_1, e_2, \ldots, e_k$ are all of distinct zeros of $P'_S(w)$. On the other hand, it holds that $\bar{N}(r, \nu_{f_j}^{e_\ell}) \leq T(r, f_j) + O(1)$ and

$$\begin{aligned}\bar{N}(r, \nu_{f'_j}^{0}) &\leq T(r, f'_j) + O(1) = m(r, f'_j) + O(1) \\ &\leq m(r, f_j) + m(r, f'_j/f_j) + O(1) \leq T(r, f_j) + S(r, f_j).\end{aligned}$$

Therefore,

$$\sum_{i=1}^{N} \bar{N}(r, \nu_{\varphi}^{\alpha'_i}) \leq (k+1) \sum_{j=1}^{N} \left(T(r, f_j) + S(r, f_j)\right).$$

Since $(q-1)T(r, f_j) \leq T(r, \varphi) + S(r, g)$ by the same reasoning as in deriving (4), we have

$$\begin{aligned}(N-2-\varepsilon)(q-1)T(r, f_j) &\leq (N-2-\varepsilon)T(r,\varphi) + S(r,g) \\ &\leq \sum_{i=1}^{N} \bar{N}(r, \nu_{\varphi}^{\alpha'_i}) + \tilde{S}(r,g) \\ &\leq (k+1) \sum_{i=1}^{N} T(r, f_i) + \tilde{S}(r,g),\end{aligned}$$

where $\tilde{S}(r,g)$ denotes a term satisfying the condition that $\tilde{S}(r,g) = o(T(r,g)) + O(1)$ for any positive number r excluding a set $E \subset (1, +\infty)$ with $\int_E d \log \log r < +\infty$. Summing up these inequalities, we obtain

$$(N-2-\varepsilon)(q-1) \sum_{j=1}^{N} T(r, f_j) \leq N(k+1) \sum_{j=1}^{N} T(r, f_j) + \tilde{S}(r,g).$$

https://doi.org/10.1017/S0027763000025769 Published online by Cambridge University Press

Dividing each term of this inequality by $\sum_{j=1}^{N} T(r, f_j)$ and letting $r \to +\infty$ outside some measurable set $E(\subset (1, +\infty))$ with $\int_E d \log \log r < +\infty$, we obtain

$$(N - 2 - \varepsilon)(q - 1) \leq N(k + 1).$$

Since we can take an arbitrarily small positive number ε, we can conclude $(N - 2)(q - 1) \leq N(k + 1)$ and hence

$$N \leq \frac{2q - 2}{q - k - 2}.$$

This contradicts the assumption. The proof of Theorem 1.3 is completed.

§3. Uniqueness polynomials for entire functions

We first discuss uniqueness polynomials for meromorphic functions (or entire functions) in a broad sense, which are defined as follows.

DEFINITION 3.1. A nonconstant monic polynomial $P(w)$ is called a *uniqueness polynomial for meromorphic functions (or entire functions) in a broad sense* if $P(f) = P(g)$ implies $f = g$ for two nonconstant meromorphic functions (or entire functions) f and g.

In [2], the author gave some sufficient conditions of uniqueness polynomials for meromorphic functions in a broad sense. Here, we study uniqueness polynomials for entire functions in a broad sense.

THEOREM 3.2. *Let $P(w)$ be a nonconstant monic polynomial without multiple zeros such that $P'(w)$ has distinct k zeros $e_1, e_2, \ldots, e_k$ with multiplicities $q_1, q_2, \ldots, q_k$, respectively, and suppose that $P(w)$ satisfies the condition* (H). *If $k \geq 2$ and $q := \deg(P) \geq 4$, then $P(w)$ is a uniqueness polynomial for entire functions in a broad sense.*

Proof. Assume that there exist distinct entire functions f and g with $P(f) = P(g)$. Consider the polynomial $Q(z, w) := (P(z) - P(w))/(z - w)$ in z, w and the associated homogeneous polynomial

$$Q^*(u_0, u_1, u_2) := u_0^{q-1} Q\left(\frac{u_1}{u_0}, \frac{u_2}{u_0}\right)$$

in u_0, u_1, u_2, where $q = \deg P$. Define the algebraic curve

$$V : Q^*(u_0, u_1, u_2) = 0$$

in $P^2(\mathbf{C})$. As was shown in [2], V is irreducible. Consider the holomorphic map $\Phi := (1 : f : g) : \mathbf{C} \to P^2(\mathbf{C})$. Obviously, the image of Φ is included in V and omits the set $V \cap \{u_0 = 0\}$. Let $\mu : \tilde{V} \to V$ be the normalization of V. Then, $\mu^{-1}(V \cap \{u_0 = 0\})$ consists of at least $q-1$ points, because we can write

$$V : (u_1^{q-1} + u_1^{q-2}u_2 + \cdots + u_2^{q-1}) + u_0 R(u_0, u_1, u_2) = 0$$

with a homogeneous polynomial $R(u_0, u_1, u_2)$ of degree $q-2$ and the first term is factorized into distinct $q-1$ linear functions. Therefore, the associated map $\tilde{\Phi} : \mathbf{C} \to \tilde{V}$ with $\Phi = \mu \cdot \tilde{\Phi}$ omits $\geq q-1$ points. Since $q-1 \geq 3$ by the assumption, the universal covering surface of $\tilde{V} \setminus \mu^{-1}(\{u_0 = 0\})$ is biholomorphic to the unit disc in the complex plane. Therefore, the map $\tilde{\Phi}$, and so Φ, is a constant. This contradicts the assumption. The proof of Theorem 3.2 is completed.

Now, we inquire into uniqueness polynomials.

In [1], the author gave the following sufficient condition of uniqueness polynomials.

Theorem 3.3. *Let $P(w)$ be a monic polynomial without multiple zeros such that $P'(w) = q\prod_{\ell=1}^{k}(w - e_\ell)^{q_\ell}$ and assume that $P(w)$ satisfies the condition* (H). *If $k \geq 4$ and*

$$P(e_1) + P(e_2) + \cdots + P(e_k) \neq 0,$$

then $P(w)$ is a uniqueness polynomial for meromorphic functions.

As was shown in [1], any polynomial $P(w)$ with $k = 1$ is not a uniqueness polynomials for entire functions. We now study uniqueness polynomials for entire functions in the cases $k = 2$ and $k = 3$.

For the case $k = 2$, we have the following.

Theorem 3.4. *Let $P(w)$ be a monic polynomial without multiple zeros such that $P'(w) = q(w - e_1)^{q_1}(w - e_2)^{q_2}$ $(e_1 \neq e_2)$. If $q \geq 4$ and $P(e_1) \neq \pm P(e_2)$, then $P(w)$ is a uniqueness polynomial for entire functions.*

For the case $k = 3$, we can prove the following.

THEOREM 3.5. *Let $P(w)$ be a monic polynomial without multiple zero such that $P'(w)$ has distinct three zeros e_1, e_2, e_3 with multiplicities q_1, q_2, q_3, respectively, and suppose that $P(w)$ satisfies the conditions* (H). *Here, we choose indices so that $q_1 \leq q_2 \leq q_3$. Then, $P(w)$ is a uniqueness polynomial for entire functions except the cases*

(i) $q_1 = q_2 = q_3 = 1$,

(ii) $q_1 = 1, q_2 = q_3 \geq 2$ *and* $P(e_2) + P(e_3) = 0$ *and*

(iii) $q_1 = q_2 = q_3 \geq 2$ *and* $P(e_1) + P(e_2) + P(e_3) = 0$.

For the proof of Theorems 3.4 and 3.5, we show the following.

LEMMA 3.6. *Let $P(w)$ be a monic polynomial without multiple zeros such that $P'(w) = q\prod_{\ell=1}^{k}(w - e_\ell)^{q_\ell}$. Assume that $P(w)$ satisfies the condition* (H) *and that there exist distinct nonconstant entire functions f, g such that $P(f) = cP(g)$ for a constant $c \neq 0, 1$. Set*

$$\Lambda := \{(\ell, m); P(e_\ell) = cP(e_m)\}.$$

Then,

(i) *If $(\ell_0, m) \notin \Lambda$ for any m or if $(m', \ell_0) \notin \Lambda$ for any m', then $q_{\ell_0} = 1$.*

(ii) *If $(\ell, m) \in \Lambda$, then $q_\ell = q_m$.*

Proof. Changing indices and exchanging the roles of f and g if necesary, we may assume that $(1, m) \notin \Lambda$ $(1 \leq m \leq k)$ for the proof of (i), and that $(1, 2) \in \Lambda$ and $q_2 \leq q_1$ for the proof of (ii). Consider the polynomials

$$Q(w) := P(w) - P(e_1),\ Q^*(w) := cP(w) - P(e_1)$$

and denote all distinct zeros of $Q(w)$ and of $Q^*(w)$ by $\alpha_1, \ldots, \alpha_M$ and by $\beta_1, \ldots, \beta_N$, respectively, where we may set $\alpha_1 = e_1$ and, furthermore, $\beta_1 = e_2$ if $(1, 2) \in \Lambda$. For convenience sake, we set $q^* := 0$ if $(1, m) \notin \Lambda$ for any $m (1 \leq m \leq k)$, and $q^* := q_2$ if $(1, 2) \in \Lambda$. As is easily seen, α_1 is a zero of $Q(w)$ with multiplicity $q_1 + 1$, and the other α_i's are its simple zeros because $P(w)$ has no multiple zero and satisfis the condition (H). Similarly, β_1 is a zero of $Q^*(w)$ with multiplicity $q^* + 1$ and the other β_j's are its simple zeros. Therefore, $M = q - q_1, N = q - q^*$. Now, we apply the second main theorem to obtain

$$(N - 1)T(r, g) \leq \sum_{j=1}^{N} \bar{N}(r, \nu_g^{\beta_j}) + S(r, g).$$

https://doi.org/10.1017/S0027763000025769 Published online by Cambridge University Press

On the other hand, if $g(z_0) = \beta_j$ for some z_0, then $P(f(z_0)) = cP(g(z_0)) = cP(\beta_j) = P(e_1)$ and so $f(z_0) = \alpha_i$ for some i. Therefore,

$$\sum_{j=1}^{N} \bar{N}(r, \nu_g^{\beta_j}) \leq \sum_{i=1}^{M} \bar{N}(r, \nu_f^{\alpha_i}) \leq MT(r,f) + S(r,g). \tag{5}$$

Since $P(f) = cP(g)$ implies

$$qT(r,f) = T(r, P(f)) + O(1) = T(r, P(g)) + O(1) = qT(r,g) + O(1),$$

we can conclude

$$(N-1)T(r,g) \leq MT(r,g) + S(r,g).$$

By dividing this inequality by $T(r,g)$ and letting $r \to +\infty$ outside a set E with $\int_E dr < +\infty$, we see $N - 1 \leq M$, namely, $q - q^* - 1 \leq q - q_1$. For the proof of (i), we recall $q^* = 0$ and get $q_1 \leq 1$, which is the desired conclusion.

For the proof of (ii), we recall $q^* = q_2$. Then, we have $(q_2 \leq) q_1 \leq q_2 + 1$. Now, assume that $q_1 \neq q_2$, whence $q_1 = q_2 + 1$. Here, for any point z_0 with $f(z_0) = \alpha_1 (= e_1)$, we claim that $\nu_{g'}^0(z_0) \geq 2$. In this case, since $Q^*(g(z_0)) = cP(g(z_0)) - P(e_1) = cP(g(z_0)) - P(f(z_0)) = 0$, we have different kinds of two cases (a) $g(z_0) = \beta_1 (= e_2)$ and (b) $g(z_0) = \beta_j$ for $j \geq 2$. We first consider the case (a). Observe the identity $P'(f)f' = cP'(g)g'$ obtained from $P(f) = cP(g)$. Comparing the order of zeros z_0 of both sides, we obtain $(q_1 + 1)\nu_f^{e_1} - 1 = (q_2 + 1)\nu_g^{e_2} - 1$ at z_0. Since $q_2 < q_1$, we have $\nu_f^{e_1} < \nu_g^{e_2}$. Then,

$$\nu_f^{e_1} = (q_1 + 1)\nu_f^{e_1} - q_1\nu_f^{e_1} = (q_2 + 1)\nu_g^{e_2} - q_1\nu_f^{e_1} = q_1(\nu_g^{e_2} - \nu_f^{e_1})$$

at z_0. This implies $\nu_g^{e_2} > \nu_f^{e_1} \geq q_1 > q_2 \geq 1$ and so $\nu_{g'}^0 \geq 2$ at z_0. We next consider the case (b). In this case, $P'(g(z_0)) \neq 0$, because $Q^*(e_j) \neq 0$ for $j > 2$ by the condition (H) and $(1,2) \in \Lambda$. Therefore, $\nu_{g'}^0 = (q_1+1)\nu_f^{e_1} - 1 \geq 2$ at z_0. In any case, $\nu_{g'}^0(z_0) \geq 2$. This implies that $\min(\nu_f^{\alpha_1}, 1) \leq (1/2)\nu_{g'}^0$ at z_0. Therefore, we can replace the first inequality of (5) by

$$\sum_{j=1}^{N} \bar{N}(r, \nu_g^{\beta_j}) \leq \frac{1}{2} N(r, \nu_{g'}^0) + \sum_{i=2}^{M} \bar{N}(r, \nu_f^{\alpha_i}),$$

and we have

$$(N-1)T(r,g) \leq \left(\frac{1}{2} + (M-1)\right) T(r,f) + S(r,g),$$

https://doi.org/10.1017/S0027763000025769 Published online by Cambridge University Press

because $N(r, \nu_{g'}^0) \le T(r,g) + S(r,g) = T(r,f) + S(r,g)$. This implies that $q - q_2 \le q - q_1 + 1/2$ and so $q_1 \le q_2 + 1/2$, which is a contradiction. The proof of the assertion (ii) is completed.

We now start the proofs of Theorems 3.4 and 3.5. By Theorem 3.2, the given polynomial $P(w)$ is a uniqueness polynomial for entire functions in a broad sense. Assume that $P(w)$ is not a uniqueness polynomial for entire functions. Then, we can apply Lemma 3.6.

Proof of Theorem 3.4. By the assumption, we see $\max(q_1, q_2) \ge 2$, say $q_2 \ge 2$. By Lemma 3.6, (i), there is some ℓ with $(2, \ell) \in \Lambda$. Then, we have necessarily $\ell = 1$ because $c \ne 1$, and hence $q_1 = q_2 \ge 2$ by Lemma 3.6, (ii). We again apply Lemma 3.6, (i) to see $(1, 2) \in \Lambda$. Therefore, we have $P(e_1)/P(e_2) = P(e_2)/P(e_1) = c$. This implies $P(e_1) = \pm P(e_2)$, which contradicts the assumption.

Proof of Theorem 3.5. Consider the case where $q_1 = q_2 = 1$. We may assume $q_3 \ge 2$, because otherwise we have the excluded case (i). Then, by Lemma 3.6, (i), there exists some ℓ with $(3, \ell) \in \Lambda$, which contradicts Lemma 3.6, (ii) because $q_3 \ne q_m$ for $m = 1, 2$. Next, consider the case where $q_1 = 1$ and $q_2 \ge 2$. Then, there are indices ℓ, m such that $(2, \ell), (3, m) \in \Lambda$ by Lemma 3.6, (i). Here, we have necessarily $\ell = 3$, $m = 2$ and $q_2 = q_3$ by Lemma 3.6, (ii). In this case, $P(e_2)/P(e_3) = P(e_3)/P(e_2) = c$, which implies the excluded case (ii). Lastly, consider the case where $q_1 \ge 2$. Then, by the assumption and Lemma 3.6, (i), there are indices ℓ_1, ℓ_2, ℓ_3 with $(1, \ell_1), (2, \ell_2), (3, \ell_3) \in \Lambda$. In this case, (ℓ_1, ℓ_2, ℓ_3) is a permutation of $(1, 2, 3)$ such that $\ell_m \ne m$ for every m by the condition (H). We then have $q_1 = q_2 = q_3$ and

$$\frac{P(e_1)}{P(e_{\ell_1})} = \frac{P(e_2)}{P(e_{\ell_2})} = \frac{P(e_3)}{P(e_{\ell_3})} = c(\ne 1)$$

by Lemma 3.6, (ii). We easily have $P(e_1) + P(e_2) + P(e_3) = 0$. The proof of Theorem 3.5 is completed.

References

[1] H. Fujimoto, *On uniqueness of meromorphic functions sharing finite sets*, Amer. J. Math., **122** (2000), 1175 – 1203.

[2] H. Fujimoto, *On uniqueness polynomials for meromorphic functions*, Nagoya Math. J., **170**(2003), 33 – 46.
[3] K. Yamanoi, *The second main theorem for small functions and related problems*, Acta Math., **192**(2004), 225 – 299.

KOKUCHU-KAI
6-19-18, Ichinoe
Edogawaku, Tokyo 132-0024
Japan
h_fujimoto@kokuchukai.or.jp