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WELL-POSEDNESS OF DETERMINING THE SOURCE TERM OF AN ELLIPTIC EQUATION

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In this paper the inverse problem for determining the source term of a linear, uniformly elliptic equation is investigated. The uniqueness of the inverse problem is proved under mild assumptions by use of the orthogonality method and an elimination method. The existence of the inverse problem is proved by means of the theory of solvable operators between Banach spaces, moreover, the continuous dependence of the solution to the inverse problem on measurement is also obtained.

1. INTRODUCTION

Identifying the coefficients, the boundary conditions, and/or the source term of a partial differential equation via use of some additional information about the solution to the partial differential equation is called an inverse problem. Inverse problems, of which most are not yet solved, remain as a challenge in applied mathematics.

In this paper the problem we deal with is to identify the source term of an elliptic equation, but for a general elliptic system this problem is ill-posed.

For instance, one hopes to get a pair (u, f) satisfying

$$\begin{aligned} &\Delta u = f(x,y), \quad (x,y) \in D \equiv (0,\pi) \times (0,\pi), \\ &\partial_n u \mid_{\partial D} = [\partial_x u \cos(n,x) + \partial_y u \cos(n,y)] \mid_{\partial D} = g, \end{aligned} \tag{1}$$

on the basis of a measurement of u at the boundary ∂D of D, that is, given

$$u \mid_{\partial D} = z.$$

The solution to the above inverse problem, if it exists, is not unique. In fact, if the solution of the problem is unique, then the problem with $z = 0$ and $g = 0$ should only have the zero solution $(u, f) = (0, 0)$. However, the function pair $(u, f) = \left(\sin^2 x \sin^2 y, 2\cos 2x \sin^2 y + 2\cos 2y \sin^2 x\right)$ satisfies (1) and (2) with $z = 0$ and $g = 0$.

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Many mathematicians have studied various inverse problems for elliptic equations. For a simple survey we refer to [**2, 3, 4, 9, 15, 16**] for identifying coefficients, [**6, 19**] for identifying boundary values, [**1, 7, 8, 12, 13, 14, 18, 20**] for identifying source terms of elliptic equations. We have not included a lot of papers concerning computational methods that solve inverse elliptic problems.

One of the main purposes in studying inverse problems is to discover adequate conditions, under which the solutions of the inverse problems exist, are unique, and/or depend continuously on measurements.

Prilepko [**13, 14**] proved that the source term of the Poisson equation can be uniquely determined if it is independent of one of the variables and is monotone.

Vabishchevich [**18**] also proved that determining the source term is unique but it must satisfy some curious conditions.

In this paper the inverse problem we address is to identify a pair (w, q) satisfying

$$\begin{aligned} &\mathcal{L}w = q(x)f(x,y) + \phi(x,y), \qquad (x,y) \in D \equiv \Omega \times (0,Y), \\ &\partial_\nu w \mid_{\partial\Omega} = \psi_1, \qquad y \in (0,Y), \\ &w \mid_{y=0} = \psi_2(x), \qquad \partial_y w \mid_{y=Y} = \psi_3(x), \qquad x \in \Omega, \end{aligned} \tag{3}$$

and

$$w \mid_{y=Y} = \psi_4(x), \qquad x \in \Omega, \tag{4}$$

where

$$\mathcal{L}w \equiv \partial_y^2 w + h(x)\partial_y w + \sum_{i,j=1}^{m} \partial_i(a_{ij}(x)\partial_j w) + \sum_{i=1}^{m} b_i(x)\partial_i w + c(x)w,$$

$$\partial_y w \equiv \partial w/\partial y, \quad \partial_i w \equiv \partial w/\partial x_i, \quad \partial_\nu w \equiv \sum_{i,j=1}^{m} a_{ij}\partial_j w \cos(n, x_i),$$

$$n \quad \text{normal to the boundary } \partial\Omega.$$

Khaïdarov showed the uniqueness for the inverse problem (3) with (4) and $h = 0$ in [**7, 8**] under the following assumptions: (1) $f(x,y)$ is strictly positive, and (2) f is monotonic with respect to y, that is, $\partial_y f(x,y) \geqslant 0$ and $\partial_y f \neq 0$.

In this paper we obtain the same results about the uniqueness under the weaker assumptions that $f(x,y)$ is allowed to take zero on a set of measure zero when $\partial_y f(x,y) \geqslant 0$ and $\partial_y f \neq 0$, or $f(x,y)$ has a derivative bounded from below, with respect to y, or f does not depend upon y, that is, $\partial_y f = 0$ when $f > 0$, using an orthogonality lemma, a simple transform, or an elimination method.

Amirov [1] obtained existence of an inverse problem identifying the source term of the Poisson equation in the square integrable function class using expansions of functions into eigenfunctions of the Laplacian operator.

The other problem we deal with is about existence in the inverse problem, that is, for any ψ_4 we hope to get (w, q) satisfying (3) and (4).

In this paper we obtain existence in the above-mentioned inverse problem in the same space as that in the investigation of uniqueness, using the theory of solvable operators between Banach spaces.

Furthermore, we obtain also the continuous dependence of the solution to the inverse problem on measurement, which is closely related to the problem of uniqueness, using the Banach inverse operator theorem.

Therefore, it is proved that the above-mentioned inverse problem is well-posed in Hadamard's sense.

2. Problem Statement

From now on, we suppose that $\phi, f, \psi_1, \psi_2, \psi_3, \psi_4, a_{ij}, b_i, h, c, \Omega$, and Y are given and make the following assumptions:

H1.

$$a_{ij}, b_i \in C^{1+\alpha}(\overline{\Omega}),\ c,\ h \in C^{\alpha}(\overline{\Omega}),\ \phi, f \in C^{\alpha}(\overline{D}),\ c(x) \leqslant 0,$$

$$\nu \left|\xi\right|^2 \leqslant \sum_{i,j=1}^{m} a_{ij}(x)\xi_i\xi_j \leqslant \mu \left|\xi\right|^2, \quad \forall \xi \in \mathbf{R}^m, \quad \forall x \in \overline{\Omega},$$

where $\mu > 0$ and $\nu > 0$ are constant, and $C^{k+\alpha}$ and C^{α} are Hölder spaces, for example, see [5].

H2. $\Omega \subset \mathbf{R}^m$ is a bounded open set with a boundary $\partial\Omega \in C^{2+\alpha}$, $Y < +\infty$, and $\overline{\Omega} \equiv \Omega \cup \partial\Omega$.

H3. $\psi_1 \in C^{1+\alpha}(\overline{D})$, $\psi_3 \in C^{1+\alpha}(\overline{\Omega})$ $\psi_2, \psi_4 \in C^{2+\alpha}(\overline{\Omega})$ and they satisfy the consistency condition of order 0, stated in [5]; $q \in C^{\alpha}(\overline{\Omega})$; $c(x) \leqslant 0$, $c(x) - \sum_i \partial_i b_i(x) \leqslant 0$, $h(x) \geqslant 0$, $\forall x \in \Omega$.

H4. $\partial_y f \geqslant 0$, $\partial_y f \not\equiv 0$, $f(x,y) > 0$ almost everywhere in D.

H5. $c(x) \leqslant -\delta < 0$, $c(x) - \sum_i \partial_i b_i(x) \leqslant -\delta$, $\forall x \in \Omega$; $f(x,y) \geqslant \eta > 0$, $\partial_y f(x,y) \geqslant -\varepsilon$, $\forall (x,y) \in D$ and $\eta\sqrt{\delta} - \varepsilon > 0$.

H6. $\partial_y f = 0$ and $f(x,y) \equiv f^*(x) > 0$, $\forall x \in \Omega$.

It is well-known that the problem

$$\mathcal{L}v = \phi(x,y),\ (x,y) \in D, \quad \partial_\nu v \mid_{\partial\Omega} = \psi_1, \quad v \mid_{y=0} = \psi_2,\ \partial_y v \mid_{y=Y} = \psi_3$$

has a unique solution $v \in V \equiv C^{2+\alpha}(\overline{D})$ by [5] if the assumptions **H1-H3** are true. Hence, let $u = w - v$ and then it follows from (3) and (4) that

$$\mathcal{L}u = q(x)f(x,y), \quad \partial_\nu u \,|_{\partial\Omega} = 0, \quad u\,|_{y=0} = 0, \quad \partial_y u\,|_{y=Y} = 0, \tag{5}$$

and

$$u\,|_{y=Y} = z(x), \quad x \in \Omega. \tag{6}$$

It is obvious by the consistency condition that $\partial_\nu z\,|_{\partial\Omega} = 0$.

By [5] there exists a unique solution of (5), $u \in V$, corresponding to $q \in C^\alpha(\overline{\Omega})$, which is denoted by $u = u(q) = u(x,y;q)$ to show the dependence of u on q.

From now on we shall deal with the inverse problem for identifying (u,q) satisfying (5) and (6).

3. Uniqueness

To begin with, we need

Lemma 3.1. *Suppose w is the solution to the problem*

$$\mathcal{L}w = F(x,y), \quad \partial_\nu w\,|_{\partial\Omega} = 0, \quad w\,|_{y=0} = 0, \quad \partial_y w\,|_{y=Y} = 0. \tag{7}$$

Then we have

$$\int_0^Y \int_\Omega F(x,y)v(x,y)dxdy = \int_\Omega [w(x,Y)\beta(x) - w(x,Y)\partial_y v(x,Y) + \partial_y w(x,0)g(x)]\,dx, \tag{8}$$

where $v \in V \equiv C^{2+\alpha}(\overline{D})$ is the solution to an adjoint system of (3), *that is,*

$$\mathcal{L}^* v = 0, \quad \partial_\nu^* v\,|_{\partial\Omega} = 0, \quad v\,|_{y=0} = -g(x), \quad v\,|_{y=Y} = \beta(x), \tag{9}$$

where $g, \beta \in C^{2+\alpha}(\overline{\Omega})$ is arbitrary and

$$\begin{aligned} \mathcal{L}^* v &\equiv \partial_y^2 v - h(x)\partial_y v + \sum_{ij} \partial_i(a_{ji}(x)\partial_j v) - \sum_i \partial_i(b_i(x)v) + c(x)v, \\ \partial_\nu^* v &\equiv \sum_i \Big(\sum_j a_{ji}\partial_j v - b_i v\Big) \cos(n, x_i). \end{aligned} \tag{10}$$

In particular, when $w(x,Y) = 0$, $\forall x \in \Omega$, one gets

$$\int_{y=0}^Y \int_\Omega v(x,y)F(x,y)\,dxdy = \int_\Omega g(x)\partial_y w(x,0)\,dx. \tag{11}$$

PROOF: Because (9) has a unique solution $v \in V$ by [5], using Green's formula we have at once that

$$\int_0^Y \int_\Omega v(x,y)F(x,y)dxdy = \int_D v\mathcal{L}w dxdy = \int_\Omega [v\partial_y w - w\partial_y v + hvw] \,|_{y=0}^{Y}\, dx$$
$$+ \int_0^Y \int_{\partial\Omega} \sum_i \Big\{ v \sum_j a_{ij}\partial_j w - w\Big(\sum_j a_{ji}\partial_j v - b_i v\Big)\Big\} \cos(n, x_i) + \int_D w\mathcal{L}^* v$$
$$= \int_\Omega [w(x,Y)\beta(x) - w(x,Y)\partial_y v(x,Y) + \partial_y w(x,0)g(x)]dx.$$

∎

THEOREM 3.2. *Suppose the assumptions* **H**1-**H**3 *and one of the assumptions* **H**4, **H**5, *and* **H**6 *hold. Then the solution to the inverse problem* (5) *and* (6) *for determining* (u,q), *if it exists, is unique.*

PROOF: Suppose that there are two solutions, (u_1, q_1) and (u_2, q_2), satisfying (5) and (6). Set

$$u \equiv u_1 - u_2, \quad q \equiv q_1 - q_2;$$

then we have

$$\mathcal{L}u = qf, \quad \partial_\nu u \,|_{\partial\Omega} = 0, \quad u\,|_{y=0} = 0, \quad \partial_y u\,|_{y=Y} = 0. \tag{12}$$

and

$$u\,|_{y=Y} = 0, \quad x \in \Omega. \tag{13}$$

Our purpose is to prove $u = 0$ and $q = 0$.

Let $\Omega \equiv \Omega_- \cup \Omega_0 \cup \Omega_+$, where $\Omega_- \equiv \{x \in \Omega;\ q(x) < 0\}$, $\Omega_0 \equiv \{x \in \Omega;\ q(x) = 0\}$, and $\Omega_+ \equiv \{x \in \Omega;\ q(x) > 0\}$.

Obviously, Ω_- and Ω_+ are both open.

If $\Omega_0 \neq \Omega$, then Ω_+ or Ω_- is not empty. Moreover, if only one of them is empty, for example, $\Omega_- = \emptyset$ and $\Omega_+ \neq \emptyset$, then $F(x,y) = q(x)f(x,y) \geqslant 0$, $\forall (x,y) \in D$ and $F \neq 0$. When take $g = 0$ and $\beta \in C^{2+\alpha}(\overline{\Omega})$ with $\beta(x) > 0$, then the solution of (9), v, is positive by the maximum principle [5], so

$$\int_0^Y \int_\Omega q(x)f(x,y)v(x,y)\,dxdy > 0,$$

which is contrary to (11) in Lemma 3.1.

Hence, Ω_+ and Ω_- are both nonempty, and then $q(x)$ changes its sign in Ω.

Set $\beta(x) = \operatorname{sign}\ q(x)$, $x \in \Omega$, and then $\beta \in L^\infty(\Omega) \subset L^p(\Omega)$ with $p > 1$ being large enough.

Let the functions $g_1 \in C_0^\infty(\mathbf{R}^{m+1})$ and $g_2 \in C_0^\infty(\mathbf{R}^m)$ satisfy $g_1(x,y) \geqslant 0$, $g_2(x) \geqslant 0$,

$$\int_{\mathbf{R}^{m+1}} g_1(x,y)\,dxdy = 1, \quad \int_{\mathbf{R}^m} g_2(x)\,dx = 1.$$

Consider the regularisation operators

$$\begin{aligned} \mathfrak{A}_n p(x,y) &= n^{m+1} \int_D g_1(n(x-\xi), n(y-\eta)) p(\xi,\eta)\,d\xi d\eta, \\ \mathfrak{B}_n k(x) &= n^m \int_\Omega g_2(n(x-\xi)) k(\xi)\,d\xi. \end{aligned} \tag{14}$$

Set $a_{ij}^{(n)} \equiv \mathfrak{B}_n a_{ij}$, $b_i^{(n)} \equiv \mathfrak{B}_n b_i$, $h_n \equiv \mathfrak{B}_n h$, $c_n \equiv \mathfrak{B}_n c$, $q_n \equiv \mathfrak{B}_n q$, $\beta_n \equiv \mathfrak{B}_n \beta$, and $f_n \equiv \mathfrak{A}_n f$, and then $a_{ij}^{(n)}, b_i^{(n)}, h_n, c_n, q_n, \beta_n \in C^\infty(\overline{\Omega})$, and $f_n \in C^\infty(\overline{D})$. Moreover, $a_{ij}^{(n)}, b_i^{(n)} \xrightarrow{s} a_{ij}, b_i$ in $C^{1+\alpha}(\overline{\Omega})$, $h_n, c_n, q_n \xrightarrow{s} h, c, q$ in $C^\alpha(\overline{\Omega})$, respectively, $f_n \xrightarrow{s} f$ in $C^\alpha(\overline{D})$, and $\beta_n \xrightarrow{s} \beta$ in $L^p(\Omega)$ with $|\beta_n(x)| \leqslant 1$, $\forall x \in \overline{\Omega}$. ("$x_n \xrightarrow{s} x$, in X" means that x_n converges strongly to x in X.) Furthermore, according to the assumption **H3** we have

$$c_n(x) \leqslant 0, \quad c_n(x) - \sum_i \partial_i b_i^{(n)}(x) \leqslant 0, \quad \forall x \in \Omega.$$

Consider the following problems:

$$\mathcal{L}_n u_n = q_n f_n, \quad B_n u_n \,|_{\partial\Omega} = 0, \quad u_n \,|_{y=0} = 0, \quad \partial_y u_n \,|_{y=Y} = 0, \tag{15}$$

and

$$\mathcal{L}_n^* v_n = 0, \quad B_n^* v_n \,|_{\partial\Omega} = 0, \quad v_n \,|_{y=Y} = \beta_n, \quad \partial_y v_n \,|_{y=0} = 0, \tag{16}$$

where

$$\begin{aligned} \mathcal{L}_n w &\equiv \partial_y^2 w + h_n \partial_y w + \sum_{i,j} \partial_i \left(a_{ij}^{(n)} \partial_j w \right) + \sum_i b_i^{(n)} \partial_i w + c_n w, \\ B_k w &\equiv \sum_{i,j} a_{ij}^{(k)} \partial_j w \cos(n, x_i), \\ \mathcal{L}_n^* v &\equiv \partial_y^2 v - h_n \partial_y v - \sum_{i,j} \partial_i \left(a_{ji}^{(n)} \partial_j v \right) - \sum_i \partial_i \left(b_i^{(n)} v \right) + c_n v, \\ B_k^* v &\equiv \sum_i \left[\sum_j a_{ji}^{(k)} \partial_j v - b_i^{(k)} v \right] \cos(n, x_i). \end{aligned}$$

It follows by [5] that there exist solutions $u_n, v_n \in C^\infty(\overline{D})$ of (15) and (16), and then by virtue of the maximum principle

$$-1 < v_n(x,y) < 1, \quad (x,y) \in D. \tag{17}$$

Moreover, consider the following problem

$$\mathcal{L}^* v = 0, \quad \partial_\nu^* v\,|_{\partial\Omega} = 0, \quad v\,|_{y=Y} = \beta, \quad \partial_y v\,|_{y=0} = 0. \tag{18}$$

There exists a unique solution $v \in W \equiv H^{1/2}(D) \subset L^2(D)$ of (18) by means of [**10**]; the definition of $H^{1/2}(D)$ can be found in [**10**]. Furthermore, by Lemma 3.3, which will be proved next, it follows that

$$\begin{aligned} u_n &\xrightarrow{s} u \quad \text{in} \quad C^{2+\alpha}(\overline{D}), \\ v_n &\xrightarrow{s} v \quad \text{in} \quad W, \end{aligned} \tag{19}$$

and then $|v(x,y)| \leqslant 1$, almost everywhere on D, where u is the solution of (12).

In addition, $\partial_y v_n \equiv w_n$ are also the solutions to the following problems:

$$\mathcal{L}_n^* w_n = 0, \quad B_n^* w_n\,|_{\partial\Omega} = 0, \quad w_n\,|_{y=Y} = \beta_n^* \equiv \partial_y v_n(\cdot, Y), \quad w_n\,|_{y=0} = 0. \tag{20}$$

obviously, $\beta_n^* \in C^\infty(\overline{\Omega})$, and by Lemma 3.3 there is an $M > 0$ such that $|\beta_n^*(x)|$, $\left|\partial_y^2 v_n(x,Y)\right| \leqslant M$, $\forall x \in \overline{\Omega}$, $\forall n$. Hence, it follows from Lemma 3.1 that

$$\begin{aligned} \int_\Omega u_n(x,Y)[\beta_n^*(x) - \partial_y w_n(x,Y)]\,dx &= \int_D q_n(x) f_n(x,y) \partial_y v_n(x,y)\,dx dy \\ &= \int_\Omega q_n(x)[f_n(x,y)v_n(x,y)]\,\Big|_{y=0}^{Y}\,dx - \int_D q_n v_n \partial_y f_n\,dx dy \\ &= \int_\Omega q_n(x)[\beta_n(x) f_n(x,Y) - f_n(x,0)v_n(x,0)]dx - \int_D q_n v_n \partial_y f_n\,dx dy. \end{aligned} \tag{21}$$

If the assumption **H4** is true and considering (19) one can get

$$u_n(\cdot, Y) \xrightarrow{s} u(\cdot, Y) = 0 \quad \text{in} \quad C^{2+\alpha}(\overline{\Omega}).$$

Therefore,

$$\left|\int_\Omega u_n(x,Y)\beta_n^*(x)\,dx\right|, \quad \left|\int_\Omega u_n(x,Y)\partial_y^2 v_n(x,Y)\,dx\right| \leqslant M\,\|u_n(\cdot,Y)\|_{C(\overline{\Omega})}\ \text{mes}\,\Omega \to 0.$$

Let n go to infinity in (21):

$$\begin{aligned}
0 &= \int_\Omega q(x)[\beta(x)f(x,Y) - v(x,0)f(x,0)]dx - \int_D qv\partial_y f dxdy \\
&= \int_{\Omega_+} q(x)[f(x,Y) - f(x,0)v(x,0)]dx - \int_0^Y \int_{\Omega_+} qv\partial_y f dxdy \\
&\quad + \int_{\Omega_-} |q(x)|\,[f(x,Y) + f(x,0)v(x,0)]dx + \int_0^Y \int_{\Omega_-} |q|\, v\partial_y f dxdy \\
&> \int_{\Omega_+} q(x)[f(x,Y) - f(x,0)]dx - \int_0^Y \int_{\Omega_+} q\partial_y f dxdy \\
&\quad + \int_{\Omega_-} |q(x)|\,[f(x,Y) - f(x,0)]dx - \int_0^Y \int_{\Omega_-} |q|\,\partial_y f dxdy = 0.
\end{aligned}$$

The above contradiction is from the hypotheses $\Omega_+ \neq \emptyset$ and $\Omega_- \neq \emptyset$. So, $q = 0$, and then $u = 0$ by [5].

Next, if the assumption **H5** is true, then instead of (12) and (13), setting $u = we^{\lambda y}$ we consider the following problem:

$$\begin{gathered}
\mathcal{L}w + (2\lambda + h)\partial_y w + \left(\lambda^2 + h\lambda\right)w = q(x)F(x,y), \\
\partial_\nu w\,|_{\partial\Omega} = 0, \quad w\,|_{y=0} = 0, \quad \partial_y w\,|_{y=Y} = 0,
\end{gathered} \tag{22}$$

and

$$w\,|_{y=Y} = 0, \quad x \in \Omega, \tag{23}$$

where $F(x,y) \equiv f(x,y)e^{-\lambda y}$ and by the assumption **H5** a negative constant λ can be chosen such that $\partial_y F = (\partial_y f - \lambda f)e^{-\lambda y} \geqslant |\lambda|\eta - \varepsilon > 0$, $\forall (x,y) \in D$ and $c(x) + \lambda^2 + h\lambda \leqslant 0$, $c(x) + \lambda^2 + h\lambda - \sum_i \partial_i b_i(x) \leqslant 0$, $\forall x \in \Omega$. (For example, take $\lambda \in \left(\varepsilon/\eta, \sqrt{\delta}\right)$ which satisfies all the above-mentioned requirements.

The inverse problem (22) with (23) satisfies **H4**. From the above conclusion we can at once get $q = 0$, $w = 0$; therefore, $u = 0$.

Finally, if the assumption **H6** is true, that is, $f(x,y) = f^*(x)$, then differentiating the two sides of (5) with respect to y and setting $w = \partial_y u$ we have

$$\mathcal{L}w = 0, \qquad \partial_\nu w\,|_{\partial\Omega} = 0, \qquad w\,|_{y=Y} = 0. \tag{24}$$

Obviously, we have to obtain a boundary value of w on a lower base. Setting $\forall x \in \overline{\Omega}$

$$\psi(x) = \begin{cases} 0 & \text{if } \partial_y u(x,0) = 0, \\ \sup\{y_1 > 0;\ \partial_y u(x,y) \neq 0,\ \forall y \in [0,y_1)\} & \text{otherwise}, \end{cases} \tag{25}$$

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by Lemma 3.4, which will be proved next, we know that $\psi \in C^{1+\alpha}(\overline{\Omega})$, $\psi(x) \in (0, Y)$, and $\partial_y u(x, \psi(x)) = 0$, $\forall x \in \overline{\Omega}$, that is,

$$w\,|_{\Gamma} = 0, \quad \Gamma \equiv \{(x, \psi(x));\ x \in \overline{\Omega}\}. \tag{26}$$

Thus, the generalised Dirichlet boundary value problem (24) with (26) has only the zero solution in the open set

$$E \equiv \{(x, y);\ \psi(x) < y < Y,\ x \in \Omega\}$$

by the maximum principle, that is, $\partial_y u(x, y) = 0$, $\forall (x, y) \in E$. Therefore, $u(x, y) = u^*(x)$, $\forall (x, y) \in \overline{E}$. But $u(x, Y) = 0$, so $u(x, y) = 0$, $\forall (x, y) \in \overline{E}$, and then $q(x) f^*(x) = 0$, $\forall x \in \Omega$ by (5). Hence $q = 0$ by the assumption **H6**. ∎

Lemma 3.3. *Let $a_{ij}^{(k)}, b_i^{(k)} \xrightarrow{s} a_{ij}, b_i$ in $C^{1+\alpha}(\overline{\Omega})$, $h_k, c_k, q_k \xrightarrow{s} h, c, q$ in $C^{\alpha}(\overline{\Omega})$, respectively, $f_k \xrightarrow{s} f$ in $C^{\alpha}(\overline{D})$, and $\beta_k \xrightarrow{s} \beta$ in $L^p(\Omega)$. Then there is a constant $M > 0$ such that*

$$\begin{aligned} &u_k \xrightarrow{s} u \text{ in } C^{2+\alpha, 1+\alpha/2}(\overline{D}), \\ &v_k \xrightarrow{s} v \text{ in } W, \end{aligned} \tag{27}$$

and

$$|\beta_k^*(x)|, \quad \left|\partial_y^2 v_k(x, Y)\right| \leqslant M, \qquad \forall x \in \overline{\Omega}, \tag{28}$$

where u_k, v_k, u, and v are determined by (15), (16), (12), and (18) respectively, and $\beta_k^*(x) \equiv \partial_y v_k(x, Y)$.

Proof: By the assumptions one gets

$$\left\|a_{ij}^{(k)}, b_i^{(k)}\right\|_{1+\alpha}, \quad \|h_k, c_k, q_k\|_{\alpha}, \quad \|f_k\|_{\alpha}, \quad \|\beta_k\|_p \leqslant M_1, \tag{29}$$

where $\|\cdot\|_{\alpha}$ denotes the norm in the space C^{α}, $\|\cdot\|_p$ is the norm in the space L^p, and the constant M_1 is independent of k. It is obvious by the theory of partial differential equations, for example, see [5], that

$$\|u_k, v_k\|_{2+\alpha} \leqslant M_2, \tag{30}$$

where the constant M_2 is also independent of k. From (30), particularly, one gets

$$|\beta_k^*(x)| \equiv |\partial_y v_k(x, Y)|, \quad \left|\partial_y^2 v_k(x, Y)\right| \leqslant \|v_k\|_{2+\alpha} \leqslant M, \qquad \forall x \in \overline{\Omega},$$

which is just (28).

Subtraction of (12) from (15) leads to:

(31)
$$\begin{aligned}\mathcal{L}(u_k - u) = q(f_k - f) + f_k(q_k - q) - \sum_{i,j} \partial_i \Big[\Big(a_{ij}^{(k)} - a_{ij}\Big)\partial_j u_k\Big] \\ - \sum_i \Big(b_i^{(k)} - b_i\Big)\partial_i u_k - (h_k - h)\partial_y u_k - (c_k - c)u_k,\end{aligned}$$

$$\partial_\nu(u_k - u)\,|_{\partial\Omega} = -\sum_{i,j}\Big(a_{ij}^{(k)} - a_{ij}\Big)\partial_j u_k \cos(n, x_i),$$

$$u_k - u\,|_{y=0} = 0, \quad \partial_y(u_k - u)\,|_{y=Y} = 0.$$

Considering (29) and (30) and then by [5]

(32)
$$\begin{aligned}\|u_k - u\|_{2+\alpha} \leqslant M_1\Big\{\sum_{i,j}\Big\|a_{ij}^{(k)} - a_{ij}\Big\|_{1+\alpha} + \sum_i \Big\|b_i^{(k)} - b_i\Big\|_{1+\alpha} \\ + \|h_k - h\|_\alpha + \|c_k - c\|_\alpha + \|q_k - q\|_\alpha + \|f_k - f\|_\alpha\Big\},\end{aligned}$$

where M_1 is independent of k. Therefore, by the assumptions one gets

$$u_k \xrightarrow{s} u \quad \text{in} \quad C^{2+\alpha}(\overline{D}).$$

Similarly, one has

$$v_k \xrightarrow{s} v \quad \text{in} \quad W.$$

□

Lemma 3.4. *ψ defined by (25) possesses the following properties:*

1. *$\psi(x)$ is defined and $\psi(x) \in [0, Y]$, $\forall x \in \overline{\Omega}$.*
2. *$\partial_y u(x, \psi(x)) = 0$, $\forall x \in \overline{\Omega}$.*
3. *$\psi \in C^{1+\alpha}(\overline{\Omega})$.*

Proof: First, we prove Property 1 for any $x \in \overline{\Omega}$.

If $\partial_y u(x,0) \neq 0$, then owing to $u(x,0) = u(x,Y) = 0$ by Rolle's theorem there exists $\widetilde{y} \in (0, Y)$ such that $\partial_y u(x, \widetilde{y}) = 0$. So, $\psi(x) \leqslant \widetilde{y} < Y$; hence, Property 1 is true.

It is obvious by the definition of ψ that Property 2 is true.

Finally, we prove Property 3.

Take any $x_1 \in \overline{\Omega}$, and set $y_1 = \psi(x_1)$. If $y_1 > 0$, then $\partial_y u(x_1, y) \neq 0$, $\forall y \in [0, y_1)$. There is no harm in supposing

(33)
$$\partial_y u(x_1, y) > 0, \qquad \forall y \in [0, y_1).$$

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Thus, $u(x_1,y_2) > u(x_1,y_3) > 0$, $\forall 0 < y_3 < y_2 < y_1$.

We assert that $\partial_y^2 u(x_1,y_1) < 0$. In fact, the Taylor's expansion of $u(x_1,y)$ at (x_1,y_1) is

$$u(x_1,y) = u(x_1,y_1) + \frac{1}{2}\partial_y^2 u(x_1,y_1)\delta y^2 + o(\delta y^2), \quad \forall y \in (y_1-\eta, y_1+\eta),$$

where $\delta y \equiv y - y_1$. Because $u(x_1,y) < u(x_1,y_1)$, $\forall y \in [0,y_1)$, $\partial_y^2 u(x_1,y_1) < 0$.

Next, we prove $\psi \in C(\overline{\Omega})$. If this were false, then there is $\widetilde{x} \in \overline{\Omega}$ such that $\widetilde{y} \equiv \psi(\widetilde{x})$ is not equal to one of $\overline{\psi}$ and $\underline{\psi}$, where

$$\overline{\psi} \equiv \limsup_{x\to\widetilde{x}} \psi(x), \qquad \underline{\psi} \equiv \liminf_{x\to\widetilde{x}} \psi(x).$$

Obviously, $0 \leqslant \underline{\psi} \leqslant \widetilde{y} \leqslant \overline{\psi} \leqslant Y$. Moreover, there are $\{x_n^1\}$, $\{x_n^2\} \subset \overline{\Omega}$ such that $x_n^1 \to \widetilde{x}$, $x_n^2 \to \widetilde{x}$, $\underline{\psi} = \lim_{n\to\infty} \psi(x_n^1)$, and $\overline{\psi} = \lim_{n\to\infty} \psi(x_n^2)$. Therefore, $\partial_y u(\widetilde{x},\underline{\psi}) = \partial_y u(\widetilde{x},\overline{\psi}) = 0$, $\partial_y^2 u(\widetilde{x},\underline{\psi}) \leqslant 0$, and $\partial_y^2 u(\widetilde{x},\overline{\psi}) \leqslant 0$.

If $\underline{\psi} < \widetilde{y}$, then by (33) we have $\partial_y u(\widetilde{x},\underline{\psi}) > 0$, which is contrary to $\partial_y u(\widetilde{x},\underline{\psi}) = 0$.

If $\overline{\psi} > \widetilde{y}$, then by $\partial_y u(\widetilde{x},\widetilde{y}) = 0$ and $\partial_y^2 u(\widetilde{x},\widetilde{y}) < 0$ there is $\delta > 0$ with $4\delta < \overline{\psi} - \widetilde{y}$ such that

$$(34) \qquad \partial_y u(x,y) < 0, \qquad \forall (x,y) \in \overline{B}(\widetilde{x},\delta) \times [\widetilde{y}+\delta, \widetilde{y}+2\delta],$$

where $\overline{B}(\widetilde{x},\delta)$ is the closed ball of radius δ about $\widetilde{x}$.

When $n > N_1(\delta)$ we have $|x_n^2 - \widetilde{x}| < \delta$, so

$$(35) \qquad \partial_y u(x_n^2, y) < 0, \qquad \forall y \in [\widetilde{y}+\delta, \widetilde{y}+2\delta].$$

On the other hand, if $n > N_2(\delta)$, $\psi(x_n^2) > \overline{\psi} - \delta > \widetilde{y} + 2\delta$, then

$$(36) \qquad \partial_y u(x_n^2, y) > 0, \qquad \forall y \in [0, \psi(x_n^2)).$$

If $n > \max(N_1, N_2)$, then we have a contradiction comparing (35) with (36).

Now, we prove $\psi \in C^1(\overline{\Omega})$. Because for any $x_1 \in \overline{\Omega}$ and $\forall x \in B_1 \subset \overline{\Omega}$

$$\partial_y u(x,\psi(x)) = 0, \qquad \partial_y^2 u(x,\psi(x)) \neq 0,$$

there is a function $p = p(x)$ by the implicit function theorem such that $p \in C^1(B_1)$, $\partial_y u(x,p(x)) = 0$, $\forall x \in B_1$, and $\psi(x_1) = p(x_1)$, where B_1 is a neighbourhood of x_1.

We assert

$$\psi(x) = p(x), \qquad \forall x \in B_2,$$

where $B_2 \subset B_1$ is a neighbourhood of x_1.

In point of fact, there is a neighbourhood of $(x_1, \psi(x_1))$, U, such that $\partial_y^2 u(x,y) \neq 0$, $\forall (x,y) \in U$. Therefore, there exists a neighbourhood of x_1, B_3, such that for any $x \in B_3$ the function $\phi(y) \equiv \partial_y u(x,y)$ is strictly monotonic, and we also have

$$\partial_y u(x,y) \neq 0, \qquad \forall x \in B_3,\ \forall y \neq \psi(x).$$

Thus, $\partial_y u(x, p(x)) = 0$, $\forall x \in B_3 \cap B_1$ is true if and only if $\psi(x) = p(x)$, $\forall x \in B_1 \cap B_3 \equiv B_2$.

Finally, we prove $\psi \in C^{1+\alpha}(\overline{\Omega})$. Indeed, by the implicit function theorem it follows that

$$\partial_i \psi(x) = -\frac{\partial_i \partial_y u(x, \psi(x))}{\partial_y^2 u(x, \psi(x))}, \qquad \forall x \in \Omega.$$

Considering $\left|\partial_y^2 u(x, \psi(x))\right| \geqslant c > 0$, $\forall x \in \overline{\Omega}$ one can get

$$\frac{|\partial_i \psi(x_1) - \partial_i \psi(x_2)|}{\left(|x_1 - x_2|^2 + |y_1 - y_2|^2\right)^{\alpha/2}} \leqslant 1/c \frac{|\partial_i \partial_y u(x_1, y_1) - \partial_i \partial_y u(x_2, y_2)|}{\left(|x_1 - x_2|^2 + |y_1 - y_2|^2\right)^{\alpha/2}} \leqslant M,$$

where $y_i = \psi(x_i)$, $(i = 1, 2)$.

Thus, $\psi \in C^{1+\alpha}(\overline{\Omega})$. □

4. Existence and continuous dependence

First of all, recall the following definition from [**17**]:

If $A : \mathcal{X} \to \mathcal{Y}$ is a linear densely defined operator on $\mathcal{X}$ into $\mathcal{Y}$, then A' is the conjugate (dual) of A on $\mathcal{Y}'$ into $\mathcal{X}'$, where $\mathcal{X}'$ and $\mathcal{Y}'$ are conjugate (dual) spaces of the Banach spaces $\mathcal{X}$ and $\mathcal{Y}$, respectively.

The annihilator, $A^\perp$, of a set $A \subset \mathcal{X}'$ is defined by $A^\perp \equiv \{x \in X;\ \langle x, x' \rangle = 0, \forall x' \in A\}$, where $\langle x, x' \rangle$ denotes the value of a functional x' at x, and the null space of A is defined by $\mathcal{N}(A) \equiv \{x \in \mathcal{X};\ Ax = 0\}$.

Furthermore, a set $F' \subset \mathcal{X}'$ is said to be total if to each $x \neq 0$ in $\mathcal{X}$ there corresponds some $x' \in F'$ such that $\langle x, x' \rangle \neq 0$.

From [**17**] one can get

Lemma 4.1. *Suppose that A is a linear closed dense defined operator. Then*

1. $\mathcal{R}(A')^\perp \cap \mathcal{D}(A) = \mathcal{N}(A)$,
2. *$\mathcal{R}(A')$ is total in $\mathcal{X}'$ if and only if A^{-1} exists,*
3. *$\mathcal{R}(A)$ is closed if and only if $\mathcal{R}(A') = \mathcal{N}(A)^\perp$,*

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where $\mathcal{R}(A')$ is the range of A' and $\mathcal{D}(A)$ is the domain of A.

Consider the operator

$$(37)\qquad \begin{gathered} P : Q \to \mathcal{K} \\ Pq = u(\cdot, Y; q) \end{gathered}$$

where $u(q)$ is the solution of (5), $Q \equiv C^{\alpha}(\overline{\Omega})$, and

$$\mathcal{K} \equiv \{v \in C^{2+\alpha}(\overline{\Omega});\ \partial_{\nu} v \mid_{\partial\Omega} = 0\}.$$

Obviously, $\mathcal{K}$ is a Banach space with the norm of $C^{2+\alpha}(\overline{\Omega})$.

LEMMA 4.2. *Suppose that the assumptions of Theorem 3.2 hold. Then*

1. *$P \in \mathcal{L}(Q, \mathcal{K})$, the space of bounded linear operators on X to Y.*
2. *$\mathcal{N}(P) = \{0\}$.*
3. *$\|Pq\|_{\mathcal{K}} \leqslant c_4 \|q\|_Q$, $\quad \forall q \in Q$, where the constant c_4 is only dependent on a_{ij}, b_i, h, c, f, Ω, and Y.*

PROOF: The result 2 is obvious by Theorem 3.2. The operator P is linear on Q into $\mathcal{K}$ by the formula

$$Pq = u(\cdot, Y; q) = \int_D G(\cdot, Y; \xi, \eta) q(\xi) f(\xi, \eta)\, d\xi d\eta,$$

where G is the Green function of (5). Besides, it follows from [11] that

$$(39)\qquad \|u\|_V \leqslant c_2(\|fq\|_{\mathcal{F}} + \|u\|_0),$$

and one also gets $\|u\|_0 \leqslant c_2 \|qf\|_{\mathcal{F}}$ under $c(x) \leqslant 0$, where $V \equiv C^{2+\alpha}(\overline{D})$, $\|u\|_0 \equiv \sup_{(x,y)\in\overline{D}} |u(x,y)|$, and $\mathcal{F} \equiv C^{\alpha}(\overline{D})$. So,

$$\|Pq\|_{\mathcal{K}} = \|u(\cdot, Y; q)\|_{\mathcal{K}} \leqslant \|u(q)\|_V \leqslant c_2 \|fq\|_{\mathcal{F}}.$$

Moreover,

$$\begin{aligned} \|fq\|_{\mathcal{F}} = {} & \sup_{(x,y)\in\overline{D}} \{|q(x)f(x,y)|\} \\ & + \sup\Big\{|q(x_1)f(x_1,y_1) - q(x_2)f(x_2,y_2)| \,/\, \big(|x_1 - x_2|^2 + |y_1 - y_2|^2\big)^{\alpha/2}\Big\} \\ \leqslant {} & \sup\{|f(x,y)q(x)|\} + \sup\{|f(x_1,y_1)|\,|q(x_1 - q(x_2))| \,/\, |x_1 - x_2|^{\alpha}\} \\ & + \sup\Big\{|q(x_2)|\,|f(x_1,y_1) - f(x_2,y_2)| \,/\, \big(|x_1 - x_2|^2 + |y_1 - y_2|^2\big)^{\alpha/2}\Big\} \\ \leqslant {} & \|f\|_0 \|q\|_0 + \|f\|_0 \|q\|_{\alpha} + \|f\|_{\alpha} \|q\|_0 \leqslant \|q\|_Q \|f\|_{\mathcal{F}}, \end{aligned}$$

where $\|f\|_0 \equiv \sup_{(x,t)\in\overline{D}} |f(x,t)|$ and $\|q\|_0 \equiv \sup_{x\in\overline{\Omega}} |q(x)|$. Hence

$$\|Pq\|_{\mathcal{K}} \leqslant c_4 \|q\|_Q ; \tag{39}$$

therefore, $P \in \mathcal{L}(Q,\mathcal{K})$. □

LEMMA 4.3. *Let the assumptions of Theorem 3.2 be true.*

Then the range of P, $\mathcal{R}(P) \equiv K$, is a closed subspace of $\mathcal{K}$. Moreover, $\forall z \in K$ the inverse problem (5) with (6) has a unique solution $(u,q) \in V \times Q$.

PROOF: Suppose that $\mathcal{X} = Q$, $\mathcal{Y} = \mathcal{K}$, and $A = P$. So, by Lemma 4.2 one can get that P is linear and continuous and that $\mathcal{N}(P) = \{0\}$. Hence, it follows by Lemma 4.1 that $\mathcal{R}(P')^{\perp} = \mathcal{N}(P) = \{0\}$, and then $\mathcal{R}(P') = \mathcal{N}(P)^{\perp} = Q'$. Thus, $\mathcal{R}(P')$ in Q' is total by the definition. Using Lemma 4.1 again one gets that K is closed in $\mathcal{K}$ and that the inverse operator of P, P^{-1}, exists, that is, $\forall z \in K$ there is a unique $q \in Q$ such that $q = P^{-1}z$. Substituting q into (5) one can obtain $u \in V$.

It is easy to check that (u,q) is the solution of the inverse problem (5) with (6). □

THEOREM 4.4. *If the assumptions of Theorem 3.2 are valid, then the inverse problem (5) with (6) is well-posed in Hadamard's sense, that is, for any $z \in \mathcal{K}$ there exists a unique solution (u,q) satisfying (5) and (6) simultaneously. Moreover, (u,q) depends continuously upon z. In fact, the estimate*

$$\|u\|_V + \|q\|_Q \leqslant c_1 \|z\|_{\mathcal{K}} \tag{40}$$

is true, where c_1 is a constant depending continuously on $a_{ij}, b_i, h, c, f, \Omega$, and Y.

PROOF: In order to prove that for any $z \in \mathcal{K}$ there is a unique pair $(u,q) \in V \times Q$ satisfying (5) and (6), obviously by Lemma 4.3, one only needs to prove $K = \mathcal{K}$ in other words, it is sufficient to prove that the operator P defined by (37) is open. If it were false, then $\forall n \in \mathbf{N}$, $\forall q \in Q$, there are $k_n \in K$ such that $k_n = Pq$ and $\|q\|_Q > n\|k_n\|_{\mathcal{K}}$. In particular, take $k_n \in K$ with $\|k_n\| = 1$ and $q_n \in Q$ with $\|q_n\| > n$ such that $k_n = Pq_n$.

By the Hahn-Banach theorem there are $v_n^* \in Q'$ such that

$$\|v_n^*\| = 1, \quad \langle q_n, v_n^*\rangle = \|q_n\|, \ n = 1,2,\cdots.$$

Set $v_n = v_n^*/\|q_n\|$, $n = 1,2,\cdots$, then

$$\|v_n\| \leqslant 1/n, \quad \langle q_n, v_n\rangle = 1, \ n = 1,2,\cdots. \tag{41}$$

On the other hand, $\mathcal{R}(P') = Q'$ and P' is linear and bounded, so, P' is open by [17]. Moreover, $v_n \in Q' = \mathcal{R}(P')$ and $v_n \to 0$ by (41), hence there exist $w_n \in \mathcal{K}' = \mathcal{D}(P')$ such that $v_n = P'w_n$, $w_n \xrightarrow{s} 0$ in $\mathcal{K}'$. Therefore,

$$|\langle q_n, v_n\rangle| = |\langle q_n, P'w_n\rangle| = |\langle Pq_n, w_n\rangle| = |\langle k_n, w_n\rangle| \leqslant \|k_n\|\,\|w_n\| = \|w_n\| \to 0,$$

which is contrary to (41).

So far we have proved that the continuous linear operator $P : Q \to \mathcal{K}$ is surjective and injective, thus the inverse operator $P^{-1} : \mathcal{K} \to Q$ is continuous by the Banach inverse operator theorem, for example, see [**17**], and then there is a constant c_2 such that $\forall z \in \mathcal{K}$, $\exists q \in Q$,

$$\|q\|_Q = \left\|P^{-1}z\right\| \leqslant c_2 \|z\|_{\mathcal{K}}. \tag{42}$$

It is obvious that c_2 depends continuously on the estimate of P. Combining (42) with (38) in Lemma 4.2 we immediately obtain (40). ∎

CONCLUSION. The results of Theorem 2 are also valid for other boundary value problems of the elliptic equation. For example, the inverse problem with Dirichlet boundary-value condition and Neumann measurement: $u(x,0) = u(x,Y) = 0$ and $\partial_y u(x,Y) = z(x)$ or with Neumann boundary-value condition and Dirichlet measurement: $\partial_y u(x,0) = \partial_y u(x,Y) = 0$ and $u(x,Y) = z(x)$ are also well-posed under the same assumptions as those in Theorem 1.

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