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A NOTE ON SUBDIRECTLY IRREDUCIBLE RINGS

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Let R be a commutative subdirectly irreducible ring, with minimal ideal M. It is shown that either R is a field, or $M^2 = 0$. A construction is given which yields commutative subdirectly irreducible rings possessing nonzero-divisors, and nonzero nilpotent elements either with a unity element, or without. Such a ring without unity has been constructed by Divinsky. The same technique enables the construction of subdirectly irreducible rings with mixed additive groups.

§1.

In [2] a complete description was given of the torsion, and torsion free abelian groups which can occur as the additive groups of subdirectly irreducible rings. The mixed case remained unsolved. In this note, some facts concerning subdirectly irreducible rings will be given, which shed some more light on the additive group structure of these rings, especially for commutative rings. The classification of the mixed additive groups of subdirectly irreducible rings remains an open problem, even for commutative rings. Examples will be given showing that many mixed groups are the additive groups of subdirectly irreducible rings.

§2.

R will denote a subdirectly irreducible ring, R^+ the additive group of R, and M the unique minimal ideal in R. For S a subset of R, (S) = the subgroup of R^+ generated by S, and $\langle S \rangle$ = the ideal in

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R generated by S. The additive group of the field of rational numbers will be denoted by Q, and a cyclic group of order n, by $Z(n)$.

LEMMA 1. *If* R^+ *is torsion free then* $M^+ = \oplus Q$, *if* R^+ *is not torsion free then* $M^+ = \oplus Z(p)$ *with* p *a fixed prime.*

Proof. Suppose that R^+ is torsion free. Let $a \in M$, $a \neq 0$. Then $M = \langle na \rangle$ for every positive integer n. It is readily seen that this implies that M^+ is divisible, so by [4, Theorem 23.1], $M^+ = \oplus Q$. Suppose that R^+ is not torsion free. Then there exist a prime p, and $a \in R$, $a \neq 0$, such that $pa = 0$. Since $M \subseteq \langle a \rangle$, $pM \subseteq p\langle a \rangle = 0$. It now follows from [4, Theorem 8.5] that $M^+ = \oplus Z(p)$.

LEMMA 2. *If* R^+ *is not torsion free then either* $R^+ = \oplus Z(p)$ *or* $M^2 = 0$.

Proof. By Lemma 1, $M^+ = \oplus Z(p)$. Suppose that $R^+ \neq \oplus Z(p)$. Then there exists $a \in R$ such that $pa \neq 0$. Let $x, y \in M$. Since $M \subseteq \langle pa \rangle$, $x = pr$ for some $r \in R$. Hence $xy = (pr)y = r(py) = 0$, and so $M^2 = 0$.

COROLLARY 3. *If* R^+ *is mixed, then* $M^2 = 0$.

THEOREM 4. *Let* R *be commutative. Then either* R *is a field, or* $M^2 = 0$.

Proof. McCoy [6] has shown that a commutative subdirectly irreducible ring must be of one of the following three types: (α) R is a field, (β) every element in R is a zero divisor, (γ) there exist both nondivisors of zero, and nonzero nilpotent elements in R. Suppose that R is of type (β). Let $x, y \in R$, $x \neq 0$, $y \neq 0$, $xy = 0$. Since $M \subseteq \langle x \rangle$, and $M \subseteq \langle y \rangle$, $M^2 \subseteq \langle x \rangle \langle y \rangle = 0$. Suppose that R is of type (γ). Let $a \in R$, $a \neq 0$, such that $a^k = 0$ for some positive integer k. Then $M^k \subseteq \langle a \rangle^k = 0$. Suppose that $M^2 \neq 0$. Then $M^2 = M$, which yields that $M^k = M \neq 0$, a contradiction.

COROLLARY 5. *Let* R *be commutative. If* $M^2 \neq 0$, *then either* $R^+ = \oplus Q$ *or* $R^+ = \oplus Z(p)$, p *a fixed prime.*

Proof. Theorem 4, and [3, Theorems 4.1.1 and 4.1.3].

COROLLARY 6. *If R is commutative and semiprime, then R is a field.*

EXAMPLE 7. Let H be an infinite rank torsion free group, and let p be a prime. Then $G = Z(p^\infty) \oplus H$ is the additive group of a subdirectly irreducible ring.

Proof. Let $Z(p^\infty) = \bigcup_{i=1}^{\infty} (a_i)$ with $(a_1) \subsetneqq (a_2) \subsetneqq \cdots \subsetneqq (a_i) \subseteq \cdots$ an ascending chain of cyclic groups, $|a_i| = p^i$, and let $b_i, i \in I$, be a maximal independent set in H. Put $F = \bigoplus_{i\in I} (b_i)$. It may be assumed that I is the disjoint union $I = N \cup J$ with N the set of natural numbers. Let $\varphi : F \otimes F \to Z(p^\infty)$ be the homomorphism induced by the maps

$$\varphi(b_i \otimes b_j) = \begin{cases} 0 & \text{for } i,j \in J \\ a_{i+j} & \text{for } i,j \in N \\ a_i & \text{for } i \in N,\ j \in J \\ a_j & \text{for } i \in J,\ j \in N \end{cases}$$

Since $F \otimes F$ is isomorphic to a subgroup of $H \otimes H$, and since $Z(p^\infty)$ is injective in the category of abelian groups, φ extends to a homomorphism $\varphi : H \otimes H \to Z(p^\infty)$. Let $x_i \in Z(p^\infty)$, $h_i \in H$, $i = 1,2$, and define $(x_1 + h_2) \cdot (x_2 + h_2) = \varphi(h_1 \otimes h_2)$. This multiplication indices a ring structure R on G. Let $g = x + h$, $x \in Z(p^\infty)$, $h \in H$, and $g \neq 0$. If $h = 0$, then $x \neq 0$, and it is readily seen that $\langle g\rangle \supseteq (a_1)$. Suppose that $h \neq 0$. Since $\{b_i | i \in I\}$ is a maximal independent set in H, there exist a positive integer n, nonzero integers $n_1,\ldots,n_k$, and $i_1,\ldots,i_k \in I$ such that $nh = n_1 b_{i_1} + \ldots + n_k b_{i_k}$. Let ℓ be a positive integer such that $p^\ell > \sum_{i=1}^{k} |n_i|$. Then $nh \cdot b_\ell$ is a nonzero element in $Z(p^\infty)$. Hence $\langle g\rangle \supseteq (nh \cdot b_\ell) \supseteq (a_1)$. Therefore every nonzero ideal in R contains (a_1), and so R is subdirectly irreducible.

A major part of Divinsky's paper [1] is the construction of a commutative subdirectly irreducible ring of type (γ) (see proof of Theorem 4) without unity. The additive group of the ring in Divinsky's example is torsion free. The following construction, similar to that in the previous example, yields infinitely many pairwise non-isomorphic commutative subdirectly irreducible rings of type (γ) both with and without unity, and with mixed additive group.

EXAMPLE 8. Let F be an infinite rank free group, $F = \bigoplus_{i \in I} (e_i)$, with (e_i) an infinite cyclic group for all $i \in I$, and let $Z(p^\infty) = \bigcup_{i=1}^{\infty} (a_i)$ as in the above example. It may be assumed that I is the disjoint union $I = N_0 \cup J$ with N_0 the set of non-negative integers. Put $G = Z(p^\infty) \oplus F$. Let q be either a prime, $q \neq p$, or let $q = 1$. Define $a_i a_j = 0$ for all positive integers i, j.

$$e_i a_j = a_j e_i = \begin{cases} q a_j & \text{for } i = 0, \; j \text{ a positive integer} \\ 0 & \text{for } j \text{ a positive integer, } i \in I - \{0\} \end{cases}$$

$$e_i e_j = e_j e_i = \begin{cases} q e_j & \text{for } i = 0, \; j \in I \\ 0 & \text{for } i, j \in J \\ a_{i+j} & \text{for } i, j \text{ positive integers} \\ a_i & \text{for } i \text{ a positive integer, } j \in J \\ a_j & \text{for } j \text{ a positive integer, } i \in I \end{cases}$$

These products induce a ring structure R on G. R is commutative, subdirectly irreducible of type (γ). If $q = 1$, then e_0 is a unity in R. If $q \neq 1$, then R is without unity.

H. Heatherly has indicated that a nilpotent subdirectly irreducible ring with mixed additive group has been constructed in [5]. The author is indebted to him for this and other important communications concerning subdirectly irreducible rings.

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