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ON THE MODULE STRUCTURE OF A GROUP ACTION ON A LIE ALGEBRA

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Abstract

Let G be a finite group, K a field, and V a finite-dimensional KG-module. Write $L(V)$ for the free Lie algebra on V; similarly, let $M(V)$ be the free metabelian Lie algebra. The action of G extends naturally to these algebras, so they become KG-modules, which are direct sums of finite-dimensional submodules. This paper explores whether indecomposable direct summands of such a KG-module (for some specific choices of G, K and V) must fall into finitely many isomorphism classes. Of course this is not a question unless there exist infinitely many isomorphism classes of indecomposable KG-modules (that is, K has positive characteristic p and the Sylow p-subgroups of G are non-cyclic) and $\dim V > 1$.

The first two results show that the answer is positive for $M(V)$ when K is finite and $\dim V = 2$, but negative when G is the Klein four-group, the characteristic of K is 2, and V is the unique 3-dimensional submodule of the regular module D. In the third result, G is again the Klein four-group, K is any field of characteristic 2 with more than 2 elements, V is any faithful module of dimension 2, and B is the unique 3-dimensional quotient of D; the answer is positive for $L(V)$ if and only if it is positive for each of $L(B)$, $L(D)$, and $L(V \otimes V)$.

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1. Introduction and notation

Let K be a field of positive characteristic p and G a group. Throughout this paper, all KG-modules considered will be right KG-modules and all tensor products are tensor products over K. If V is a vector space over K (or, briefly, K-space), we write $A(V)$ for the free associative algebra (with identity element) on V: thus $A(V)$ is the free associative algebra over K with the property that $A(V)$ contains V as a subspace, and every basis for V generates $A(V)$ freely as an algebra. If V is a KG-module, the

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action of G on V extends uniquely to $A(V)$ subject to $A(V)$ becoming a KG-module on which the elements of G act as algebra automorphisms. Similarly, $R(V)$ denotes the free restricted Lie algebra on V and $L(V)$ denotes the free Lie algebra on V. It is well-known that if $A(V)$ is regarded as a restricted Lie algebra under the operations given by $[a, b] = ab - ba$ and $a^{[p]} = a^p$, then the Lie subalgebra generated by V and the restricted Lie subalgebra generated by V may be identified with $L(V)$ and $R(V)$, respectively. In this sense, we consider $L(V)$ and $R(V)$ submodules of $A(V)$. For a non-negative integer n, let $A^n(V)$ be the nth homogeneous component of $A(V)$: it is the subspace of $A(V)$ spanned by all monomials $v_1 \cdots v_n$, with $v_1, \ldots, v_n \in V$, and each $A^n(V)$ is a KG-module. It is well-known that $A(V)$ has the following direct decomposition as K-space: $A(V) = \bigoplus_{n\geq 0} A^n(V)$ with $A^0(V) = K$. For $n \geq 1$, the homogeneous components of degree n in $L(V)$ and $R(V)$ are given by $L^n(V) = L(V) \cap A^n(V)$ and $R^n(V) = R(V) \cap A^n(V)$. The free metabelian Lie algebra on V is defined by $M(V) = L(V)/L(V)''$, where $L(V)''$ is the second derived algebra of $L(V)$. Furthermore, we write $S(V)$ for the symmetric algebra on V. These algebras will be regarded as KG-modules in the obvious way and their homogeneous components of degree n will be denoted by $M^n(V)$ (for $n \geq 1$) and $S^n(V)$ (for $n \geq 0$, and $S^0(V) = K$), respectively. Each $M^n(V)$ and $S^n(V)$ is a KG-module. For any Lie algebra L over K (or, briefly, Lie algebra), we write $[u, v]$ for the Lie product with $u, v \in L$, and expressions of the form $[u_1, u_2, \ldots, u_n]$ are taken as left-normed so that $[u_1, u_2, \ldots, u_n] = [[u_1, \ldots, u_{n-1}], u_n]$ for $n \geq 3$.

For a finite-dimensional KG-module V, let $\mathscr{I}(V)$ denote the set of isomorphism classes of indecomposable direct summands of V. A *graded KG-module* is a KG-module V with a distinguished decomposition $V = V_1 \oplus V_2 \oplus \cdots$ where each V_n is a finite-dimensional KG-module. For a graded KG-module V, we define $\mathscr{I}(V) = \bigcup_{n\geq 1} \mathscr{I}(V_n)$. All the aforementioned algebras on a finite-dimensional KG-module are graded KG-modules. By a result of Higman (see [10, Chapter VII, Theorem 5.4]), a necessary and sufficient condition for the existence of infinitely many isomorphism classes of indecomposable KG-modules is that *the field K has positive characteristic p and the group G has a non-cyclic Sylow p-subgroup.*

Our main purpose in this paper is to study whether $\mathscr{I}(L(V))$ or $\mathscr{I}(M(V))$ is finite for certain K, G and V. Let K be a finite field, $G = \mathrm{SL}(2, K)$ and V the natural 2-dimensional KG-module. It follows from a result of Alperin and Kovács [1] that $\mathscr{I}(S(V))$ is finite. In Section 2, we prove the analogous result for the free metabelian Lie algebra $M(V)$ (see Proposition 2.1). Karagueuzian and Symonds (see [11]) generalized the aforementioned result for symmetric algebras as follows: *for any finite group G (not necessarily a p-group) and any KG-module with* $\dim_K V \leq 3$, *$\mathscr{I}(S(V))$ is finite.* In the case of the free metabelian Lie algebra $M(V)$, with $\dim_K V = 3$ and K a field of characteristic 2, the situation is different. In particular, let $U_3(K)$ be the group of 3×3 upper unitriangular matrices over K and let V be the natural

3-dimensional $KU_3(K)$-module. Then $\mathscr{I}(M(V))$ is not finite (see Corollary 2.6).

In Section 3, we consider free Lie algebras. For any finite-dimensional V, the Lie subalgebra of $R(V)$ generated by $R^2(V) \oplus R^3(V)$ is freely generated by this submodule, so we may denote it by $L(R^2(V) \oplus R^3(V))$. Write

$$L^n_{\text{grad}}(R^2(V) \oplus R^3(V)) = R^n(V) \cap L(R^2(V) \oplus R^3(V))$$

for its nth homogeneous component in the grading it inherits from $R(V)$. We prove that if K is any field of characteristic 2, G is any group, V is any 2-dimensional KG-module, and $n \geq 3$, then $L^n(V) = L^n_{\text{grad}}(R^2(V) \oplus R^3(V))$. This fact turns out to be very useful in the case when K has more than 2 elements, G is the Klein four-group, and C is any faithful 2-dimensional KG-module, for then it enables us to show that $\mathscr{I}(L(C))$ is finite if and only if $\mathscr{I}(L(D))$, $\mathscr{I}(L(B))$ and $\mathscr{I}(L(C \otimes C))$ are finite, where D is the regular KG-module and B is the unique 3-dimensional quotient of D. We note that Michos has given an example of a 6-dimensional decomposable KG-module V, where K has characteristic 2 and G is the Klein four-group, such that $\mathscr{I}(L(V))$ is not finite (see Example 1).

2. Free metabelian Lie algebras

Let G be any group, K an arbitrary field and V a KG-module. We identify $M^1(V)$ with V, so that V is regarded as a subspace of $M(V)$. We note the standard fact that if $a_1, \ldots, a_n \in M(V)$ and $n \geq 3$, then the products $[a_1, \ldots, a_n]$ are symmetric with respect to the entries $a_3, \ldots, a_n$. If $\mathscr{V}$ is an ordered basis of V then the products $[v_1, \ldots, v_n]$, where $n \geq 1$, $v_1, \ldots, v_n \in \mathscr{V}$ and $v_1 > v_2 \leq v_3 \leq \cdots \leq v_n$, form a basis of $M(V)$ (see [3, Section 4.7]) and, for each n, those of degree n form a basis of $M^n(V)$ which is called *the standard basis* of $M^n(V)$.

PROPOSITION 2.1. *For a finite field K and a finite group G, let V be a 2-dimensional KG-module. Then $\mathscr{I}(M(V))$ is finite.*

PROOF. In general, $M^2(V)$ is the exterior square of V. Since the left-normed metabelian Lie product $[v_1, \ldots, v_n]$ as n-variable function $V \times \cdots \times V \to M^n(V)$ is multilinear, alternating in the first two variables and symmetric in the others, the defining universal properties of exterior powers, symmetric powers and tensor products (see, for example, [8, Appendix B]) guarantee that there is a KG-homomorphism $M^2(V) \otimes S^{n-2}(V) \to M^n(V)$ whose image contains all the $[v_1, \ldots, v_n]$ and is therefore $M^n(V)$ itself. This much holds even if $\dim V > 2$. Given that $\dim V = 2$, dimension comparison shows that the surjective homomorphism in question is in fact an isomorphism. Further, since $M^2(V)$ is 1-dimensional, $M^2(V)^* \otimes M^2(V)$ (where

$M^2(V)^*$ denotes the contragredient of $M^2(V)$) is the 1-dimensional trivial module, and therefore one has both

$$M^2(V) \otimes S^{n-2}(V) \cong M^n(V) \quad \text{and} \quad M^2(V)^* \otimes M^n(V) \cong S^{n-2}(V).$$

It follows that $X \mapsto M^2(V) \otimes X$ and $Y \mapsto M^2(V)^* \otimes Y$ provide a bijective correspondence between the set of the submodules X of $S^{n-2}(V)$ and the set of the submodules Y of $M^n(V)$, such that corresponding submodules are isomorphic. Thus $\mathscr{I}(M(V))$ is finite if and only if $\mathscr{I}(S(V))$ is finite.

Suppose now that K is finite. Let $\Gamma = \mathrm{GL}(2, K)$, $\Sigma = \mathrm{SL}(2, K)$, and let U be the natural module for $K\Gamma$. It follows from Alperin and Kovács [1] that in this case $\mathscr{I}(S(\mathrm{Res}_\Sigma U))$ is finite. Since the index $|\Gamma : \Sigma|$ is prime to the characteristic of K, any $K\Gamma$-module W is a direct summand of the induced module $\mathrm{Ind}^\Gamma \mathrm{Res}_\Sigma W$ (see, for example, [2, Theorem 9.2]). In view of this, and of $S(\mathrm{Res}_\Sigma U) = \mathrm{Res}_\Sigma S(U)$, the finiteness of $\mathscr{I}(S(U))$ also follows.

It can be assumed without loss of generality that G acts faithfully on V; equivalently, that G is a subgroup of Γ and $V = \mathrm{Res}_G U$. Since $S(V) = S(\mathrm{Res}_G U) = \mathrm{Res}_G S(U)$, we may now conclude that $\mathscr{I}(S(V))$ is finite, and hence so is $\mathscr{I}(M(V))$. □

The homomorphism from G to the trivial group $\{1\}$ extends to an algebra homomorphism ε from KG to the group algebra of $\{1\}$, which we may identify with the group algebra K. The kernel of ε is the *augmentation ideal* Δ of KG and so consists of all elements $\sum \alpha_g g$ of KG with $\sum \alpha_g = 0$. In particular, Δ is a submodule of KG. Moreover, the quotient KG/Δ is isomorphic with the trivial KG-module K. We write $V^G = \{v \in V : vg = v \text{ for all } g \in G\}$.

LEMMA 2.2. *For all $n \geq 3$ there is a pair of KG-homomorphisms*

$$\phi_n : M^n(V) \to M^3(V) \otimes S^{n-3}(V), \quad \psi_n : M^3(V) \otimes S^{n-3}(V) \to M^n(V)$$

such that the composite $\psi_n \phi_n$ is multiplication by $n(n-2)$ on $M^n(V)$.

PROOF. By [9, Theorem 3.3], there is a pair of KG-homomorphisms

$$\chi_{n,3} : M^n(V) \to A^3(V) \otimes S^{n-3}(V), \quad \lambda_{n,3} : A^3(V) \otimes S^{n-3}(V) \to M^n(V)$$

such that the composite $\lambda_{n,3}\chi_{n,3}$ is multiplication by $n(n-2)$ on $M^n(V)$. The definition of the second homomorphism in [9] factors through $M^3(V) \otimes S^{n-3}(V)$, so the lemma follows. □

COROLLARY 2.3. *If K is of characteristic 2 and n is odd, then $M^n(V)$ is a direct summand of $M^3(V) \otimes S^{n-3}(V)$.*

LEMMA 2.4. *Let* $\{x_1, \dots, x_m, f\} = X \cup \{f\}$ *be a basis of* V, *ordered by* $x_1 < \cdots < x_m < f$, *and assume that* $f \in V^G$. *Then, for all* $n \geq 3$, *the map* $u \mapsto [u, f]$, *where* $u \in M^{n-1}(V)$, *is an injective* KG*-homomorphism* $\mu_n : M^{n-1}(V) \to M^n(V)$. *Moreover, the elements*

$$(2.1) \qquad [u_1, u_2, \dots, u_n] + \operatorname{Im}\mu_n \quad (u_i \in X,\ u_1 > u_2 \leq \cdots \leq u_n)$$

and

$$(2.2) \qquad [f, v_1, v_2, \dots, v_{n-1}] + \operatorname{Im}\mu_n \quad (v_i \in X,\ v_1 \leq v_2 \leq \cdots \leq v_{n-1})$$

form a basis of the quotient $M^n(V)/\operatorname{Im}\mu_n$.

PROOF. Since f is fixed by G, the map μ_n agrees with the G-actions, and hence it is a KG-homomorphism. Moreover, μ_n maps the standard basis of $M^{n-1}(V)$ onto part of the standard basis of $M^n(V)$, and the remaining elements of the standard basis of $M^n(V)$ are precisely the ones listed in (2.1) and (2.2). □

Now let K be of characteristic 2, $G = \langle \alpha, \beta \mid \alpha^2 = \beta^2 = (\alpha\beta)^2 = 1 \rangle$, $V = \Delta$, and let B_{n-1} denote the $(2n-1)$-dimensional indecomposable KG-module with basis $y_1, \dots, y_{n-1}, z_0, z_1, \dots, z_{n-1}$ and G-action given by $z_i\alpha = z_i\beta = z_i$, $0 \leq i \leq n-1$, $y_i\alpha = y_i + z_{i-1}$ and $y_i\beta = y_i + z_i$, $1 \leq i \leq n-1$. Recall that Δ is the unique 3-dimensional indecomposable submodule of the regular KG-module.

THEOREM 2.5. *For all odd* n, *with* $n \geq 3$, $M^n(\Delta)$ *is a free* KG*-module. For all even* n, $M^n(\Delta)$ *is isomorphic to a direct sum of (one copy of)* B_{n-1} *and a free* KG*-module.*

PROOF. The elements $u = 1 + \alpha$, $v = 1 + \beta$ and $f = 1 + \alpha + \beta + \alpha\beta$ form a basis of the module Δ, and the action of G on these basis elements is given by $u\alpha = u$, $u\beta = u + f$, $v\alpha = v + f$, $v\beta = v$ and $f\alpha = f\beta = f$. By Lemma 2.4, applied to $M^n(\Delta)$ and the basis $\{v, u, f\}$ of Δ with $v < u < f$, the quotient $M^n(\Delta)/\operatorname{Im}\mu_n$ has a basis consisting of the elements

$$y_i = [u, \underbrace{v, \dots, v}_{i}, \underbrace{u, \dots, u}_{n-i-1}] + \operatorname{Im}\mu_n \quad (i = 1, \dots, n-1)$$

and

$$z_i = [f, \underbrace{v, \dots, v}_{i}, \underbrace{u, \dots, u}_{n-i-1}] + \operatorname{Im}\mu_n \quad (i = 0, \dots, n-1).$$

An easy calculation shows that in $M^n(\Delta)/\operatorname{Im}\mu_n$ the elements $z_0, \dots, z_{n-1}$ are fixed by G, while $y_i\alpha = y_i + z_{i-1}$ and $y_i\beta = y_i + z_i$ $(1 \leq i \leq n-1)$. Consequently,

$$(2.3) \qquad M^n(\Delta)/\operatorname{Im}\mu_n \cong B_{n-1}$$

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for all $n \geq 3$. One easily calculates that $M^2(\Delta) \cong B_1$ and $M^3(\Delta)$ is free of rank 2. Since $M^3(\Delta)$ is free, so is the tensor product $M^3(\Delta) \otimes S^{n-3}(\Delta)$. By Corollary 2.3, for odd n, $M^n(\Delta)$ is a direct summand of $M^3(\Delta) \otimes S^{n-3}(\Delta)$, and hence it is also free. This proves the first part of the theorem. If $n \geq 4$ is even, then $\operatorname{Im} \mu_n \cong M^{n-1}(\Delta)$ is free, and since any free module is injective, (2.3) gives the second part of the theorem. □

COROLLARY 2.6. *Let K be a finite field of characteristic 2, $U_3(K)$ the group of 3×3 upper unitriangular matrices over K and V the natural 3-dimensional $KU_3(K)$-module. Then $\mathscr{I}(M(V))$ is not finite.*

PROOF. Let $g = (g_{ij})$ and $h = (h_{ij})$ be the elements of $U_3(K)$ for which $g_{12} = g_{23} = 0$ and $g_{13} = 1$, while $h_{12} = h_{13} = 0$ and $h_{23} = 1$. It is easily seen that $G = \langle g, h \rangle$ is a Klein four-group such that $\operatorname{Res}_G V \cong \Delta$. Of course $\operatorname{Res}_G M(V) = M(\operatorname{Res}_G V)$, so if $\mathscr{I}(M(V))$ were finite, then so would be $\mathscr{I}(M(\operatorname{Res}_G V)) = \mathscr{I}(M(\Delta))$, contrary to Theorem 2.5. □

3. Free Lie algebras

Let G be any group and K any field. A *graded K-space* is a K-space with a distinguished decomposition $V = V_1 \oplus V_2 \oplus \cdots$ where each V_n is finite-dimensional. For each positive integer n, $L^n_{\text{grad}}(V)$ denotes the subspace of $L(V)$ spanned by all products $[v_1, v_2, \ldots, v_k]$, with $k \geq 1$, such that, for $i = 1, \ldots, k$, $v_i \in V_{n(i)}$ for some $n(i) \geq 1$ with $n(1) + \cdots + n(k) = n$. In this way $L(V)$ becomes a graded K-space:

$$L(V) = L^1_{\text{grad}}(V) \oplus L^2_{\text{grad}}(V) \oplus \cdots .$$

If V is a finite-dimensional K-space regarded as a graded K-space with decomposition $V = V \oplus 0 \oplus 0 \oplus \cdots$, then $L^n_{\text{grad}}(V) = L^n(V)$ for all n. A *graded KG-module* is a KG-module V with a distinguished decomposition $V = V_1 \oplus V_2 \oplus \cdots$ where each V_n is a finite-dimensional KG-module. The homogeneous components $L^n_{\text{grad}}(V)$, as defined previously, are easily seen to be KG-submodules of $L(V)$. Thus $L(V)$ becomes a graded KG-module in the natural grading. We write $L_{\text{grad}}(V)$ instead of $L(V)$ when we work with a graded KG-module V.

Let $\{I_\lambda : \lambda \in \Lambda\}$ be a set consisting of one representative I_λ from each isomorphism class of finite-dimensional indecomposable KG-modules, and consider a vector space Γ_{KG} over the complex field $\mathbb{C}$ with this set as basis. For each finite-dimensional KG-module V, write $[V]$ for the element $\sum \alpha_\lambda I_\lambda$ of Γ_{KG} where each coefficient α_λ is the number of summands isomorphic to I_λ in an unrefinable direct sum decomposition of V. Define a multiplication on Γ_{KG} as the linear extension of $I_\lambda I_\mu = [I_\lambda \otimes I_\mu]$. Then Γ_{KG} becomes a $\mathbb{C}$-algebra known as the Green algebra of G.

We write $\Gamma_{KG}[[t]]$ for the algebra of all formal power series in an indeterminate t with coefficients from Γ_{KG}, and $\Gamma_{KG}[[t]]^\circ$ for the ideal consisting of the power series with zero constant terms. For any graded KG-module V, we write

$$[[V]] = \sum_{n\geq 1}[V_n]t^n.$$

This gives a convenient way of describing the graded isomorphism type of V. The map $V \mapsto [[V]]$ intertwines direct sums and tensor products of graded modules with addition and multiplication in $\Gamma_{KG}[[t]]$, and $[[L_{\text{grad}}(V)]] = \sum_{n\geq 1}[L^n_{\text{grad}}(V)]t^n$. Bryant (see [4]) studied functions $\Gamma_{KG}[[t]]^\circ \to \Gamma_{KG}[[t]]^\circ$ that always take $[[V]]$ to $[[L_{\text{grad}}(V)]]$. To be able to quote one of his results, we need yet another definition.

For positive integers r, s, we define $w(r, s)$ by

$$w(r,s) = \frac{1}{r+s}\sum_{d|(r,s)}\mu(d)\binom{(r+s)/d}{r/d},$$

where μ is the Möbius function and the sum is over all positive integers d, which divide both r and s. Note that the $w(r, s)$ are positive integers, because by Witt's formulae ([4, equation (3.6)]) they are dimensions of homogeneous components in free Lie algebras.

For the proof of the following result, we refer to [4, Theorem 2.4 and Theorem 4.2].

LEMMA 3.1. *There is a function* $\mathscr{L}_{KG} : \Gamma_{KG}[[t]]^\circ \to \Gamma_{KG}[[t]]^\circ$ *such that*

(I) *for every graded* KG*-module* V,

$$\mathscr{L}_{KG}([[V]]) = [[L_{\text{grad}}(V)]];$$

(II) *whenever* $f_1, f_2 \in \Gamma_{KG}[[t]]^\circ$,

$$\mathscr{L}_{KG}(f_1+f_2) = \mathscr{L}_{KG}(f_1) + \mathscr{L}_{KG}(f_2) + \sum w(r,s)\mathscr{L}_{KG}(f_1^r f_2^s)$$

where the summation is taken over all positive integers r, s.

Let U and V be any finite-dimensional KG-modules. For positive integers r and s, we write U^rV^s for the tensor product $U \otimes \cdots \otimes U \otimes V \otimes \cdots \otimes V$ where U and V are repeated r and s times, respectively. We write

$$f_1 = [[U]] = [U]t \quad \text{and} \quad f_2 = [[V]] = [V]t.$$

Note that $[L^n(U)]$ and $[L^n(V)]$ are the coefficients of t^n in $\mathscr{L}_{KG}(f_1)$ and $\mathscr{L}_{KG}(f_2)$, respectively. For a positive integer k, let $\Theta^k : \Gamma_{KG}[[t]]^\circ \to \Gamma_{KG}[[t]]^\circ$ be the substitution of t^k for t. It has been noted in [4, page 181] that $\Theta^k \circ \mathscr{L}_{KG} = \mathscr{L}_{KG} \circ \Theta^k$ for

$k \geq 1$. It is now easy to see that

$$\mathscr{L}_{KG}(f_1^r f_2^s) = \Theta^{r+s}(\mathscr{L}_{KG}([U^r V^s]t)) = \sum_{n\geq 1}[L^n(U^r V^s)]t^{n(r+s)}. \tag{3.1}$$

By Lemma 3.1 and equation (3.1), we obtain that

$$L(U \oplus V) \cong L(U) \oplus L(V) \oplus \bigoplus_T L(T), \tag{3.2}$$

where T ranges through all tensor products $U^r V^s$ with r, s positive integers, taking each such value at least once, but many of them more than once. We write $\mathscr{T}$ for the set of the aforementioned tensor products. The following result has been proved in [6, Theorem 4.1].

LEMMA 3.2. *Let G be any group, K any field and V any finite-dimensional KG-module. Then for any positive integer n, $L^n(V)$ is isomorphic to a direct sum of modules, each of which has the form $L^k(W)$ for some divisor k of n and some indecomposable direct summand W of $A^{n/k}(V)$.*

If T is an element of $\mathscr{T}$, then so is each tensor power $A^{n/k}(T)$ of T. Thus if W is an indecomposable direct summand of some $A^{n/k}(T)$, then W is a direct summand of some element of $\mathscr{T}$. By Lemma 3.2, we conclude that

$$\bigcup_{T\in\mathscr{T}} \mathscr{I}(L(T)) \subseteq \bigcup_{W\in\mathscr{W}} \mathscr{I}(L(W)),$$

where $\mathscr{W} = \bigcup_{T\in\mathscr{T}} \mathscr{I}(T)$. Conversely, the Elimination Theorem (see [5, Lemma 2.2]) shows that if $W \in \mathscr{I}(T)$ then $L(W)$ is a direct summand of $L(T)$, and so $\mathscr{I}(L(W)) \subseteq \mathscr{I}(L(T))$. Therefore, by (3.2), we obtain that

$$\mathscr{I}(L(U \oplus V)) = \mathscr{I}(L(U)) \cup \mathscr{I}(L(V)) \cup \bigcup_{W\in\mathscr{W}} \mathscr{I}(L(W)). \tag{3.3}$$

Suppose now that U and V are graded modules that are concentrated in degrees k and ℓ, respectively (so $U = U_k$ while $U_i = 0$ if $i \neq k$, and $V = V_\ell$ while $V_j = 0$ if $j \neq \ell$). Repeating the previous argument with $f_1 = [[U]] = [U]t^k$ and $f_2 = [[V]] = [V]t^\ell$ yields first a version of (3.1) in which both occurrences of $r + s$ are replaced by $rk+s\ell$, but then proceeds as before; the only change being required in (3.2) and (3.3) is to replace each L by L_{grad}. On the right hand side of the version of (3.3) so obtained, we prefer to retain 'the other' gradings. To this end, we note that $L^r(U) = L_{\text{grad}}^{rk}(U)$, and so on: if W is a direct summand of $U^r V^s$, then $L^m(W) = L_{\text{grad}}^{m(rk+s\ell)}(W)$. In these terms, the conclusions may be put as follows.

PROPOSITION 3.3. *Let K be any field, G any group, and U, V graded KG-modules concentrated in degrees k, ℓ, respectively. For each positive integer q,*

$$\mathscr{I}(L^q_{\text{grad}}(U \oplus V)) = \mathscr{I}(L^{q/k}(U)) \cup \mathscr{I}(L^{q/\ell}(V)) \cup \bigcup_{d|q} \bigcup_{W} \mathscr{I}(L^{q/d}(W)),$$

where Lie powers to non-integers are read as 0, the range of d is the set of positive divisors of q, and W runs over the union of the $\mathscr{I}(U^r V^s)$ with $r, s \geq 1$ and $rk+s\ell = d$.

For the proof of the following result, we refer to [12, Section 2].

LEMMA 3.4. *Let K be a field of characteristic 2 and V any 2-dimensional K-space. Let $\{x, y\}$ be a K-basis of V and let $\mathscr{E}$ be the subset of $R(V)$ defined by $\mathscr{E} = \{x^2, [x, y], y^2, [x, y, x], [x, y, y]\}$. Then the restricted Lie subalgebra E^* of $R(V)$ generated by $\mathscr{E}$ is free on $\mathscr{E}$.*

Let E be the Lie subalgebra of E^* generated by $\mathscr{E}$. It is a direct consequence of Lemma 3.4 that E is freely generated by $\mathscr{E}$. Since $\mathscr{E}$ is a K-basis of $R^2(V) \oplus R^3(V)$, we may write $E = L(R^2(V) \oplus R^3(V))$.

THEOREM 3.5. *Let K be a field of characteristic 2, G any group and V any 2-dimensional KG-module. Then $L^n(V) = L^n_{\text{grad}}(R^2(V) \oplus R^3(V))$ for $n \geq 3$.*

PROOF. Since $[a, b^2] = [a, b, b]$ for all $a, b \in A(V)$, we obtain that $[a, b^2] \in L(V)$ for all $a, b \in R(V)$ and so it is easily verified that

$$L^n_{\text{grad}}(R^2(V) \oplus R^3(V))) \subseteq L^n(V) \quad \text{for all } n \geq 3. \tag{3.4}$$

Next, we shall use induction on n to show that

$$L^n(V) \subseteq L^n_{\text{grad}}(R^2(V) \oplus R^3(V)) \quad \text{for all } n \geq 2. \tag{3.5}$$

For $n = 2$ and $n = 3$, our claim is trivially true, so we may assume that $n \geq 4$. Let $\{x, y\}$ be a K-basis of V. To prove our claim, it is enough to show that $[x_1, \dots, x_n] \in L^n_{\text{grad}}(R^2(V) \oplus R^3(V))$ whenever $x_1 = x$, $x_2 = y$ and $x_3, \dots, x_n \in \{x, y\}$. Our inductive hypothesis implies that $L^n(V) \cap L(V)'' \subseteq L^n_{\text{grad}}(R^2(V) \oplus R^3(V))$ and so, working modulo $L^n(V) \cap L(V)''$, we obtain that

$$[x_1, \dots, x_n] = [x, y, \underbrace{y, \dots, y}_{r}, \underbrace{x, \dots, x}_{s}] + w$$

for some $w \in L^n(V) \cap L(V)''$ and some r, s with $r + s = n - 2$. Since K has characteristic 2, we have

$$[a, \underbrace{b, \dots, b}_{2m}] = [a, \underbrace{b^2, \dots, b^2}_{m}]$$

for all non-negative integers m and all $a, b \in A(V)$. Using this repeatedly, we see that

$$[x, y, \underbrace{y, \dots, y}_{2k}, \underbrace{x, \dots, x}_{2\ell}] = [[x, y], \underbrace{y^2, \dots, y^2}_{k}, \underbrace{x^2, \dots, x^2}_{\ell}] + w_1,$$
$$[x, y, \underbrace{y, \dots, y}_{2k}, \underbrace{x, \dots, x}_{2\ell+1}] = [[x^2, y], \underbrace{y^2, \dots, y^2}_{k}, \underbrace{x^2, \dots, x^2}_{\ell}] + w_2,$$
$$[x, y, \underbrace{y, \dots, y}_{2k+1}, \underbrace{x, \dots, x}_{2\ell}] = [[x, y^2], \underbrace{y^2, \dots, y^2}_{k}, \underbrace{x^2, \dots, x^2}_{\ell}] + w_3,$$
$$[x, y, \underbrace{y, \dots, y}_{2k+1}, \underbrace{x, \dots, x}_{2\ell+1}] = [[x^2, y^2], \underbrace{y^2, \dots, y^2}_{k}, \underbrace{x^2, \dots, x^2}_{\ell}] + w_4,$$

where $w_1, \dots, w_4$ lie in $L^n(V) \cap L(V)''$ and hence are contained in $L(R^2(V) \oplus R^3(V))$. This proves statement (3.5) and so completes the proof of the theorem. □

Let G be the Klein four-group and C a faithful 2-dimensional KG-module, with K a field of characteristic 2. For such a module to exist, K must have more than 2 elements: one can see from Conlon [7] that there exists precisely one isomorphism type for each element of K different from 0 and 1. Let D be the regular KG-module, B the unique 3-dimensional quotient of D, and A the 1-dimensional trivial KG-module. It is easy to see that $L^2(C) \cong A$, $R^2(C) \cong B$ and $L^3(C) = R^3(C) \cong C$. The multiplication rules given by Conlon [7, page 89] yield that, for $m, n \geq 1$, $\mathscr{I}(B^m C^n)$ is either $\{C, D\}$ or $\{C \otimes C, D\}$, depending on whether n is odd or even. From Theorem 3.5 and Proposition 3.3 we now get that, for $q \geq 4$,

$$\mathscr{I}(L^q(C)) = \mathscr{I}(L^{q/2}(B)) \cup \mathscr{I}(L^{q/3}(C)) \cup \bigcup_{d|q} \bigcup_{W} \mathscr{I}(L^{q|d}(W))$$

where the range of W is the union of the $\mathscr{I}(B^m C^n)$ with $m, n \geq 1$ and $2m + 3n = d$. This range is readily seen to be empty when $d \leq 4$ or $d = 6$, it is $\{C, D\}$ when d is an odd number greater than 3, and it is $\{C \otimes C, D\}$ when d is an even number greater than 6. Equivalently, if $q \geq 4$, then

(3.6) $$\mathscr{I}(L^q(C)) = \mathscr{I}(L^{q/2}(B)) \cup \bigcup_{r} \mathscr{I}(L^{q/r}(C)) \cup \bigcup_{s} \mathscr{I}(L^{q/s}(C \otimes C)) \cup \bigcup_{t} \mathscr{I}(L^{q/t}(D)),$$

where r is odd and $r \geq 3$, s is even and $s \geq 8$, and either $t = 5$ or $t \geq 7$, and r, s, t range through the divisors of q subject only to these conditions. It follows immediately that $\mathscr{I}(L(C)) \supseteq \mathscr{I}(L(B)') \cup \mathscr{I}(L(C \otimes C)) \cup \mathscr{I}(L(D))$, where $L(B)'$ is the derived algebra of $L(B)$. Let X be any indecomposable in $\mathscr{I}(L(C))$, and choose q minimal with respect to $X \in \mathscr{I}(L^q(C))$. If $q \geq 4$, we may apply (3.6); by the minimality of q,

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no $\mathscr{I}(L^{q/r}(C))$ can contain X, so X must lie in $\mathscr{I}(L(B)') \cup \mathscr{I}(L(C \otimes C)) \cup \mathscr{I}(L(D))$. In view of $L^3(C) \cong L^1(C) = C$ and $L^2(C) \cong A$, we have proved the following relation $\mathscr{I}(L(C)) = \{A, C\} \cup \mathscr{I}(L(B)') \cup \mathscr{I}(L(C \otimes C)) \cup \mathscr{I}(L(D))$.

THEOREM 3.6. *Let G be the Klein four-group, K a field of characteristic 2 containing more than two elements and C any faithful 2-dimensional KG-module. Let D be the regular KG-module, B the unique 3-dimensional quotient of D and A the 1-dimensional trivial KG-module. Then, $\mathscr{I}(L(C))$ is finite if and only if $\mathscr{I}(L(B))$, $\mathscr{I}(L(D))$ and $\mathscr{I}(L(C \otimes C))$ are finite.*

EXAMPLE 1 (Michos). Let K be a field of characteristic 2 and G a Klein four-group. Conlon [7] described an infinite sequence $A_1, A_2, \ldots$ of indecomposable KG-modules such that A_1 is the augmentation ideal of KG, $\dim A_n = 2n + 1$, and A_{r+s} is a direct summand of $A_r \otimes A_s$. An easy induction on n shows that therefore A_n is a direct summand of the n-fold tensor power of A_1. By (3.2) with $U = V = A_1$, it follows that each A_n is a direct summand of $L(A_1 \oplus A_1)$, and so $\mathscr{I}(L(A_1 \oplus A_1))$ is infinite.

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