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# CORRIGENDUM

# On the regularity of the Green–Naghdi equations – CORRIGENDUM

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This note corrects several factual errors in the study of Dritschel & Jalali (2019). That study compared the hydrostatic rotating shallow-water (SW) model with its non-hydrostatic counterpart, the Green–Naghdi (GN) model (Green & Naghdi 1976). Based on numerical simulations, evidence for a lack of regularity was found in the GN model, which appeared to exhibit an upturn in both height and velocity divergence spectra at high wavenumbers (small scales).

Since publishing this work, we have derived a new reformulation of the GN equations which, in numerical simulations, does not exhibit any evidence of a lack of regularity. This new reformulation (the subject of a forthcoming paper) explicitly uses the non-hydrostatic pressure, which is found from a robustly convergent linear elliptic equation. We call this reformulation the 'vertically averaged' (VA) model, to distinguish it from the original, implicit (and commonly used) form of the GN model.

The implicit GN model requires nonlinear iteration to solve for one of the field tendencies; in our case we iterated over the acceleration divergence, $\gamma = \nabla \cdot (\mathrm{D}\boldsymbol{u}/\mathrm{D}t)$. Moreover, we must iterate to find the height field $h$ from the definition of the potential vorticity (see appendix B of Dritschel & Jalali (2019), for full details). In the course of reformulating the GN model into an explicit system of equations (with no iteration required on any field tendency) we discovered two mistakes that made the lack of regularity in GN appear to be more serious than it actually is. In fact, there appears to be no lack of regularity, but as discussed below, the implicit GN model, with these mistakes corrected, still exhibits a lack of regularity at high wavenumbers; the VA model does not.

The two mistakes discovered were, in order of importance: (1) the simplified form of the semi-implicit time-stepping procedure employed; and (2) incomplete dealiasing of nonlinear products. The simplified semi-implicit time-stepping procedure came from appendix B in Mohebalhojeh & Dritschel (2004), and was used in (C14) of Dritschel & Jalali (2019). This, however, does not fully control the high-wavenumber portions of the fields. In the SW model, this does not matter, as spectra tend to be steeper. Also, the SW model needs virtually no dealiasing, whereas it was found to be essential in the GN model which is much more nonlinear.

With these mistakes corrected, the original implicit GN model still exhibits a lack of regularity at small scales, but comparison with the VA model (which is mathematically equivalent) show that this is numerical (the GN and VA models agree in detail at early times when the fields all have steep spectra). Figure 1 shows this apparent lack of regularity by comparing difference spectra at a late time for the dimensionless height anomaly

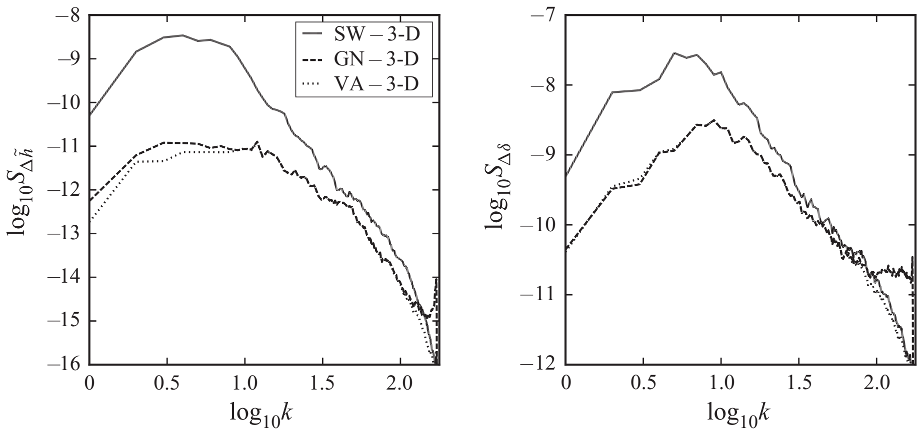


FIGURE 1. Spectra of dimensionless height anomaly and vertically averaged divergence differences, $\Delta\tilde{h}$ and $\Delta\delta$, respectively, at the final time of a series of model simulations. The differences of the SW, GN and VA model fields are computed relative to a full three-dimensional (3-D) simulation of the rotating shallow-water flow in a horizontally doubly periodic domain represented by $512^2$ uniformly spaced grid points and 64 vertical layers. Note the $\log_{10}$ scaling of each axis.

$\tilde{h} = (h - H)/H$ (where $H$ is the mean fluid depth) and the horizontal divergence $\delta = \boldsymbol{\nabla} \cdot \boldsymbol{u}$, where the differences are relative to a full 3-D simulation of a rotating shallow-water fluid. The results were obtained using the field differences between the two-dimensional and the 3-D models (for divergence, this was first vertically averaged in the 3-D model; details are provided in a forthcoming paper). The specific case shows the final time ($t = 25$) in a simulation identical to that carried out in Dritschel & Jalali (2019), except for a mean depth $H = 0.4$ (twice as large), and using a numerical algorithm entirely based on the pseudo-spectral method so that all models use the same numerical approach and parameter settings. Figure 1 shows that the SW model is generally the least accurate, except at high wavenumbers where there is still a significant upturn in the spectra in the GN model (as found in Dritschel & Jalali (2019), albeit it is more pronounced there). The new explicit form of GN, the VA model, exhibits steep difference spectra like those found in the SW model. This indicates that the GN model, mathematically, is well posed, but that, numerically, the implicit formulation is problematic.

We believe that the rise in field spectra in the GN model at high wavenumbers comes from the nonlinear iteration used by Dritschel & Jalali (2019) to solve for the tendency of the acceleration divergence, $\gamma_t$, in (2.11). This is a highly nonlinear implicit equation, significantly with equal numbers of spatial derivatives on both sides of the equation. This is solved by moving the constant-coefficient linear terms to the left-hand side and all remaining terms to the right-hand side (see (B4) in Dritschel & Jalali (2019)). One can then invert the linear operator appearing on the left-hand side to update the estimate of $\gamma_t$. While this procedure converges numerically, it is likely that it does so only because of the strong dealiasing used throughout.

## Declaration of interests

The authors report no conflict of interest.

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