Can. J. Math., Vol. XXXVIII, No. 6, 1986, pp. 1281-1298

SPACES ON WHICH EVERY CONTINUOUS MAP INTO A GIVEN SPACE IS CONSTANT

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1. Introduction. This paper is concerned with topological spaces whose continuous maps into a given space R are constant, as well as with spaces having this property locally. We call these spaces R-monolithic and locally R-monolithic, respectively.

Spaces with such properties have been considered in **[1]**, **[5]**-**[7]**, **[10]**, **[11]**, **[22]**, **[28]**, **[31]**, where with the exception of **[10]**, the given space R is the set of real-numbers with the usual topology. Obviously, for a countable space, connectedness is equivalent to the property that every continuous real-valued map is constant. Countable connected (locally connected) spaces have been constructed in papers **[2]**-**[4]**, **[8]**, **[9]**, **[11]**-**[21]**, **[23]**-**[26]**, **[30]**.

In the present paper (as in **[10]**, **[11]**) the construction of an R-monolithic (locally R-monolithic) space is based on the following steps: 1) The construction of an auxiliary space T having two points p^-, p^+, for which $f(p^-) = f(p^+)$, for every continuous map f of T into R, 2) The embedding of a given space X into a given space Y_1 (which in the sequel will be denoted by $I^1(X, \Lambda_0)$), for which $f(a) = f(b)$, for every $a, b \in X$ and for every continuous map f of Y_1 into R, 3) The construction of a sequence of spaces $Y_0 = X, Y_1, Y_2, \dots, Y_n, \dots$ such that $Y_n \subseteq Y_{n+1}$, $n = 1, 2, \dots$, and having the property $f(a) = f(b)$, for every $a, b \in Y_n$ and for every continuous map of Y_{n+1}, into the space R, 4) The determination of a topology on the set

$$Y = \bigcup_{i=1}^{\infty} Y_i$$

(this space will be denoted by $I(X)$ or $I(X, R)$).

Many of the common properties of spaces T and X are carried on the resulting space Y. For example, if these spaces are both regular, countable, Moore or spaces with the first axiom of countability, then Y is correspondingly regular, countable, Moore or a space with the first axiom of countability. Thus, the above method yields R-monolithic, locally R-monolithic regular spaces, as well as countable connected, locally connected spaces with additional properties.

It is to be noted that Herrlich's work **[10]**, is the only one where the space R is an arbitrary Hausdorff space (actually a T_1 space). However,

Received June 14, 1982, and in revised form June 22, 1983.

the construction of the space Y_1, in **[10]** (corresponding to the second step) forces the resulting space not to be locally R-monolithic. Furthermore, some of its cardinal invariants are "very large" with respect to those of the space R. Finally, the auxiliary space (of the first step) is not countable and hence it can not be used for the construction of countable connected spaces.

2. Definitions. All spaces in this paper are assumed to be Hausdorff.

Let R be a space. A space X is called *R-monolithic* if every continuous map of X into R is constant. A space X is called *locally R-monolithic* if for every $x \in X$ and for every open neighbourhood U of x, there exists an open neighbourhood V of x, $V \subseteq U$, which is an R-monolithic space.

A space X is said to be: 1) *Urysohn*, if for every two distinct points x, y of X, there exist open neighbourhoods V, U of the points x, y such that $\bar{U} \cap \bar{V} = \emptyset$, 2) *Almost regular* if there exists a dense subset of X, at every point of which the space X is regular.

Let X be a space. We denote 1) by $|X|$ the cardinality of X, 2) by $w(X)$ the weight of X or the minimal cardinality of a basis of X, 3) by $\chi(X, p)$, $p \in X$, the character of X at the point p or the minimal cardinality of a basis at p, 4) by $\chi(X)$ the character of X, i.e.,

$$\chi(X) = \sup\{\chi(X, p): p \in X\}$$

5) by $d(X)$ the density of X, or the minimal cardinality of a dense subset of X, 6) by $\psi(X, p)$, $p \in X$ the pseudocharacter of X at the point p or the minimal cardinality of pseudobases of p, (the set $\{U_\alpha\}$ consisting of open neighbourhoods of the point p, is called a pseudobasis if $\cap U_\alpha = \{p\}$) 7) by $\psi(X)$ the pseudocharacter of the space X, i.e.,

$$\psi(X) = \sup\{\psi(X, p): p \in X\}$$

and by $\psi^+(X)$ the smallest cardinal number which is greater than $\psi(X)$.

For every space X we denote by $\Delta(X)$ the diagonal of X i.e.,

$$\Delta(X) = \{ (a, b) \in X \times X: a = b\}.$$

Finally, we denote by $\aleph_0$ the first infinite cardinal number and by $\aleph_1$ the first uncountable cardinal number.

3. The space $I^1(X, \Lambda_0)$. Let X be a topological space and $B(X)$ a basis of open sets of X. Let, also, T be a topological space and p^-, p^+ two distinct non-isolated points of T.

By $B^0(p^-)$ (resp. $B^0(p^+)$) we denote a basis of open neighbourhoods of the point p^- (resp. p^+), in the space T and by $B(p^-)$ (resp. $B(p^+)$) the set whose elements are sets of the form $V \setminus \{p^-\}$ (resp. $V \setminus \{p^+\}$), where $V \in B^0(p^-)$ (resp. $V \in B^0(p^+)$). Also, we denote by $B(J)$ a basis of open sets of the space

$$J = T\backslash\{p^-, p^+\}.$$

In the sequel, the letters H, G (with or without indices) will be used to denote elements of the sets $B(p^-)$ and $B(p^+)$, respectively.

Let Λ_0 be a subset of the set $X \times X\backslash\Delta(X)$. For every open set U of X, we set

$$\Lambda_0^-(U) = \{\lambda = (a, b) \in \Lambda_0 : a \in U, b \notin U\}$$

$$\Lambda_0^+(U) = \{\lambda = (a, b) \in \Lambda_0 : a \notin U, b \in U\}$$

$$\Lambda_0(U) = \{\lambda = (a, b) \in \Lambda_0 : a \in U, b \in U\}.$$

For every $\lambda \in \Lambda_0$ we denote by J^λ a space homeomorphic to J and by i^λ a homeomorphism of J onto J^λ. We assume that

$$J^\lambda \cap X = \emptyset$$

for every $\lambda \in \Lambda_0$ and if $\lambda_1 \neq \lambda_2$ then

$$J^{\lambda_1} \cap J^{\lambda_2} = \emptyset.$$

For every subset M of J and for every $\lambda \in \Lambda_0$, the set $i^\lambda(M) \subseteq J^\lambda$, is denoted by M^λ. If $N = \{M_\alpha\}$ is a family of subsets of J, then the family $\{M_\alpha^\lambda\}$ of subsets of J^λ is denoted by N^λ.

Set

$$I^1(X, \Lambda_0) = X \cup \bigcup_{\lambda \in \Lambda_0} J^\lambda.$$

For every open set U of X we set,

$$(1) \quad O^1(U, H, G) = U \cup \bigcup_{\lambda \in \Lambda_0(U)} J^\lambda \cup \bigcup_{\lambda \in \Lambda_0^-(U)} H^\lambda \cup \bigcup_{\lambda \in \Lambda_0^+(U)} G^\lambda.$$

The family of all sets of the form (1), where U belongs to a family of open sets C, is denoted by $O^1(C, H, G)$.

We set

$$B_1^0 = \cup O^1(B(X), H, G),$$

where H runs over $B(p^-)$ and G over $B(p^+)$.

Finally, we set

$$B_1^1 = \bigcup_{\lambda \in \Lambda_0} B^\lambda(J)$$

and

$$B_1 = B_1^0 \cup B_1^1.$$

Since

$$O^1(U_1, H_1, G_1) \cap O^1(U_2, H_2, G_2)$$

$$\supseteq O^1(U_1 \cap U_2, H_1 \cap H_2, G_1 \cap G_2)$$

the set B_1 can be considered as a basis for a topology on the set $I^1(X, \Lambda_0)$. Obviously the space $I^1(X, \Lambda_0)$ has the following properties:

i) The space X is a closed subspace of $I^1(X, \Lambda_0)$.

ii) The set J^λ, $\lambda \in \Lambda_0$ is an open subset of $I^1(X, \Lambda_0)$ and if $\lambda = (a, b)$, then

$$\overline{J^\lambda} = J^\lambda \cup \{a, b\}.$$

iii) The map $\overline{i}^\lambda : T \rightarrow \overline{J^\lambda}$, defined by

$$\overline{i}^\lambda(p^-) = a, \ i^\lambda(p^+) = b \quad \text{and} \quad \overline{i}^\lambda(p) = i^\lambda(p)$$

for every $p \in J$, is a homeomorphism.

LEMMA 1. *The space $I^1(X, \Lambda_0)$ is Hausdorff. If X and T are both Urysohn (or regular) then $I^1(X, \Lambda_0)$ is Urysohn (or regular).*

Proof. First we prove that

$$(2) \quad \overline{O^1(U, H, G)} = \overline{U}^X \cup \bigcup_{\lambda \in \Lambda_0(U)} J^\lambda \cup \bigcup_{\lambda \in \Lambda_0^-(U)} (\overline{H}^J)^\lambda \cup \bigcup_{\lambda \in \Lambda_0^+(U)} (\overline{G}^J)^\lambda.$$

In fact, the set on the right side of (2) is, obviously, contained in the left. We show that if a point x does not belong to the left-hand side of (2), then it can not belong to the set of the right-hand side of (2), either. This is easily verified by considering the following cases: 1) $x \in X$, 2) $x \in J^\lambda$ and

$$\lambda \notin \Lambda_0^-(U) \cup \Lambda_0^+(U) \cup \Lambda_0(U),$$

3) $x \in J^\lambda$ and

$$\lambda \in \Lambda_0^-(U) \cup \Lambda_0^+(U).$$

Now we shall prove that $I^1(X, \Lambda_0)$ is Hausdorff (Urysohn, if X and T are both Urysohn). Let x, y be distinct points of $I^1(X, \Lambda_0)$. We consider the following cases: 1) one of the points, let it be x, belongs to X and

$$y \in I^1(X, \Lambda_0)\backslash X,$$

2) both x and y belong to $I^1(X, \Lambda_0)\backslash X$, 3) both x, y belong to X.

In case 1) there exists $\lambda = (a, b) \in \Lambda_0$ such that $y \in J^\lambda$. Let $U \in B(X)$, $x \in U$ and at least one of the points a, b is not contained in U. There exist sets $V \in B(J)$, $H \in B(p^-)$, $G \in B(p^+)$ such that $y \in V^\lambda$, $V \cap H = \emptyset$ and $V \cap G = \emptyset$, ($\overline{V}^J \cap \overline{H}^J = \emptyset$ and $\overline{V}^J \cap \overline{G}^J = \emptyset$, for the Urysohn case). Hence the sets $O^1(U, H, G)$ and V^λ are open neighbourhoods of the points x, y respectively, such that

$$O^1(U, H, G) \cap V^\lambda = \emptyset$$

(and $\overline{O^1(U, H, G)} \cap \bar{V}^\lambda = \emptyset$ for the Urysohn case).

The proof is similar for cases 2) and 3).

Finally, let X, T be both regular. Obviously, property ii) of $I^1(X, \Lambda_0)$ implies that at every point of the set

$$\bigcup_{\lambda \in \Lambda_0} J^\lambda,$$

the space $I^1(X, \Lambda_0)$ is regular. Let

$$x \in X, U \in B(X) \quad \text{and} \quad x \in O^1(U, H, G).$$

There exist sets $U_1 \in B(X)$, $H_1 \in B(p^-)$, $G_1 \in B(p^+)$ such that $x \in U_1$, $\bar{U}_1^X \subseteq U$, $\bar{H}_1^J \subseteq H$, $\bar{G}_1^J \subseteq G$. Then

$$x \in O^1(U_1, H_1, G_1)$$

and, as follows by (2),

$$\overline{O^1(U_1, H_1, G_1)} \subseteq O^1(U, H, G).$$

Consequently, the space $I^1(X, \Lambda_0)$ is regular at the point x.

4. The space $I(X)$. In the same manner that we constructed the space $I^1(X, \Lambda_0)$, we can construct, by induction, the spaces $I^n(X, \Lambda_{n-1})$, $n = 2, 3, \ldots$, where Λ_{n-1} is an arbitrary subset of the set

$$I^{n-1}(X, \Lambda_{n-2}) \times I^{n-1}(X, \Lambda_{n-2}) \setminus \Delta(I^{n-1}(X, \Lambda_{n-2}))$$

and where

$$I^n(X, \Lambda_{n-1}) = I^1(I^{n-1}(X, \Lambda_{n-2}), \Lambda_{n-1}).$$

We set

$$I(X) = \bigcup_{n=1}^{\infty} I^n(X, \Lambda_{n-1}).$$

Obviously, the set $I(X)$ depends not only on the spaces X, T but also on the sets Λ_{n-1}, $n = 1, 2 \ldots$.

We set

$$B_0^0(H, G) = B(X), B_1^0(H, G) = O^1(B(X), H, G),$$
$$B_1^1(H, G) = \bigcup_{\lambda \in \Lambda_0} B^\lambda(J)$$

and we define, by induction, the sets

$$B_n^k(H, G) = O^1(B_{n-1}^k(H, G), H, G), 0 \leqq k < n,$$

and

$$B_n^n(H, G) = \bigcup_{\lambda \in \Lambda_{n-1}} B^\lambda(J).$$

Finally we set

$$B_n^k = \cup B_n^\kappa(H, G),\ n = 0, 1, 2, \dots, \kappa = 0, 1, \dots, n$$

$$B_n = \bigcup_{k=0}^{n} B_n^\kappa,\ n = 0, 1, 2, \dots.$$

We proved before that the set B_1 is a basis of the space $I^1(X, \Lambda_0)$. In order to prove that the set B_2 is a basis of the space $I^2(X, \Lambda_1)$ observe that if $H \subseteq H_1 \cap H_2$, $G \subseteq G_1 \cap G_2$ and $U \in B(X)$, then

$$O^1(O^1(U, H, G), H, G) \subseteq O^1(O^1(U, H_1, G_1), H_2, G_2).$$

It can be proved by induction that the set B_n is a basis of the space $I^n(X, \Lambda_{n-1})$.

Now we define a topology on the set $I(X)$: For every $U \in B_n^n$, $n = 0, 1, \dots$ we set

$$O^2(U, H, G) = O^1(O^1(U, H, G), H, G)$$

and by induction

$$O^k(U, H, G) = O^1(O^{k-1}(U, H, G), H, G).$$

We also set

$$O(U, H, G) = \bigcup_{k=1}^{\infty} O^k(U, H, G).$$

If $U \in B_n^k$, $n = 1, 2, \dots, k < n$, then obviously there exists $V \in B_k^k$ such that

$$U = O^{n-k}(V, H, G).$$

In this case we set $O(U, H, G) = O(V, H, G)$. We denote by $B^n(H, G)$, $n = 0, 1, 2, \dots,$ the family of all sets of the form $O(U, H, G)$, where $U \in B_n^n$.

We set

$$B^n = \cup B^n(H, G) \quad \text{and} \quad B = \bigcup_{n=0}^{\infty} B^n.$$

It can be easily proved that the set B can be considered as a basis for a topology on the set $I(X)$.

The spaces $I^n(X, \Lambda_{n-1})$ and $I(X)$ have the following properties:

a) The space $I^n(X, \Lambda_{n-1})$, $n \geqq 1$, is Hausdorff and if X, T are both Urysohn (or regular), then the space $I^n(X, \Lambda_{n-1})$ is also Urysohn (or regular).

b) The space $I^{n-1}(X, \Lambda_{n-2})$, $n \geqq 2$, is a closed subspace of both spaces $I^n(X, \Lambda_{n-1})$ and $I(X)$.

c) The set J^λ, $\lambda \in \Lambda_{n-1}$, is open in $I^n(X, \Lambda_{n-1})$.

d) For every $\lambda = (a, b) \in \Lambda_{n-1}$,

$$\overline{J^\lambda} = \overline{J^\lambda}^{I(X)} = \overline{J^\lambda}^{I^n(X,\Lambda_{n-1})} = J^\lambda \cup \{a, b\}.$$

e) The map

$$\overline{i^\lambda}: T \to \overline{J^\lambda} \subseteq I(X), \lambda = (a, b),$$

defined by $\overline{i^\lambda}(p^-) = a$, $\overline{i^\lambda}(p^+) = b$ and $\overline{i^\lambda}(p) = i^\lambda(p)$, for every $p \in J$ is a homeomorphism.

LEMMA 2. *The space $I(X)$ is Hausdorff. If the spaces X and T are both Urysohn (or regular) then the space $I(X)$ is Urysohn (or regular).*

Proof. First we observe that if $U_1, U_2 \in B_n$, $n = 0, 1, \dots$ and $U_1 \cap U_2 = \emptyset$ then

(3) $O(U_1, H_1, G_1) \cap O(U_2, H_2, G_2) = \emptyset$

(4) $\overline{O(U, H, G)} = \bigcup_{k=1}^{\infty} \overline{O^k(U, H, G)}^{I^{n-k}(X,\Lambda_{n-k-1})}$

for every $U \in B_n^n$.

Relations (2) and (4) imply that, if

$$U_1, U_2 \in B_n, \overline{U_1}^{I^n}(X, \Lambda_{n-1}) \cap \overline{U_2}^{I^n}(X, \Lambda_{n-1}) = \emptyset$$

and

$$\overline{H_1 \cup H_2}^J \cap \overline{G_1 \cup G_2}^J = \emptyset,$$

then

(5) $\overline{O(U_1, H_1, G_1)} \cap \overline{O(U_2, H_2, G_2)} = \emptyset.$

Also by (2) and (4) is implied that if $U_1, U_2 \in B_n$,

$$\overline{U_1}^{I^n(X,\Lambda_{n-1})} \subseteq U_2, \quad \overline{H}_1^J \subseteq H_2, \quad \text{and} \quad \overline{G}_1^j \subseteq G_2$$

then

(6) $\overline{O(U_1, H_1, G_1)} \subseteq \overline{O(U_2, H_2, G_2)}.$

By the above properties of the spaces $I(X)$, $I^n(X, \Lambda_{n-1})$, $n = 1, 2, \dots$ and by (3), it follows that $I(X)$ is Hausdorff. Similarly by (5) it follows that if X, T are Urysohn then $I(X)$ is Urysohn. And likewise, by (6) if X, T are both regular then $I(X)$ is regular.

5. Results. The following theorem is obviously applied to the construction of not only regular R-monolithic, locally R-monolithic spaces but also of countable connected locally connected spaces.

THEOREM 1. *For every Hausdorff space R and for every Hausdorff (or Urysohn, or regular) space X there exists an almost regular R-monolithic, locally R-monolithic Hausdorff (or Urysohn, or regular, respectively) space $I(X, R)$ containing X as a closed subspace and for which*

$$(7) \quad \alpha(I(X, R)) = \max\{\alpha(X), \psi^{+}(R), \aleph_1\},$$

where $\alpha(I(X, R))$ and $\alpha(X)$ are for spaces $I(X, R)$ and X respectively any of the following: their cardinality, their weight, their density and their character or pseudocharacter.

If the space R is the set of real-numbers and X is non-finite Hausdorff or Urysohn, then relation (7) *becomes*

$$\alpha(I(X, R)) = \max\{\alpha(X), \aleph_0\}.$$

Proof. Let us suppose that $T_1 = T_1(R)$ is a regular space having the following properties:

i) There exist two distinct non-isolated points p^-, p^+ of T_1 such that $f(p^-) = f(p^+)$, for every continuous map f of T_1 into R

$$\text{ii)} \quad w(T_1) = |T_1| = \chi(T_1) = \psi(T_1) = d(T_1) = \max\{\psi^{+}(R), \aleph_1\}.$$

Now let X be a Hausdorff (or Urysohn or regular) space not consisting of a single point. We construct as in Section 4 the space $I(X)$ using instead of T, the space T_1 and at the same time, considering that

$$\Lambda_{n-1} = L_{n-1} \times L_{n-1} \backslash \Delta(L_{n-1} \times L_{n-1}), \quad n = 1, 2, \dots,$$

where the set L_0 is dense in X, $|L_0| = d(X)$ and that the set L_{n-1}, $n = 2, 3, \dots$, is a dense subset of $I^{n-1}(X, \Lambda_{n-2})$, such that

$$|L_{n-1}| = d(I^{n-1}(X, \Lambda_{n-2})).$$

We now set $I(X) = I(X, R)$ and prove that $I(X, R)$ is the required space. In fact X is a closed subspace of $I(X, R)$ (by properties i) of $I^1(X, \Lambda_0)$ and b) of $I(X)$. By Lemma 2, $I(X, R)$ is Hausdorff (or Urysohn or regular).

Observe that $I(X, R)$ is almost regular at every point of the dense subset

$$\bigcup_{i=0}^{\infty} \left(\bigcup_{\lambda \in \Lambda_0} J^{\lambda} \right)$$

which is proved in a similar way as Lemma 2.

Now we prove that $I(X, R)$ is R-monolithic. Let f be a continuous map of $I(X, R)$ into R. If $\lambda = (a, b) \in \Lambda_{n-1}$, $n = 1, 2, \dots$ then by property e) of $I(X)$, the map $f \circ \overline{i}^{\lambda}$ of T_1 into R is continuous and, by property i) of T_1,

$$(f \circ \overline{i}^{\lambda})(p^-) = (f \circ \overline{i}^{\lambda})(p^+).$$

Hence $f(a) = f(b)$, i.e., the map f is constant on the set L_{n-1}. Since the set L_{n-1} is dense in $I^{n-1}(X, \Lambda_{n-2})$, $n = 2, 3, \ldots$, the map f is constant on the set $I^{n-1}(X, \Lambda_{n-2})$ and therefore on $I(X, R)$, that is, the space $I(X, R)$ is R-monolithic.

In order to show that the space $I(X, R)$ is locally R-monolithic, it suffices to prove that every element of the basis B of $I(X, R)$ is an R-monolithic subspace. Let

$$O(U, H, G) \in B,$$

where $U \in B_n^n$, $n = 0, 1, \ldots$ and let f be a continuous map of $O(U, H, G)$ into R. If $\lambda = (a, b) \in \Lambda_n$ and $a, b \in U$ then by property d) of $I(X)$ and the definition of $O^1(U, H, G)$ we have

$$\overline{I^{\lambda}}^{I(X,R)} = J^{\lambda} \cup \{a, b\} \subseteq O^1(U, H, G)$$

and hence $f(a) = f(b)$. Since the set L_n is dense in $I^n(X, \Lambda_{n-1})$, the set $L_n \cap U$ is dense in U. Consequently, the map f is constant on U. Similarly, since

$$O^{k-1}(U, H, G) = O^1(O^k(U, H, G), H, G)$$

the map f is constant on the set $O(U, H, G)$.

Relation (7) is implied by the construction of $I(X, R)$ and by property (b) of T_1. We only prove the case where $\alpha(I(X, R))$ and $\alpha(X)$ are the densities of $I(X, R)$ and X, respectively.

Let M be a dense subset of $I(X, R)$ such that

$$|M| = d(I(X, R)).$$

For every point $x \in I(X, R)$ we define a set $a(x)$ as follows: If $x \in X$, then we set $a(x) = \{x\}$. If $x \in I(X, R)\setminus X$ and n is the minimal integer for which $x \in I^{n+1}(X, \Lambda_n)$ then there exists $\lambda = (c, d)$ belonging to Λ_n such that $x \in J^{\lambda}$. We set $a(x) = \{c, d\}$.

Now we define the subsets $A_0(x)$ and $A_1(x)$ of X and $I^1(X, \Lambda_0)$, respectively. The point y of X, or of $I^1(X, \Lambda_0)$ belongs to $A_0(x)$, or to $A_1(x)$, respectively, if and only if there exist points $x_1, x_2, \ldots, x_m$ such that $x_1 = x$, $x_m = y$ and $x_{i+1} \in a(x_i)$, for every $i = 1, 2, \ldots, m - 1$.

Finally, we set

$$A_0(M) = \bigcup_{x \in M} A_0(x) \quad \text{and} \quad A_1(M) = \bigcup_{x \in M} A_1(x).$$

We observe that

$$|A_0(M)| = |A_1(M)| = |M|.$$

The set $A_0(M)$ is dense in X. Indeed, let U be an open subset of X. The set $O(U, H, G)$ is an open subset of $I(X, R)$ and hence

https://doi.org/10.4153/CJM-1986-065-1 Published online by Cambridge University Press

$$M \cap O(U, H, G) \neq \emptyset.$$

Let $x_1 \in M \cap O(U, H, G)$.

By the definition of the set $O(U, H, G)$ we have

$$a(x) \cap O(U, H, G) \neq \emptyset.$$

Let $x_2 \in a(x) \cap O(U, H, G)$. Then

$$a(x_2) \cap O(U, H, G) \neq \emptyset.$$

Let $x_3 \in a(x_2) \cap O(U, H, G)$. Continuing in this manner, we finally get a point z which belongs to the set U. Similarly, the set $A_1(M)$ is dense in $I^1(X, \Lambda_0)$. Since J^λ, $\lambda \in \Lambda_0$ is an open subset of $I^1(X, \Lambda_0)$, the set $A_1(M) \cap J^\lambda$ is dense in J^λ. By the above we have

$$d(I(X, R)) = |M| \geqq d(X) \quad \text{and} \quad d(I(X, R)) = |M| \geqq d(T),$$

i.e.,

$$d(I(X, R)) \geqq \max\{d(X), d(T)\} = \max\{d(X), \psi^+(R), \aleph_1\}.$$

On the other hand, since

$$d(X) = |L_0| \text{ and } d(T) = \max\{\psi^+(R), \aleph_1\}$$

it follows that

$$|L_1| = \max\{d(X), \psi^+(R), \aleph_1\}$$

and by induction

$$|L_n| = \max\{d(X), \psi^+(R), \aleph_1\}.$$

Hence

$$d(I(X, R)) = \left| \bigcup_{n=0}^{\infty} L_n \right| = \max\{d(X), \psi^+(R), \aleph_1\},$$

and therefore

$$d(I(X, R)) = \max\{d(X), \psi^+(R), \aleph_1\}.$$

For the second part of the theorem, where R is the space of real-numbers, we repeat the previous process using an almost regular Urysohn space $T_2 = T_2(R)$, in place of T_1 with the following properties:

i) There exist two points p^-, p^+ of T_2 such that $f(p^-) = f(p^+)$, for every continuous real-valued function f of T_2.

ii) The space T_2 is regular at p^-, p^+.

iii) $|T_2| = w(T_2) = \aleph_0$.

To complete the proof of the theorem, it suffices to construct the spaces $T_1(R)$ and $T_2(R)$.

We first construct $T_1(R)$. Let T_{4n+2}, $n = 0, \pm 1, \pm 2, \dots$ and T^m_{2q+1}, $q = 0, \pm 1, \pm 2, \dots, m = 1, 2, \dots$ be pairwise disjoint sets, such that

$$|T_{4n+2}| = |T^m_{2q+1}| = \max\{\psi^+(R), \aleph_1\}.$$

Let

$$\varphi^m_{4n+3}:T_{4n+2} \to T^m_{4n+3},$$
$$\varphi^m_{4n+1}:T_{4n+2} \to T^m_{4n+1}$$

be maps which are one-to-one and onto. Also, let p^-, p^+, a^m_{4n}, $m = 1, 2, \dots, n = 0, \pm 1, \pm 2, \dots$ be distinct points not belonging to any of the above sets. The set $T_1(R)$ consists of the sets T_{4n+2}, T^m_{2q+1} and of the points a^m_{4n}, p^-, p^+.

The topology on $T_1(R)$ is defined as follows: Every point of T^m_{2q+1} is isolated. For every point $x \in T_{4n+2}$, a basis of open neighbourhoods are the sets of the form $A \cup \{x\}$, where A consists of all but finite number of elements of the set

$$\{\varphi^m_{4n+3}(x):m = 1, 2, \dots\} \cup \{\varphi^m_{4n+1}(x):m = 1, 2, \dots\}.$$

A basis of open neighbourhoods of a^m_{4t} are the sets of the form $C \cup \{a^m_{4t}\}$, where C contains all but finite number of elements of the set

$$T^m_{4t-1} \cup T^m_{4t+1}.$$

Finally, a basis of open neighbourhoods for the points p^+, p^- are the sets

$$\bigcup_{k>4n+2} T_k \cup \bigcup_{k>4n+2} T^m_k \cup \{a^m_k:k > 4n + 2, m = 1, 2, \dots\} \cup \{p^+\}$$

$$\bigcup_{k<-4n-2} T_k \cup \bigcup_{k<-4n-2} T^m_k \cup \{a^m_k:k < -4n - 2, m = 1, 2, \dots\} \cup \{p^-\}$$

where $n = 1, 2, \dots,$ respectively.

The proof of property i) is similar to that of the corresponding property of the space X in **[27]**, in which the space R is the set of real numbers with the usual topology. Indeed, let f be a continuous map of T_1 into R. Obviously for every open neighbourhood U of the point $f(a^m_{4t})$, the set $T^m_{4t+1}\setminus f^{-1}(U)$ is finite. Since the singleton $\{f(a^m_{4t})\}$ is the intersection of $\psi(R)$ open neighbourhoods of the point $f(a^m_{4t})$ it follows that for the set

$$A^m_{4t+1} = T^m_{4t+1}\setminus f^{-1}(f(a^m_{4t}))$$

it holds that

$$|A^m_{4t+1}| \leqq \max\{\psi(R), \aleph_0\}.$$

Consequently,

$$|(\varphi^m_{4t+1})^{-1}(A^m_{4t+1})| \leqq \max\{\psi(R), \aleph_0\}.$$

Similarly for the set

$$A^m_{4t-1} = T^m_{4t-1} \setminus f^{-1}(f(a^m_{4t}))$$

it holds that

$$|A^m_{4t-1}| \leqq \max\{\psi(R), \aleph_0\}.$$

Consequently,

$$|(\varphi^m_{4t-1})^{-1}(A^m_{4t-1})| \leqq \max\{\psi(R), \aleph_0\}.$$

For the set

$$B_{4t+2} = T_{4t+2} \setminus \bigcup_{m=1}^{\infty} ((\varphi^m_{4t+1})^{-1}(A^m_{4t+1}) \cup (\varphi^m_{4t+3})^{-1}(A^m_{4t+3}))$$

it holds that

$$|B_{4t+2}| = \max\{\psi^+(R), \aleph_1\}.$$

Let $x_{4t+2} \in B_{4t+2}$. Obviously,

$$f(x_{4t+2}) = \lim_{m\to\infty} f(a^m_{4t}) = \lim_{m\to\infty} f(a^m_{4(t+1)})$$

and hence the point $f(x_{4t+2})$ does not depend on t. Since

$$\lim_{t\to\infty} x_{4t+2} = p^+ \quad \text{and} \quad \lim_{t\to-\infty} x_{4t+2} = p^-,$$

it follows that $f(p^+) = f(p^-)$.

The other properties of $T_1(R)$ are easily verified. A space analoguous to $T_1(R)$ can also be constructed using the method of [**10**].

Now we construct the space $T_2(R)$. Let E^2 denote the plane i.e., the set of ordered pairs of real numbers. We set

$$D = \{(x, y) \in E^2 : 0 < x < 1, 0 < y < 1\}.$$

Let $[A, B]$ be the linear segment joining the points A, B of E^2.

It can be easily proved that there exists a subset Q of D with the following properties: 1) if $(x, y) \in Q$, then the numbers x, y are rationals, 2) if $(x_1, y_1), (x_2, y_2) \in Q$ and $(x_1, y_1) \neq (x_2, y_2)$, then $y_1 \neq y_2$, 3) the set Q is dense in D, 4) the set $Q \cap [A^1_k, B^1_k]$ is dense in $[A^1_k, B^1_k]$, where

$$A^1_k = \left(\frac{1}{k}, 0\right) \quad \text{and} \quad B^1_k = \left(\frac{1}{k}, 1\right), \quad k = 2, 3, \dots,$$

5) the set $Q \cap [A^2_k, B^2_k]$ is dense in $[A^2_k, B^2_k]$, where

$$A_k^2 = \left(\frac{k}{k+1}, 0\right) \quad \text{and} \quad B_k^2 = \left(\frac{k}{k+1}, 1\right), \quad k = 2, 3, \dots$$

Set $p^- = (0, 0)$, $p^+ = (1, 0)$ and $T_2(R) = Q \cup \{p^-, p^+\}$. The topology on $T_2 = T_2(R)$ is defined as follows: The points (x, y) of Q for which $x \neq 1/k$, $x \neq k/(k + 1)$, $k = 2, 3, \dots$ are isolated. The sets

$$O_n(p^-) = \left\{ (x, y) \in T_2 : 0 \leqq x < \frac{1}{n}, 0 \leqq y < \frac{1}{n} \right\}, \quad n = 1, 2, \dots$$

are a basis of open neighbourhoods of p^-.

The sets

$$O_n(p^+) = \left\{ (x, y) \in T_2 : 1 - \frac{1}{n} < x \leqq 1, 0 \leqq y < \frac{1}{n} \right\}, \quad n = 1, 2, \dots,$$

are a basis of open neighbourhoods of p^+.

For every point $(1/(2k + 1), r) \in Q$, $k = 1, 2, \dots$ the sets

$$O_n\left(\frac{1}{2k + 1}, r\right) = \left\{ (x, y) \in T_2 : \frac{1}{2k + 2} < x < \frac{1}{2k}, r - \frac{1}{n} < y < r + \frac{1}{n} \right\},$$

$n = 1, 2, \dots$, are a basis of open neighbourhoods of $(1/(2k + 1), r)$. For every point $(2k/(2k + 1), r) \in Q$, $k = 1, 2, \dots$ the sets

$$O_n\left(\frac{2k}{2k + 1}, r\right) = \left\{ (x, y) \in T_2 : \frac{2k - 1}{2k} < x < \frac{2k + 1}{2k + 2}, r - \frac{1}{n} < y < r + \frac{1}{n} \right\},$$

$n = 1, 2, \dots$ are a basis of open neighbourhoods of $(2k/(2k+1), r)$. As a basis of open neighbourhoods for the points $(1/2k, r)$ and $((2k- 1)/2k, r)$ of Q, $k = 1, 2, \dots$ we take the sets $Q \cap U$, where U is a sufficiently small open neighbourhood of the points $(1/2k, r)$ and $((2k - 1)/2k, r)$ respectively, in E^2.

It can be easily verified that $T_2(R)$ has all the previously mentioned properties.

The following corollaries are direct consequences of Theorem 1.

COROLLARY 1. *For every countable Hausdorff (or Urysohn) space X, there exists a connected, locally connected countable Hausdorff (or Urysohn) almost regular space, containing X as a closed subspace.*

It should be noted that this corollary answers the question, whether it is possible to construct a connected, locally connected, countable Urysohn and almost regular space, posed by G. X. Ritter [**24**].

COROLLARY 2. *For every regular space X, there exists a regular space Y, containing X as a closed subspace and having the following properties*:

a) *Every continuous real-valued function of Y is constant.*

b) *For every point $y \in Y$ and for every open neighbourhood U of y, there exists an open neighbourhood V of y such that $V \subseteq U$ and every continuous real-valued function of V is constant.*

Definition. A point a of a connected space X is said to be a *dispersion point* if the space $X \setminus \{a\}$ is totally disconnected.

THEOREM 2. *For every regular (or Hausdorff, or Urysohn) totally disconnected space X and for every point a of X, there exists a connected regular (or Hausdorff, or Urysohn) space $I(X)$, containing X as a closed subspace, having the point a as the dispersion point and*

$$|I(X)| = \max\{ |X|, \aleph_1\}$$

(*or* $|I(X)| = |X|$, *for Hausdorff, or Urysohn spaces*).

Proof. Let R be the space of real-numbers and $T_1(R)$, (or $T_2(R)$, in case X is Hausdorff, or Urysohn) the space constructed in Theorem 1. We construct the space $I(X, R)$ choosing the set Λ_0 to be all pairs (x, y) for which $x \in X \setminus \{a\}$, $y = a$ and the set Λ_n, $n = 1, 2, \ldots$, all pairs (x, y) for which

$$x \in I^n(X, \Lambda_{n-1}) \setminus \{a\}, \quad y = a.$$

Set $I(X) = I(X, R)$. By Theorem 1, the space $I(X)$ is R-monolithic regular (or Hausdorff, or Urysohn), containing X as a closed subspace, and at the same time,

$$|I(X)| = \max\{ |X|, \aleph_1\},$$

(or $|I(X)| = |X|$). The space $I(X)$ is connected since it is R-monolithic. Hence in order to prove the theorem, it suffices to show that the space $I(X) \setminus \{a\}$ is totally disconnected.

First, we construct a continuous map f of $I(X) \setminus \{a\}$ into $X \setminus \{a\}$ such that $f(x) = x$, for every $x \in X \setminus \{a\}$. Let

$$x \in I(X) \setminus \{a\}.$$

If $x \in I(X) \setminus X$, then there exists a sequence of pairs $\lambda_0 = (x_0, a), \ldots, \lambda_n = (x_n, a)$, such that $\lambda_i \in \Lambda_i$, $i = 0, 1, \ldots, n$ and

$$x_i \in J^{\lambda_{i-1}}, \quad i = 1, \ldots, n, \quad x_0 \in X, x \in J^{\lambda_n}.$$

Obviously, this sequence is uniquely determined. We set $f(x) = x_0$, for every $x \in I(X)\backslash X$ and $f(x) = x$, for every $x \in X\backslash\{a\}$. From the definition of the topology on $I(X)$, it easily follows that the map is continuous.

Similarly, for every $n = 1, 2, \ldots$, we construct the continuous map f_n of $I(X)\backslash\{a\}$ into $I^n(X, \Lambda_{n-1})\backslash\{a\}$ such that

$$f(x) = x \quad \text{for every } x \in I^n(X, \Lambda_{n-1})\backslash\{a\}.$$

We observe that the space $T_1(R)$ (as well as the space $T_2(R)$) is totally disconnected. From this it is easy to prove that the space $I^1(X, \Lambda_0)$ is totally disconnected. Hence, for every $n = 1, 2, \ldots$, the space $I^n(X, \Lambda_{n-1})$ is totally disconnected.

We now prove that $I(X)\backslash\{a\}$ is totally disconnected. Let M be a connected subspace of $I(X)\backslash\{a\}$. If $x, y \in M$, $x \neq y$ then there exists an integer n such that

$$x, y, \in I^n(X, \Lambda_{n-1})\backslash\{a\}.$$

But then the set $f_n(M)$ is a connected subspace of the space $I^n(X, \Lambda_{n-1})$ containing at least two distinct points, which is a contradiction.

COROLLARY 3. *Every countable totally disconnected Hausdorff, (or Urysohn) space can be embedded in a countable connected almost regular, Hausdorff, (or Urysohn) space with a dispersion point.*

COROLLARY 4. *Every regular totally disconnected space can be embedded in a connected regular space with a dispersion point.*

THEOREM 3. *For every Moore space X, there exists an R-monolithic, locally R-monolithic Moore space $I(X)$, containing X as a closed subspace, where R is the set of real-numbers with the usual topology.*

Proof. Consider the Moore space A constructed in **[1]**, which has two distinct points a, b such that $f(a) = f(b)$, for every continuous real-valued function f of A. We construct the space $I(X)$, (see Section 4) considering: 1) $T = A$, $a = p^-$, $b = p^+$, 2) The set Λ_0 to be dense in $X \times X\backslash\Delta(X)$ and the set Λ_n, $n = 1, 2, \ldots$, dense in

$$I^n(X, \Lambda_{n-1}) \times I^n(X, \Lambda_{n-1})\backslash\Delta(I^n(X, \Lambda_{n-1})).$$

By Theorem 1, the space $I(X)$ is R-monolithic, locally R-monolithic, containing X as a closed subspace. In order to prove that $I(X)$ is Moore, we will use the following necessary and sufficient condition which characterizes such a space, **[29]**. "A space Y is Moore, if and only if, it is regular and there exists a sequence $w_1, w_2, \ldots$, of open coverings of Y, such that if U is an open set and $y \in U$, then there exists an integer n for which $\mathrm{st}_y w_n \subseteq U$, where $\mathrm{st}_y w_n = \{U \in w_n: y \in U\}$" We can assume that if $m \leqq n$, then w_n is inscribed in w_m.

Let $\sigma_1, \sigma_2, \ldots$ and $\pi_1, \pi_2, \ldots$ be sequences of open coverings of X, T, respectively, satisfying the above condition. Obviously, for the sequence

$r_1, \rho_2, \ldots$, of open coverings of

$$J = T\backslash\{p^-, p^+\},$$

where

$$\rho_n = \{W\backslash\{p^-, p^+\}: W \in \pi_n\},$$

the condition is satisfied. We will construct a sequence $w_1, w_2, \ldots$ of open coverings of $I(X)$, also satisfying the condition.

Set

$$H_n = \mathrm{st}_{p^-}\pi_n\backslash\{p^-\}, \quad G_n = \mathrm{st}_{p^+}\pi_n\backslash\{p^+\}.$$

We assume that the covering w_n, $n = 1, 2, \ldots$ consists of sets of the form $O(U, H_n, G_n)$, where U belongs to σ_n, or to ρ_n^λ,

$$\lambda \in \bigcup_{i=0}^{\infty} \Lambda_i.$$

We will prove that $w_1, w_2, \ldots$ is the required sequence. Let $y \in I(X)$, U be an open neighbourhood of y and

$$y \in O(V, H, G) \subseteq U,$$

where $V \in B_n^n$ (see Section 4). First, suppose that $y \in X$. Then there exists an integer n such that:

$$1)\ \mathrm{st}_y\sigma_n \subseteq V, \quad 2)\ H_n \subseteq H, \quad 3)\ G_n \subseteq G,$$

which imply that

$$\mathrm{st}_y w_n \subseteq O(V, H, G) \subseteq U.$$

Now suppose that

$$y \in I^1(X, \Lambda_0)\backslash X.$$

Hence there exists $\lambda = (c, d) \in \Lambda_0$, such that $y \in J^\lambda$. Therefore, there exists an integer n such that:

$$1)\ \mathrm{st}_y(\rho_n)^\lambda \subseteq V, \quad 2)\ H_n \subseteq H, \quad 3)\ G_n \subseteq G,$$

$$4)\ d \notin \mathrm{st}_c\sigma_n, \quad 5)\ y \notin H_n \cup G_n,$$

which imply that

$$\mathrm{st}_y w_n \subseteq O(V, H, G) \subseteq U.$$

The existence of an integer n such that

$$\mathrm{st}_y w_n \subseteq O(V, H, G) \subseteq U,$$

in case

$$y \in I^{k+1}(X, \Lambda_k)\backslash I^k(X, \Lambda_{k-1}),$$

can be shown by induction. Consequently, $I(X)$ is a Moore space.

Problem. Let R be a space with topological property P. Does an R-monolithic, (locally R-monolithic) space, with property P exist?

The problem seems to be interesting in case P is for example: 1) "The space is countable", 2) "The space is Moore", 3) "The space has the first axiom of countability", 4) "The space has the second axiom of countability".

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