A Note on the Capelli Operators associated with a Symmetric Matrix

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Introduction.

In the *Proceedings of the Edinburgh Mathematical Society*, 1948, there appear two papers by Lars Gårding and Turnbull respectively (Gårding [1], Turnbull [2]) which formulate the theory of Cayley and Capelli operators associated with symmetric matrices. Turnbull derives the modification, appropriate to symmetric matrices, of Capelli's Theorem, which states that (taking a third order operator for the sake of ease in writing)

$$\left(xyz \,\middle|\, \frac{\partial}{\partial x}\,\frac{\partial}{\partial y}\,\frac{\partial}{\partial z}\right) \equiv \sum_{i,j,k} (xyz)_{ijk}\left(\frac{\partial}{\partial x}\,\frac{\partial}{\partial y}\,\frac{\partial}{\partial z}\right)_{ijk} = \begin{vmatrix} x\dfrac{\partial}{\partial x}+2 & x\dfrac{\partial}{\partial y} & x\dfrac{\partial}{\partial z} \\ y\dfrac{\partial}{\partial x} & y\dfrac{\partial}{\partial y}+1 & y\dfrac{\partial}{\partial z} \\ z\dfrac{\partial}{\partial x} & z\dfrac{\partial}{\partial y} & z\dfrac{\partial}{\partial z} \end{vmatrix},$$

where the symbol $(xyz)_{ijk}$ stands for the determinant

$$\begin{vmatrix} x_i & y_i & z_i \\ x_j & y_j & z_j \\ x_k & y_k & z_k \end{vmatrix},$$

with a similar meaning for the determinantal differential operator, while the symbols $x\dfrac{\partial}{\partial x}$, $x\dfrac{\partial}{\partial y}$, ... are polarisations (Capelli [1]; *cf.* Turnbull [1], p. 116). Gårding's theorem deals with the effect of such modified Capelli operators on powers of the determinant of the symmetric matrix in question. The subject of this note is an alternative derivation of the modified Capelli theorem and of Gårding's theorem.

§ 1. *Factorisation of a Symmetric Matrix.*

Let $X = [x_{ij}]$ be an arbitrary symmetric matrix of m rows whose elements are independent variables, apart from the condition $x_{ij} = x_{ji}$. It is always possible to find a square matrix Y of order $m \times m$ such that $Y'Y = X$. In fact, if H is an orthogonal matrix such that $H'XH = \Lambda$ is a diagonal matrix, the elements of H and Λ being algebraic functions of the x_{ij}, then

$Y = HM\sqrt{\Lambda}H'$ is a matrix of the required form, where M is an arbitrary orthogonal matrix. The y_{ij} are thus expressed as functions of the $\frac{m^2+m}{2}$ variables x_{ij} along with the $m^2 - \frac{m^2+m}{2}$ independent elements of M, m^2 independent variables in all. Conversely the equations $X = Y'Y$ and $M = H'YH\sqrt{\Lambda^{-1}}$ express the x_{ij} and the independent elements of M as functions of the y_{ij}. The y_{ij} are therefore m^2 independent variables. § 2 below will be concerned with the relation between partial differentiation with respect to the x_{ij} and partial differentiation with respect to the y_{ij}. The matrix M does not come into the discussion again, having been introduced simply to check the independence of the y_{ij}.

Let $y_{m+1}, y_{m+2}, \ldots, y_{m+k}$ be a further set of m-ary vectors having as elements mk further independent variables and suppose that the variables x_{ij} $(i, j = 1, 2, \ldots, m+k)$ are defined by the equations

$$x_{ij} = \sum_{h=1}^{m} y_{hi} y_{hj}.$$

The x_{ij} for i and j not greater than m are, of course, simply the elements of X, but it is obvious that the x_{ij} for i and j exceeding m may not, in general, be treated as independent variables. For example

$$\begin{vmatrix} x_{11} & x_{12} & \cdots & x_{1r} \\ x_{21} & x_{22} & \cdots & x_{2r} \\ \vdots & \vdots & & \vdots \\ x_{r1} & x_{r2} & \cdots & x_{rr} \end{vmatrix}$$

is a zero determinant if $r > m$, but would certainly not vanish if all the x_{ij} appearing were independent variables.

On the other hand if $\phi(x)$ is a polynomial in the x_{ij} which contains the elements of each vector $y_{m+1}, \ldots, y_{m+k}$ linearly, then the vanishing of $\phi(x)$ identically in the y_{ij} implies its vanishing identically in the x_{ij}, regarded as independent variables. To prove this, define the polar operations

$$D_{ij} = \left(y_i \frac{\partial}{\partial y_j}\right) = \sum_{h=1}^{m} y_{hi} \frac{\partial}{\partial y_{hj}}$$

and consider the expression

$$\begin{vmatrix} D_{11}+m+k-1 & D_{12} & D_{13} & \cdots & D_{1,m+k} \\ D_{21} & D_{22}+m+k-2 & D_{23} & \cdots & D_{2,m+k} \\ \vdots & \vdots & \vdots & & \vdots \\ D_{m+k,1} & D_{m+k,2} & D_{m+k,3} & \cdots & D_{m+k,m+k} \end{vmatrix} \phi(x). \quad (1)$$

(1) is, by Capelli's theorem, a linear combination with polynomials in the x_{ij} as coefficients of $(m+k)$-rowed determinants of the form

$$\begin{vmatrix} x_{1i} & x_{1j} & \dots & x_{1h} \\ x_{2i} & x_{2j} & \dots & x_{2h} \\ \vdots & \vdots & & \vdots \\ x_{m+k,i} & x_{m+k,j} & \dots & x_{m+k,h} \end{vmatrix}.$$

These determinants all vanish identically in the y_{ij}, since each x_{ij} is an inner product of two m-ary vectors, but, as pointed out above, they do not vanish identically in the x_{ij} unless, of course, they contain repeated columns. The only determinant of this form which does not have repeated columns is obtained by putting $i, j, \dots, h$ equal to $1, 2, \dots, m+k$. But the resulting determinant would be quadratic in the elements of y_{m+k}, whereas (1) is certainly linear in these variables, and so this determinant must have zero coefficient. The expression (1) therefore vanishes identically in the x_{ij}.

Now expand the determinantal operator in (1), keeping the individual operators in the same order from left to right as the columns from which they are selected. The effect of the leading term

$$(D_{11}+m+k-1)(D_{22}+m+k-2)\dots(D_{m+k-1,m+k-1}+1)D_{m+k,m+k}$$

on $\phi(x)$ is simply to multiply it by a non-zero numerical factor. The other terms in the expansion all have as last factor to the right (ignoring elements from the main diagonal, which are equivalent in effect to numerical factors) an operator D_{ij} from above the diagonal, which decreases the degree to which the suffix j occurs in $\phi(x)$ by one, while increasing the degree to which i occurs by one. Thus the statement that (1) vanishes identically in the x_{ij} is equivalent to the statement that $\phi(x)$ is equal, identically in the x_{ij}, to a linear combination of polynomials in the x_{ij}, and each of these new polynomials has the property that the degree to which a certain suffix j occurs is less than that to which it occurs in $\phi(x)$, while the degree of occurrence of an earlier suffix i is higher than in $\phi(x)$. The new polynomials, having been derived from $\phi(x)$ by polarisation, must vanish when $\phi(x)$ vanishes. If the Capelli theorem is now applied to these new polynomials, and then to the further polynomials so introduced, and so on, a stage will finally be reached where $\phi(x)$ will be expressed as an aggregate of polars of polynomials in the x_{ij} $(i, j = 1, 2, \dots, m)$, *i.e.* in the elements of X, each of which is itself a polar of $\phi(x)$, and so each of which vanishes when $\phi(x)$ vanishes. But polynomials in the elements of X which vanish must do so identically in the x_{ij}, and since, moreover, the new expression for $\phi(x)$ has been obtained by a step by step process in each step of which the transformation holds identically in the x_{ij}, it follows that if $\phi(x)$ vanishes, it does so identically in the x_{ij}.

The foregoing proof is really a special case of the proof of the second fundamental theorem for orthogonal invariants (*cf.* Weyl [1], p. 75). And so the result which has been explicitly derived above could be regarded as a consequence of this theorem.

§ 2. *Differential Operators Associated with a Symmetric Matrix.*

Let X be an arbitrary m-rowed symmetric matrix written as $Y'Y$, and let f be a polynomial in its elements $x_{ij} = \Sigma\, y_{hi} y_{hj}$. Then

$$\frac{\partial f}{\partial y_{pq}} = \sum_{i \leqslant j} \frac{\partial f}{\partial x_{ij}} \frac{\partial x_{ij}}{\partial y_{pq}} = \sum_{i<q} \frac{\partial f}{\partial x_{iq}} y_{pi} + \sum_{j>q} \frac{\partial f}{\partial x_{qj}} y_{pj} + 2 \frac{\partial f}{\partial x_{qq}} y_{pq}$$

$$= \sum_{i=1}^{m} y_{pi}(1+\delta_{qi}) \frac{\partial f}{\partial x_{qi}} \tag{2}$$

where δ_{qi} is the Kronecker δ. Define the operator $\dfrac{\bar{\partial}}{\partial x_{ij}}$ by the equation

$$\frac{\bar{\partial}}{\partial x_{ij}} = \tfrac{1}{2}(1+\delta_{ij}) \frac{\partial}{\partial x_{ij}}.$$

Then (2) can be put in the form

$$\frac{\partial}{\partial y_{pq}} = 2 \sum_i y_{pi} \frac{\bar{\partial}}{\partial x_{iq}}.$$

If $p \neq r$, $q \neq s$, then a second differentiation gives

$$\frac{\partial^2}{\partial y_{pq} \partial y_{rs}} = 4 \sum_{i,j} y_{pi} y_{rj} \frac{\bar{\partial}^2}{\partial x_{qi} \partial x_{sj}},$$

and forming the determinant

$$\left(\frac{\partial}{\partial y_q} \frac{\partial}{\partial y_s}\right)_{p,r} = \frac{\partial^2}{\partial y_{pq} \partial y_{rs}} - \frac{\partial^2}{\partial y_{ps} \partial y_{rq}}$$

we find that

$$\left(\frac{\partial}{\partial y_q} \frac{\partial}{\partial y_s}\right)_{p,r} = 4 \sum_{i,j} (y_i y_j)_{p,r} \left(\frac{\bar{\partial}}{\partial x_q} \frac{\bar{\partial}}{\partial x_s}\right)_{i,j}. \tag{3}$$

Multiply by the determinant $(y_h y_k)_{p,r}$ and sum with respect to the pair p, r:

$$\sum_{p,r} (y_h y_k)_{p,r} \left(\frac{\partial}{\partial y_q} \frac{\partial}{\partial y_s}\right)_{p,r} = 4 \sum_{i,j} (x_h x_k)_{i,j} \left(\frac{\bar{\partial}}{\partial x_q} \frac{\bar{\partial}}{\partial x_s}\right)_{i,j},$$

i.e. in the usual bi-determinant notation

$$\left(y_h y_k \,\middle|\, \frac{\partial}{\partial y_q} \frac{\partial}{\partial y_s}\right) = 2^2 \left(x_h x_k \,\middle|\, \frac{\bar{\partial}}{\partial x_q} \frac{\bar{\partial}}{\partial x_s}\right).$$

Similarly the equation for the third order operators is

$$\left(y_i y_j y_k \,\middle|\, \frac{\partial}{\partial y_q}\frac{\partial}{\partial y_r}\frac{\partial}{\partial y_s}\right) = 2^3\left(x_i x_j x_k \,\middle|\, \frac{\bar\partial}{\partial x_q}\frac{\bar\partial}{\partial x_r}\frac{\bar\partial}{\partial x_s}\right),$$

and so on for any order, the equation for the operators of r-th order involving the numerical factor 2^r on the right-hand side. In particular the first order case is

$$D_{hq} = 2\bar{D}_{hq}$$

where D_{hq} is defined as $\sum_{p=1}^{m} y_{ph}\frac{\partial}{\partial y_{pq}}$ and $\bar{D}_{hq}$ as $\sum_{p=1}^{m} x_{ph}\frac{\bar\partial}{\partial x_{pq}}$.

§3. *The Modified Capelli Theorem.*

Capelli's theorem, taking the case of third order operators for simplicity, states that

$$\left(y_1 y_2 y_3 \,\middle|\, \frac{\partial}{\partial y_1}\frac{\partial}{\partial y_2}\frac{\partial}{\partial y_3}\right) = \begin{vmatrix} D_{11}+2 & D_{12} & D_{13} \\ D_{21} & D_{22}+1 & D_{23} \\ D_{31} & D_{32} & D_{33} \end{vmatrix}.$$

Expressing these differential operators in terms of the x_{ij} and using the results of §2, we have

$$\left(x_1 x_2 x_3 \,\middle|\, \frac{\bar\partial}{\partial x_1}\frac{\bar\partial}{\partial x_2}\frac{\bar\partial}{\partial x_3}\right) = \begin{vmatrix} \bar{D}_{11}+1 & \bar{D}_{12} & \bar{D}_{13} \\ \bar{D}_{21} & \bar{D}_{22}+\frac{1}{2} & \bar{D}_{23} \\ \bar{D}_{31} & \bar{D}_{32} & \bar{D}_{33} \end{vmatrix},$$

and this is the modified Capelli theorem as required; in general, for a Capelli operator of r-th order, the numbers $\frac{r-1}{2}, \frac{r-2}{2}, \ldots, \frac{3}{2}, 1, \frac{1}{2}$ are to be added to the 1st, 2nd, ..., $(r-1)$-th diagonal elements, respectively, in the determinant on the right.

§4. *Gårding's Theorem.*

To establish Gårding's Theorem, I shall work out the proof for a fourth order determinant operated on by a second order operator: this makes quite clear the general principle. Let X be an arbitrary 4-rowed symmetric matrix, and let $X = Y'Y$ where Y is an arbitrary matrix of order 4×4 whose elements are independent variables. Let

$$|X| = (x_1 x_2 x_3 x_4), \quad |Y| = (y_1 y_2 y_3 y_4).$$

It is required to evaluate $\left(\frac{\bar\partial}{\partial x_1}\frac{\bar\partial}{\partial x_2}\right)_{i,j}(x_1 x_2 x_3 x_4)^r$, where i, j are any two distinct numbers from the set 1, 2, 3, 4. Introduce two new quaternary column vectors whose elements are independent variables, namely

$$z_1 = \{z_{11} z_{21} z_{31} z_{41}\}, \quad z_2 = \{z_{12} z_{22} z_{32} z_{42}\},$$

and define

$$t_{ij} = \sum_{h=1}^{4} y_{hi} z_{hj}.$$

Then, multiplying (3) by $(z_1 z_2)_{p,r}$ and summing with respect to the pairs p, r, we find that

$$\left(z_1 z_2 \middle| \frac{\partial}{\partial y_q} \frac{\partial}{\partial y_s}\right) = 2^2 \left(t_1 t_2 \middle| \frac{\bar{\partial}}{\partial x_q} \frac{\bar{\partial}}{\partial x_s}\right).$$

Hence

$$\begin{aligned} 2^2 \left(t_1 t_2 \middle| \frac{\bar{\partial}}{\partial x_1} \frac{\bar{\partial}}{\partial x_2}\right) (x_1 x_2 x_3 x_4)^r &= \left(z_1 z_2 \middle| \frac{\partial}{\partial y_1} \frac{\partial}{\partial y_2}\right) (y_1 y_2 y_3 y_4)^{2r} \\ &= 2r(2r+1)(y_1 y_2 y_3 y_4)^{2r-1} (z_1 z_2 y_3 y_4) \quad \text{(Turnbull [1], p. 115, ex. 2)} \\ &= 2r(2r+1)(x_1 x_2 x_3 x_4)^{r-1} (y_1 y_2 y_3 y_4)(z_1 z_2 y_3 y_4) \\ &= 2r(2r+1)(x_1 x_2 x_3 x_4)^{r-1} (t_1 t_2 x_3 x_4). \qquad (4) \end{aligned}$$

Since (4) is linear in the elements of each of the vectors z_i, which here correspond to the vectors $y_{m+1}, \ldots, y_{m+k}$ of § 1, it follows from the reasoning in that paragraph that the t_{ij} may be treated as independent variables in (4). Equate the coefficients of the minor $(t_1 t_2)_{i,j}$ on both sides of (4):

$$\left(\frac{\bar{\partial}}{\partial x_1} \frac{\bar{\partial}}{\partial x_2}\right)_{i,j} (x_1 x_2 x_3 x_4)^r = r(r+\tfrac{1}{2})(x_1 x_2 x_3 x_4)^{r-1} (x_3 x_4)_{h,k}$$

where $(x_3 x_4)_{h,k}$ is the Laplacian cofactor of $(x_1 x_2)_{i,j}$ in $(x_1 x_2 x_3 x_4)$.

The general result for an arbitrary m-rowed symmetric matrix X is

$$\left(\frac{\bar{\partial}}{\partial x_i} \frac{\bar{\partial}}{\partial x_j} \cdots \frac{\bar{\partial}}{\partial x_k}\right)_{p,q,\ldots,s} |X|^r = r(r+\tfrac{1}{2})(r+1) \ldots \left(r + \frac{l-1}{2}\right) |X|^{r-1} . \Delta$$

where the differential operator is an l-rowed determinant ($l \leqslant m$), and Δ is the Laplacian cofactor in $|X|$ of $(x_i x_j \ldots x_k)_{p,q,\ldots,s}$; this last equation is the statement of Gårding's Theorem.

REFERENCES.

Capelli [1] : " Über die Zurückführung des Cayleyschen Operatoren Ω auf gewöhnliche Polaroperatoren ", *Math. Ann.*, 29 (1887), 331–338.

Gårding [1]: "Extension of a Formula by Cayley to Symmetric Determinants", *Proc. Edinburgh Math. Soc.* (2), 8 (1948), 73–75.

Turnbull [1] : *Theory of Determinants, Matrices and Invariants* (Second Edition, 1945).

——— [2] : " Symmetric Determinants and the Cayley and Capelli Operators", *Proc. Edinburgh Math. Soc.* (2), 8 (1948), 76–86.

Weyl [1] : *The Classical Groups* (Princeton, 1939).

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