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ON UNIONS OF METRIZABLE SUBSPACES

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1. Introduction and definitions. In this paper we study the question "which generalized metric spaces can be written as the union of κ (closed) metrizable subspaces, where κ is a cardinal number with $\kappa \leqq c$?" Questions of this type first arose in [**16**] where J. Nagata asked for examples of certain generalized metric spaces which could not be written as the union of countably many closed metrizable subspaces. Using Baire Category arguments, Fitzpatrick provided the required examples in [**12**]. We begin this paper by sharpening Fitzpatrick's examples, showing in Section 2 that there is a Moore space which is not the union of countably many metrizable subspaces of any kind. Then in Section 3 we present a positive result, proving that any σ-space, and a fortiori any Moore space, can be written as the union of $c = 2^{\omega_0}$ closed metrizable subspaces. (A related result is obtained for Bennett's quasi-developable spaces.) In Section 4 we show that Section 3 contains the best positive result. More precisely, we show that in any model of Martin's Axiom plus $\omega_1 < c$, neither the Moore space given by Fitzpatrick [**12**] nor the Lasnev space given by van Doren in [**7**] can be the union of κ metrizable closed subspaces for any $\kappa < c$. In Section 5 we present examples which show that members of two other generalized metric classes—the Σ-spaces of Nagami and the BCO spaces of Wicke and Worrell—cannot always be written as the union of a continuum of metrizable subspaces. In the final section, we show that some generalized metric structure is necessary for the positive results of Section 3 by describing a perfect space (= closed sets are G_δ's) which is first-countable and hereditarily paracompact, and yet which is not the union of a continuum of metrizable subspaces. In addition, our example is a generalized ordered space in the sense of [**15**].

Our terminology generally follows that of [**6**]. By a *Moore space* we mean a regular T_1-space X which admits a sequence $\langle \mathcal{G}(n) \rangle$ of open coverings such that if U is open in X and if $p \in U$, then for some n, $p \in \operatorname{St}(p, \mathcal{G}(n)) \subset U$, where

$$\operatorname{St}(p, \mathcal{G}(n)) = \bigcup \{G \in \mathcal{G}(n) : p \in G\}.$$

If the collections $\mathcal{G}(n)$ in that definition are not required to cover all of X, then the preceding sentence describes a *quasi-developable space* in the sense of

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Bennett [**4**]. By a *Lasnev space* we mean any image of a metric space under a closed, continuous mapping. Both Moore spaces and Lasnev spaces are examples of σ-spaces, where by a *σ-space* we mean a T_1-space X admitting a collection $\mathscr{N}$ of closed sets satisfying:

a) if U is open and $p \in U$, than $p \in N \subset U$ for some $N \in \mathscr{N}$ (such an $\mathscr{N}$ is called a *network* for X);
b) the collection $\mathscr{N}$ is σ-discrete in X.

Definitions of *semistratifiable spaces*, *Σ-spaces* and *semi-metric spaces*, with relevant references, appear in [**6**].

The results announced above suggest two questions which we cannot yet settle:

(1) Can the main positive result in Section 3 be proved for semistratifiable spaces or for semi-metric spaces? (The semistratifiable spaces lie somewhere between the perfect spaces and the σ-spaces, and semimetric spaces are just first-countable semistratifiable spaces.)

(2) Is it true that every regular space with the point-countable base can be written as the union of a continuum of (closed) metrizable subspaces? (See Corollary 3.3.)

We record, without proof, some easy results about the stability of the class of spaces which can be written as the union of c (closed) metrizable subspaces. These results shed some light on where not to look for counterexample solutions to the above problems.

a) If X can be written as the union of c closed metrizable subspaces and if $f : X \rightarrow Y$ is a closed continuous surjection, then Y can also be written as the union of c closed metrizable subspaces. (This follows from (3.1).)

b) If each space $X(n)$, for $n \in \omega_0$, can be written as the union of a continuum of (closed) metrizable subspaces, then so can the product space $Y = \prod \{X(n) : n \in \omega_0\}$.

c) If X can be written as the union of a continuum of (closed) metrizable subspaces, then so can each subspace of X.

Convention. In this paper, we assume that each space is at least a T_1-space.

2. An example concerning countable unions.

EXAMPLE 2.1. *There is a separable Moore space M which is the union of ω_1 closed, metrizable subspaces and yet which is not the union of any collection of countably many metrizable subspaces.*

Proof. Let X be any separable, non-metrizable Moore space having $\text{card}(X) = \omega_1$. (For example, let X be a dense subspace of the usual Niemytski

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plane having cardinality ω_1 and having ω_1 points on the x-axis.) Then X must contain an uncountable discrete subspace E. Our example will be the space $M = X^{\omega_0}$, the usual product of countably many copies of X, endowed with the usual product topology $\mathscr{P}$.

To see that X is the union of ω_1 closed, metrizable subspaces, enumerate X as $X = \{x(\alpha) : \alpha < \omega_1\}$. For each $\beta < \omega_1$, let

$$A(\beta) = \{x(\alpha) : \alpha < \beta\}^{\omega_0}.$$

Clearly each $A(\beta)$ is a closed metrizable subspace of M and $M = \bigcup \{A(\beta) : \beta < \omega_1\}$.

Next, suppose that $\mathscr{A}$ is a countable collection of metrizable subspaces of M and that $M = \bigcup \mathscr{A}$. We introduce two auxilliary spaces. Let D be the set X equipped with the discrete topology and let the product topology of D^{ω_0} be denoted by $\mathscr{Q}$. Then $\mathscr{P} \subset \mathscr{Q}$. Since $D^{\omega_0} = \bigcup \mathscr{A}$ and since $(D^{\omega_0}, \mathscr{Q})$ is a Baire space, some $A \in \mathscr{A}$ must fail to be nowhere dense in D^{ω_0}. Let B be a nonvoid $\mathscr{Q}$-basic open set in D^{ω_0} such that $A \cap B$ is $\mathscr{Q}$-dense in $(B, \mathscr{Q}_B)$. We observe that $(B, \mathscr{Q}_B)$ is homeomorphic to D^{ω_0} so that the proof will be complete once we prove

(*) If S is a dense subset of $(D^{\omega_0}, \mathscr{Q})$, then $(S, \mathscr{P}_S)$ is not metrizable.

To prove (*), note that if S is dense in D^{ω_0}, then S is also dense in $(M, \mathscr{P})$ so that, $(M, \mathscr{P})$ being separable and first countable, $(S, \mathscr{P}_S)$ is also separable. Next, letting

$$q : D^{\omega_0} \rightarrow D$$

be projection onto the first coordinate, we see that since S is dense in D^{ω_0}, $q[S]$ must be dense in D. But then $q[S] = D$. Let $p : (M, \mathscr{P}) \rightarrow X$ be first coordinate projection. As functions between sets, $q = p$ so that $p[S] = X$. Let F be any subset of S which contains exactly one point from each member of the collection $\{p^{-1}[\{x\}] : x \in E\}$, where E is the uncountable discrete subspace of X found above. Then $(F, \mathscr{P}_F)$ is an uncountable discrete subspace of $(M, \mathscr{P})$. But $(F, \mathscr{P}_F)$ is also a subspace of $(S, \mathscr{P}_S)$ so that the latter space, being separable but not hereditarily separable, cannot be metrizable.

2.2 *Remarks.* (a) If we assume Martin's Axiom plus $\omega_1 < c$, then the space M in the above example can be chosen in such a way that M^{ω_0} is normal.

(b) We do not know whether, assuming Martin's Axiom plus $\omega_1 < c$, there is a Moore space that is not the union of any family of fewer than c metrizable subspaces. The obvious candidate for an example is $M = X^{\omega_0}$ where X is a separable Moore space having a closed discrete subset of cardinality c. Unfortunately, the above argument breaks down, since if D is any uncountable discrete space, then D^{ω_0} is the union of a family $\mathscr{A}$ of nowhere dense sets, where $\text{card}(\mathscr{A}) = \omega_1$.

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3. Positive results.

THEOREM 3.1. *Any σ-space is the union of a collection of c closed metrizable subspaces.*

Proof. Let $\mathscr{F} = \bigcup \{\mathscr{F}(n) : n \in \omega_0\}$ be a σ-discrete network for X. For each n, the set $X - \bigcup \mathscr{F}(n)$ is an open set and hence an F_σ-subset of X, say

$$X - \bigcup \mathscr{F}(n) = \bigcup \{C(n, k) : k \in \omega_0\},$$

each $C(n, k)$ being closed in X.

Let J be any subset of ω_0. We say that a point $p \in X$ is a *J-point* provided $J = \{n \in \omega_0 : p \in \bigcup \mathscr{F}(n)\}$. Fix $J \subset \omega_0$ and consider the set

$$X(J) = \{p \in X : p \text{ is a } J\text{-point}\}.$$

For each $n \in J$, $X(J) \subset \bigcup \mathscr{F}(n)$ so that the collection

$$\mathscr{F}'(n) = \{X(J) \cap F : F \in \mathscr{F}(n)\}$$

is a discrete relatively closed cover of $X(J)$. But then $\mathscr{F}'(n)$ is also a collection of relatively open subsets of $X(J)$.

We assert that the collection $\bigcup \{\mathscr{F}'(n) : n \in J\}$ is a base for the subspace $X(J)$. For suppose $p \in X(J) \cap G$ where G is an open subset of X. Then for some m, some member $F \in \mathscr{F}(m)$ has $p \in F \subset G$. But then $m \in J$ so that

$$F \cap X(J) \in \bigcup \{\mathscr{F}'(n) : n \in J\}$$

as required. Thus $X(J)$ has a σ-discrete base of open sets so that metrizability of $X(J)$ will follow from the Bing-Nagata-Smirnov Theorem once we have proved that $X(J)$ is regular. But that is immediate since $X(J)$ is T_1 and $\bigcup \{\mathscr{F}'(n) : n \in J\}$ is a base of relatively closed and open sets for $X(J)$.

Since it is clear that $X = \bigcup \{X(J) : J \subset \omega_0\}$ we can complete the proof by showing that each $X(J)$ is the union of c subspaces, each closed in X. To that end, for each function $\sigma : \omega_0 - J \to \omega_0$, we define

$$X(J, \sigma) = X(J) \cap (\bigcap \{C(n, \sigma(n)) : n \in \omega_0 - J\}).$$

We first show that each $X(J, \sigma)$ is closed in X. So suppose $q \in X - X(J, \sigma)$. The case where $q \notin \bigcap \{C(n, \sigma(n)) : n \in \omega_0 - J\}$ is easy, so assume $q \in \bigcap \{C(n, \sigma(n)) : n \in \omega_0 - J\}$. Then $q \notin X(J)$ so that $J \neq J_q$ where we have written $J_q = \{n \in \omega_0 : q \in \bigcup \mathscr{F}(n)\}$. Since

$$q \in \bigcap \{C(n, \sigma(n)) : n \in \omega_0 - J\}$$

we see that $\omega_0 - J \subset \omega_0 - J_q$ and hence $J_q \neq J$ implies $J - J_q \neq \emptyset$. Choose $m \in J - J_q$; then the open set $V = X - \bigcup \mathscr{F}(m)$ contains q and is disjoint from $X(J, \sigma)$. Therefore, each $X(J, \sigma)$ is closed in X.

Finally we show that $X(J) = \bigcup \{X(J, \sigma) : \sigma : \omega_0 - J \to \omega_0\}$. Obviously it is enough to show that each $p \in X(J)$ belongs to some $X(J, \sigma)$. Now for each

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$n \in \omega_0 - J$,

$$p \in X - \bigcup \mathscr{F}(n) = \bigcup \{C(n, k) : k \in \omega_0\}.$$

Define $\sigma(n)$ to be the least k having $p \in C(n, k)$. Then $\sigma : \omega_0 - J \rightarrow \omega_0$ and $p \in X(J, \sigma)$ as required.

THEOREM 3.2. *Any quasi-developable T_1-space is the union of a family of c metrizable subspaces.*

Proof. In the light of [**5**, Lemma 4], there is a quasi-development $\langle \mathscr{G}(n) \rangle$ for X such that if p is a point of an open set U, then for some n, p belongs to exactly one member of $\mathscr{G}(n)$ and that member is a subset of U. Now for each $J \subset \omega_0$, we say that p is a type J point provided $J = \{n \in \omega_0 : p$ belongs to exactly one member of $\mathscr{G}(n)\}$. (Note that this is not the same definition as we used in (3.1).) Let $X(J) = \{p \in X : p$ is a type J point$\}$. Clearly $X = \bigcup \{X(J) : J \subset \omega_0\}$ so that it is enough to show that each $X(J)$ is metrizable. Fix $J \subset \omega_0$ and write $Y = X(J)$. For each $n \in J$ let

$$\mathscr{G}'(n) = \{Y \cap G : G \in \mathscr{G}(n)\}.$$

Each $\mathscr{G}'(n)$ is a disjoint relatively open cover of Y and is, therefore, a discrete collection in the subspace Y. And since $\langle \mathscr{G}(n) \rangle$ is a quasi-development for X, $\bigcup \{\mathscr{G}'(n) : n \in J\}$ is a base for Y. Since Y is T_1 and has a σ-discrete base of relatively closed and open sets, Y is metrizable.

COROLLARY 3.3. *Any T_1-space having a σ-point-finite base is the union of c metrizable subspaces.*

Proof. It is known that any space having a σ-point-finite base is quasi-developable [**3**].

This corollary suggests Question 2 of the Introduction.

The reader will note that Theorem 3.2 does not guarantee that quasi-developable spaces can be written as the union of a continuum of closed metrizable subspaces. That there are quasi-developable spaces that cannot be represented as such a union is the point of our next example.

EXAMPLE 3.4. *There is a locally compact quasi-developable space which is not the union of a continuum of closed metrizable subspaces.*

Proof. We outline a general construction. To obtain the example we begin with a set K of cardinality c^+.

Let K be any uncountable set. Let $\mathscr{D}$ be a maximal almost disjoint collection of countably infinite subsets of K. Let $\Psi(\mathscr{D})$ be the set $K \cup \mathscr{D}$ topologized in such a way that each point of K is isolated and such that basic neighborhoods of a point $D \in \mathscr{D}$ have the form $N(D, F) = \{D\} \cup (D - F)$ where F is a finite subset of D. It is easy to see that the space $\Psi(\mathscr{D})$ is a locally compact quasi-developable space which cannot be a Moore space. That last assertion

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can be deduced from the observation, needed later, that if J is an uncountable subset of K, then the collection

$$\mathscr{D}' = \{D \cap J : D \in \mathscr{D} \quad \text{and } D \cap J \text{ is infinite}\}$$

is a maximal almost disjoint collection of countably infinite subsets of J.

Now consider a closed subset A of $\Psi(\mathscr{D})$ for which $A \cap K$ is uncountable. Write $J = A \cap K$ and consider the maximal almost disjoint collection $\mathscr{D}'$ defined above. Define a function $h : \Psi(\mathscr{D}') \to \Psi(\mathscr{D})$ by the rule that $h(x) = x$ for each $x \in J$ and $h(D \cap J) = D$ for each $D \cap J \in \mathscr{D}'$. That function is well defined, since two distinct members of $\mathscr{D}$ cannot have identical infinite intersections with J. Further, it is easy to see that h is a topological embedding and that, if $D \cap J \in \mathscr{D}'$, then $D = h(D \cap J)$ is a limit point of A and therefore belongs to A since A is closed. But then A cannot be metrizable since its subspace $\Psi(\mathscr{D}')$ is not even a Moore space.

Finally, suppose $\mathscr{A}$ is a collection of closed metrizable subspaces of $\Psi(\mathscr{D})$ having $\operatorname{card}(\mathscr{A}) < \operatorname{card}(K)$. If $\bigcup \mathscr{A} = \Psi(\mathscr{D})$ then $K = \bigcup \{A \cap K : A \in \mathscr{A}\}$ so that for at least one $A \in \mathscr{A}$ the set $A \cap K$ must be uncountable. But then A is not metrizable. That contradiction shows that $\bigcup \mathscr{A} \neq \Psi(\mathscr{D})$, as claimed.

4. The inadequacy of ω_1. Throughout this section Martin's Axion plus the negation of the Continuum Hypothesis is abbreviated ($\text{MA} + \neg\,\text{CH}$). Martin's Axiom states that no compact Hausdorff space satisfying the countable chain condition (= no family of pairwise disjoint open sets can be uncountable) can be written as the union of fewer than c nowhere dense subspaces. It is easy to deduce the same conclusion for certain non-compact spaces. For example, a completely regular space is *Čech-complete* if it is a G_δ-subset of its Čech-Stone compactification. Any complete metric space, and any complete Moore space, is Čech complete, as is any countable product of Čech complete spaces (see [**1**] for definitions and references).

LEMMA 4.1. ($\text{MA} + \neg\,\text{CH}$) *Let X be a space which satisfies the countable chain condition and which contains a dense, Čech-complete subspace. Then X cannot be written as the union of fewer than c nowhere dense subspaces.*

We now present two examples, one a Moore space and the other a Lasnev space, which cannot be written as the union of κ metrizable subspaces for any $\kappa < c$. The first example utilizes a Moore space presented by Fitzpatrick in [**12**] and the second uses a Lasnev space constructed by van Doren in [**7**].

EXAMPLE 4.2. ($\text{MA} + \neg\,\text{CH}$) *There is a Moore space which is not the union of κ closed metrizable subspaces for any $\kappa < c$.*

Let P be the familiar tangent-disk space (Niemytzki plane). Then P is a complete, separable, non-metrizable Moore space and is therefore Čech complete. Let $X = P^{\omega_0}$ be the product of countably many copies of P. Then X

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is separable and Čech complete and no non-void open subspace of X is metrizable. If $\kappa < c$ and if $X = \bigcup \{X(\alpha) : \alpha < \kappa\}$ where each $X(\alpha)$ is a closed metrizable subspace of X, then (3.1) would force $\text{INT}(X(\alpha)) \neq \emptyset$ for some $\alpha < \kappa$. But then $\text{INT}(X(\alpha))$ would be a metrizable non-empty open subset of X, which is impossible.

The Fitzpatrick technique used in (4.2) cannot be applied to find an analogous example in the class of Lasnev spaces since it is known that if both X and $X \times X$ are Lasnev spaces, then X must be metrizable. Instead we use several results of van Doren [**7**] which we reproduce here for the reader's convenience.

THEOREM 4.3. *If X is a closed continuous image of a complete metric space, then X has a dense subspace which is completely metrizable.*

THEOREM 4.4. *There is a space X which satisfies:*
a) *X is a closed continuous image of a complete separable metric space;*
b) *the set $X^{\#}$ of all points of X at which X is not first-countable is dense in X.*

4.5. *Remark.* Van Doren's theorem [**7**] does not mention that the metric space of which X is a closed continuous image is separable. However, separability of the domain space is easily seen since the domain is a product of countably many closed subsets of Euclidean spaces.

EXAMPLE 4.6. (MA + $\neg$ CH) *There is a Lasnev space which is not the union of fewer than c closed metrizable subspaces.*

For consider van Doren's space X described in (4.4). If $\kappa < c$ and if $X = \bigcup \{X(\alpha) : \alpha < \kappa\}$ where each $X(\alpha)$ is a closed metrizable subspace of X, then (4.3) and (4.1) may be applied to conclude that for some α, $\text{INT}(X(\alpha)) \neq \emptyset$. But then

$$X^{\#} \cap \text{INT}(X(\alpha)) \neq \emptyset$$

which is impossible because X is first-countable at each point of $\text{INT}(X(\alpha))$.

5. Examples concerning other generalized metric classes. In this section, if κ is a cardinal number, then κ^+ denotes the first cardinal number greater than κ. A subset $S \supset [0, \kappa)$ is *stationary* in κ if $S \cap C \neq \emptyset$ whenever C is a closed cofinal subset of $[0, \kappa)$ (which is endowed with the open-interval topology of the usual ordering). It is easily proved ([**8**], [**11**]) that

a) if $\lambda < \kappa$ and if $\bigcup \{X(\alpha) : \alpha < \lambda\}$ is stationary in κ, then some $X(\alpha)$ is stationary in κ;
b) in its topology as a subspace of $[0, \kappa)$, no stationary subset is paracompact.

In [**16**] Nagami introduced the class of Σ-spaces as a simultaneous and natural generalization of Okuyama's σ-spaces and the (countably) compact spaces. It is not necessary to reproduce the rather technical definition of the class of Σ-spaces; for our purposes, it is enough to know that every compact space is a Σ-space. Our next example conclusively shows that no analogue of Theorem 2.1 can be proved for the class of Σ-spaces.

EXAMPLE 5.1. *For each cardinal λ there is a compact Hausdorff space which is not the union of any family of $\leqq \lambda$ metrizable subspaces.*

We let $\kappa = \lambda^+$. Then the space $X = [0, \kappa]$ is a compact Hausdorff space, and if $X = \bigcup \{X(\alpha) : \alpha < \lambda\}$, then some set $X(\alpha) - \{\kappa\}$ would be stationary in $[0, \kappa)$ and hence not even paracompact, so $X(\alpha)$ could not be metrizable.

We next consider the class of BCO spaces introduced by Worrell and Wicke in [**18**]. Recall that a base $\mathscr{B}$ for a space X is a *base of countable order* (abbreviated BCO) if whenever $B(1) \supset B(2) \supset B(3) \supset \ldots$ is a sequence of distinct members of $\mathscr{B}$ each containing a point p, then $\{B(n) : n \in \omega_0\}$ is a neighborhood base at p. Our example requires a couple of easy lemmas.

LEMMA 5.2. *If Y is a first countable regular space and if for some $p \in Y$ the subspace $X = Y - \{p\}$ has a* BCO, *then Y has a* BCO.

LEMMA 5.3. *Let S be a first-countable subspace of ordinals. Then S has a* BCO.

Proof. If (5.3) is false, let α be the least ordinal such that some first-countable subspace S of $[0, \alpha)$ does not have a BCO. Then α must be a limit ordinal. For if $\alpha = \beta + 1$, then $[0, \alpha) = [0, \beta]$ and, by minimality of α, $\beta \in S$. Again by minimality of α, the space $T = S \cap [0, \beta)$ has a BCO. Now, applying (5.2), we see that the space $S = T \cup \{\beta\}$ also has a BCO, contrary to our choice of S. Therefore α is a limit ordinal. For each $\beta < \alpha$ let $S(\beta) = S \cap [0, \beta)$. Each $S(\beta)$ is an open subspace of S, $S = \bigcup \{S(\beta) : \beta < \alpha\}$, and by minimality of α, each $S(\beta)$ has a BCO. But then by [**18**, Theorem 1] the space S has a BCO, and this contradiction completes the proof.

EXAMPLE 5.4. *For each cardinal λ there is a collectionwise normal space X which has a* BCO *and yet which is not the union of a continuum of metrizable subspaces.*

For let $\kappa = \lambda^+$ and let $X = \{\alpha < \kappa : cf(\alpha) \leqq \omega_0\}$. Then X is a first-countable space of ordinals so that X has a BCO. Furthermore, X is stationary in $[0, \kappa)$ and hence if $X = \bigcup \{X(\alpha) : \alpha < \lambda\}$, then at least one of the sets $X(\alpha)$ is also stationary in $[0, \kappa)$ and therefore not even paracompact.

5.5. *Remark.* Hajnal and Juhász [**14**] have proved that if S is a first-countable space of ordinals such that for each limit ordinal λ the set $S \cap [0, \lambda)$ is not stationary in $[0, \lambda)$, then S is metrizable. The result in (5.3), combined with the characterization of paracompactness given in [**11**] show that any such space is paracompact and has a BCO. Now a theorem of Arhangel'skiĭ [**2**] may be applied to deduce metrizability of S. (A more convenient reference for that result is [**18**].)

6. A perfect space which is not the union of a continuum of metrizable subspaces. In this section we present an example which shows the need for some generalized metric structure, above and beyond such properties as paracompactness, perfectness (= closed sets are G_δ's) and first-countability, if Theorem 3.1 is to hold.

Let κ be a cardinal number such that $\kappa^{\omega_0} > \kappa \geqq c$ and let $(D, \leqq)$ be a linearly ordered set such that

a) D has cardinality κ;
b) the open interval topology of $(D, \leqq)$ is the discrete topology;
c) $(D, \leqq)$ has neither first nor last points.

Such sets are easy to find: let D be the lexicographic product set $[0, \kappa) \times Z$, where Z denotes the set of integers with the usual ordering.

The product set $X = D^{\omega_0}$ carries two topologies—the Tychonoff product topology $\mathscr{P}$ and the usual open-interval topology $\mathscr{I}$ of the lexicographic ordering of X. It is a theorem of Faber, Maurice and Wattel [**13**] (see also [**10**]) that $\mathscr{P} = \mathscr{I}$. Hence $(X, \mathscr{I})$ is a metrizable linearly ordered topological space whose density (= smallest cardinality of a dense subset) is κ [**9**, Theorem 2.3.15]. Furthermore, since D has no end points, the set X is densely ordered by the lexicographic order (i.e., if $a < c$ in X, then for some $b \in X$, $a < b < c$).

The next step is to modify the interval topology $\mathscr{I}$ using the collection $\{[x, y) : x < y \text{ in } X\}$ as a base for a new topology $\mathscr{S}$ on X. Because X is densely ordered and $(X, \mathscr{I})$ is perfect, so is the new space $(X, \mathscr{S})$. In the terminology of [**15**], $(X, \mathscr{S})$ is a generalized ordered space so that, being perfect, $(X, \mathscr{S})$ is both hereditarily paracompact and first-countable (see [**15**] or [**11**]). It remains only to show that $(X, \mathscr{S})$ is not the union of c metrizable subspaces. To that end, suppose that $(Y, \mathscr{S}_Y)$ is a metrizable subspace of $(X, \mathscr{S})$. Then we have [**9**]

$$\begin{aligned} \operatorname{card}(Y) = \operatorname{weight}(Y) = \operatorname{density}(Y) &\leqq \operatorname{density}(X, \mathscr{S}) \\ &= \operatorname{density}(X, \mathscr{I}) = \kappa \end{aligned}$$

where the first equality holds in any space with a "Sorgenfrey-type" topology, the second follows from metrizability of Y, the third follows from [**15**, Theorem 2.10] and the fourth follows from the fact that X is densely ordered. But then the union of a continuum of metrizable subspaces of $(X, \mathscr{S})$ can have cardinality at most $c \cdot \kappa = \kappa < \kappa^{\omega_0} = \operatorname{card}(X)$ and so $(X, \mathscr{S})$ cannot be the union of c metrizable subspaces.

We remark that the space $(X, \mathscr{S})$ is a generalized ordered space and not a linearly ordered space. However, afficianados of linearly ordered topological spaces will recognize that by using the lexicographic product $X \times \{0, 1\}$, we could have obtained a perfect linearly ordered space which is not the union of a continuum of metrizable subspaces.

References

1. J. Aarts and D. Lutzer, *Completeness properties designed for recognizing Baire spaces*, Dissertationes Math. CXVI (1974).

2. A. Arhangel'skiĭ, *Some metrization theorems*, Uspehi Mat. Nauk *18* (1963), 139–145 (Russian).

3. C. Aull, *Topological spaces with a σ-point-finite base*, Proc. Amer. Math. Soc. *29* (1971), 411–416.

4. H. Bennett, *On quasi-developable spaces*, Gen. Top. Appl. *1* (1971), 253–261.

5. H. Bennett and D. Lutzer, *A note on weak θ-refinability*, Gen. Top. Appl. *2* (1972), 49–54.

6. D. Burke and D. Lutzer, *Recent advances in the theory of generalized metric spaces*, Proc. Memphis State Topology Conf., Lecture Notes in Pure and Applied Mathematics, *24* (Marcel Dekker, 1977), 1–70.

7. K. van Doren, *Closed, continuous images of complete metric spaces*, Fund. Math. *80* (1973), 47–50.

8. E. van Douwen and D. Lutzer, *On the classification of stationary sets*, Michigan Math. J. *26* (1979), 47–64.

9. R. Engelking, *General topology* (Polish Scientific Publishers, 1977).

10. R. Engelking, R. Heath, and E. Michael, *Topological well-ordering and continuous selections*, Inventiones Math. *6* (1968), 150–158.

11. R. Engelking and D. Lutzer, *Paracompactness in ordered spaces*, Fundamenta Math. *94* (1976).

12. B. Fitzpatrick, *Some topologically complete spaces*, Gen. Top. Appl. *1* (1971), 101–103.

13. M. Faber, M. Maurice, and E. Wattel, *Order pre-images under one-to-one mappings*, in *Theory of Sets and Topology* (Berlin, 1972), 109–119.

14. A. Hajnal and I. Juhász, *On spaces in which every small subspace is metrizable*, Bull. Polish Acad. Sci. *24* (1976), 727–731.

15. D. Lutzer, *On generalized ordered spaces*, Dissertationes Math. *89* (1971).

16. J. Nagata, *Some problems on generalized metric spaces*, Proc. Emory Univ. Topology Conference (March, 1970), 63–70.

17. K. Nagami, *Σ-spaces*, Fundamenta Math. *65* (1969), 169–192.

18. J. Worrell and H. Wicke, *Characterizations of developable topological spaces*, Can. J. Math. *17* (1965), 820–830.

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