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ON AN INEQUALITY OF S. BERNSTEIN

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1. Introduction and statement of results. Let $R > 1$ and denote by $\mathscr{E}_R$ the ellipse

$$(1)\qquad \left\{ z = x + iy : \frac{x^2}{\left(\frac{R+R^{-1}}{2}\right)^2} + \frac{y^2}{\left(\frac{R-R^{-1}}{2}\right)^2} = 1 \right\}.$$

If P_n is a polynomial of degree at most n such that

$$(2)\qquad \max_{-1 \leqq x \leqq 1} |P_n(x)| \leqq 1,$$

then ([**2**]; also see [**13**, p. 337] and [**9**, p. 158, Prob. No. 270])

$$(3)\qquad \max_{z \in \mathscr{E}_R} |P_n(z)| \leqq R^n.$$

The standard proof of this well known result runs as follows. The function

$$(4)\qquad f(z) := z^n P_n\left(\frac{z+z^{-1}}{2}\right)$$

is entire and in view of (2) we have

$$\max_{|z|=1} |f(z)| \leqq 1.$$

Hence by the maximum modulus principle

$$(5)\qquad \max_{|z|=1/R<1} |f(z)| \leqq 1,$$

which is clearly equivalent to (3).

Here we wish to discuss how sharp the estimate (3) happens to be and to prove some results about polynomials satisfying $z^n p(1/z) \equiv p(z)$ which are relevant in this connection. In fact, the above function f is a polynomial satisfying

$$(6)\qquad z^{2n} f(1/z) \equiv f(z).$$

On the other hand, if n $(=2m)$ is even, then to every polynomial f satisfying $z^n f(1/z) \equiv f(z)$ there corresponds a polynomial p of degree at most m such that

$$f(z) \equiv z^m p\left(\frac{z+z^{-1}}{2}\right).$$

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A polynomial $f \not\equiv 0$ satisfying (6) cannot be a constant and so in (5) it is not possible that $\max_{|z|=1/R<1} |f(z)|$ be equal to 1. The sign of equality in (3) is therefore ruled out. But then, what is the best that we can say? This and some related questions have been considered in the past but apparently all the known results concern polynomials which are, in addition, real for real values of z.

It was shown by Duffin and Schaeffer [**6**] that if P_n is a polynomial of degree at most n satisfying (2) and $P_n(z)$ is real for real z, then

$$\max_{z \in \mathscr{E}_R} |P_n(z)| \leqq \tfrac{1}{2}(R^n + R^{-n}). \tag{7}$$

This inequality is sharp in the sense that the nth Chebyshev polynomial of the first kind

$$T_n(z) := 2^{n-1} \prod_{\nu=1}^{n} \{z - \cos((\nu - \tfrac{1}{2})\pi/n)\}$$

which assumes real values for real z, satisfies (2) whereas

$$|T_n(\zeta)| = \tfrac{1}{2}(R^n + R^{-n})$$

at precisely $2n$ points $\zeta \in \mathscr{E}_R$ namely

$$\zeta := \tfrac{1}{2}\{\omega R + (\omega R)^{-1}\}$$

where ω is any of the $2n$-th roots of unity.

Erdös [**7**, Theorem 7] proved the remarkable fact that if P_n is a polynomial of degree at most n satisfying (2), then

$$|P_n(z)| \leqq |T_n(z)| \quad \text{for } |z| \geqq 1 \tag{7'}$$

provided $P_n(z)$ is real for real z.

If P_n is a polynomial of degree at most n satisfying (2) then Voronovskaja and Zinger [**15**] determined max $|\text{Re } P_n(z)|$ and max $|\text{Im } P_n(z)|$ for a given complex z under the assumption that $P_n(z)$ is real for real z, and Zinger [**16**], determined the corresponding maxima for the derivatives of P_n.

None of the above results seems to have a trivial extension to the case when $P_n(z)$ is not necessarily real for real z. Inequality (7) does remain true if $n = 1$ (see [**12**, pp. 229–230]) but may not hold for $n \geqq 2$. In fact, we shall show:

THEOREM 1. *There exists a polynomial P_n of degree n such that*

$$\max_{-1 \leqq x \leqq 1} |P_n(x)| = 1$$

whereas

$$\max_{z \in \mathscr{E}_R} |P_n(z)| \geqq \tfrac{1}{2}R^n + \frac{\sqrt{2} - 1}{2} R^{n-2}. \tag{8}$$

There is no reason to believe that the coefficient $(\sqrt{2}-1)/2$ of R^{n-2} is the best possible. In fact, in the case $n = 1$ it can be replaced by $\frac{1}{2}$.

In the other direction we prove

THEOREM 2. *If P_n is a polynomial of degree at most n such that*

$$\max_{-1\leqq x\leqq 1} |P_n(x)| \leqq 1,$$

then

$$\max_{z\in\mathcal{E}_R} |P_n(z)| \leqq \tfrac{1}{2}R^n + \frac{5+\sqrt{17}}{4} R^{n-2}. \tag{9}$$

Here again, it appears to be possible to improve upon the coefficient $(5+\sqrt{17})/4$ of R^{n-2}. Since in the case $n = 1$ the precise answer is $\frac{1}{2}$ one might wonder if $(5+\sqrt{17})/4$ can, in general, be replaced by $\frac{1}{2}$. We are not able to decide this but we can prove the following

THEOREM 3. *Under the conditions of Theorem* 2

$$\max_{z\in\mathcal{E}_R} |P_n(z)| < \frac{1}{2}(R^n + R^{n-2}) + \frac{11}{4} R^{n-4}. \tag{10}$$

Theorem 2 is a simple consequence of the following

THEOREM 4. *If p is a polynomial satisfying*

$$z^n p(1/z) \equiv p(z) \quad \text{for all } z \in \mathbf{C}, \tag{11}$$

and $\max_{|z|=1} |p(z)| \leqq 1$, *then*

$$\max_{|z|=\rho<1} |p(z)| \leqq \frac{1}{2} + \frac{5+\sqrt{17}}{4}\rho^2 \quad \text{if } n \geqq 2. \tag{12}$$

Instead of proving Theorem 4 we shall prove the following equivalent result.

THEOREM 4′. *Under the conditions of Theorem* 4

$$\max_{|z|=R>1} |p(z)| \leqq \frac{1}{2}R^n + \frac{5+\sqrt{17}}{4} R^{n-2} \quad \text{if } n \geqq 2. \tag{12′}$$

Remark 1. For $n = 2$, the coefficient $(5+\sqrt{17})/4$ of R^{n-2} in (12′) can be replaced [**12**, pp. 229–230] by $\frac{1}{2}$. The same remark applies to the coefficient $(5+\sqrt{17})/4$ of ρ^2 in (12).

The sharp version of inequality (10) is already known in the case $n = 1$ whereas for $n \geqq 2$ it (inequality (10)) follows from the following

THEOREM 5. *Under the conditions of Theorem* 4

$$\max_{|z|=R>1} |p(z)| < \frac{1}{2}(R^n + R^{n-2}) + \frac{11}{4} R^{n-4} \quad \text{if } n \geqq 4. \tag{13}$$

From our proofs of (12′) and (13) it will be clear that we are not able to use the full force of the hypothesis $z^n p(1/z) \equiv p(z)$. An essentially similar conclusion holds under a much weaker hypothesis.

THEOREM 6. *If the geometric mean of the moduli of the zeros of a polynomial p of degree at most n is $\geqq 1$, and* $\max_{|z|=1} |p(z)| \leqq 1$, *then*

$$(14)\quad \max_{|z|=R>1} |p(z)| \leqq \begin{cases} \dfrac{1}{2} R + \dfrac{1}{2} & \text{if } n = 1 \\ \dfrac{1}{2} R^n + \dfrac{3 + 2\sqrt{2}}{2} R^{n-2} & \text{if } n \geqq 2. \end{cases}$$

Theorem 6 may be compared with the following result of Ankeny and Rivlin [**1**].

THEOREM A. *If the moduli of the zeros of a polynomial p of degree at most n are all $\geqq 1$, and* $\max_{|z|=1} |p(z)| \leqq 1$, *then*

$$(15)\quad \max_{|z|=R>1} |p(z)| \leqq \tfrac{1}{2}R^n + \tfrac{1}{2}.$$

Polynomials p satisfying (11) were studied by Govil, Jain and Labelle [**8**] who proved that, if in addition, p has all its zeros either in the left half plane or in the right half plane, then

$$(16)\quad \max_{|z|=1} |p'(z)| \leqq \frac{n}{\sqrt{2}} \max_{|z|=1} |p(z)|,$$

$$(17)\quad \max_{|z|=R>1} |p(z)| \leqq \frac{R^n + \sqrt{2} - 1}{\sqrt{2}} \max_{|z|=1} |p(z)|,$$

and

$$(18)\quad \max_{|z|=1} |p'(z)| \geqq \frac{n}{2} \max_{|z|=1} |p(z)|.$$

Inequalities (12′), (13) and (14) may be compared with (17).

Dewan and Govil [**5**] have shown that inequality (18) which is sharp holds for all polynomials p satisfying (11). It is not known if the same can be said about (16). However, the following theorem and its corollary seem to be of interest in this connection.

THEOREM 7. *If p is a polynomial satisfying* (11), *then for $R \geqq 1$ and $0 \leqq \theta < 2\pi$*

$$(19)\quad |p'(Re^{i\theta})| + |p'(Re^{-i\theta})| \leqq nR^{n-1} \max_{|z|=1} |p(z)|.$$

For all $R \geqq 1$, equality holds in (19) *for polynomials of the form $c(z^n + 1)$. If n is even, then polynomials of the form $cz^{n/2}$ are also extremal for $R = 1$.*

COROLLARY 1. *If p is a polynomial satisfying* (11), *then for $R \geqq 1$ and $\alpha > 0$*

(19′) $|p'(\pm R)| \leqq \frac{n}{2} R^{n-1} \max_{|z|=1} |p(z)|$,

(19″) $\frac{1}{2\alpha} \int_{-\alpha}^{\alpha} |p'(Re^{i\theta})| d\theta \leqq \frac{n}{2} R^{n-1} \max_{|z|=1} |p(z)|$,

(19‴) $\frac{1}{2\alpha} \int_{\pi-\alpha}^{\pi+\alpha} |p'(Re^{i\theta})| d\theta \leqq \frac{n}{2} R^{n-1} \max_{|z|=1} |p(z)|$.

In (19′)–(19‴) *equality holds for all $R \geqq 1$ for polynomials of the form $c(z^n + 1)$. In the case of even n, polynomials of the form $cz^{n/2}$ are also extremal for* (19′) *if $R = 1$.*

The following result bears the same relationship to (17) or to (12′), (13) and (14) as (19) does to (16).

THEOREM 8. *If p is a polynomial satisfying* (11), *then for all $\rho \geqq 0$ and $0 \leqq \theta < 2\pi$*

(20) $|p(\rho e^{i\theta})| + |p(\rho e^{-i\theta})| \leqq (\rho^n + 1) \max_{|z|=1} |p(z)|$.

The example $p(z) := c(z^n + 1)$ shows that in (20) equality is possible for all $\rho \geqq 0$ and some values of θ.

COROLLARY 2. *If p is a polynomial satisfying* (11), *then for all $\rho \geqq 0$*

(20′) $|p(\pm\rho)| \leqq \frac{1}{2}(\rho^n + 1) \max_{|z|=1} |p(z)|$.

For even n, equality holds in (20′) for $p(z) := c(z^n + 1)$. The same example shows that the estimate for $|p(\rho)|$ is sharp also for odd n. Unfortunately, the estimate for $|p(-\rho)|$ is not sharp in that case and we can easily replace it by

(20*) $|p(-\rho)| \leqq \frac{1}{2}|\rho^n - 1| \max_{|z|=1} |p(z)|$.

We also prove

THEOREM 9. *Let*

$$\psi(t) := \frac{1 + 2 \sum_{m=1}^{\infty} t^{-2m^2}}{2 \sum_{m=0}^{\infty} t^{-(2m+1)^2/2}} \quad \textit{for } t > 1.$$

If p is a polynomial satisfying (11) *and*

$$\max_{|z|=1} |p(z)| = 1,$$

then

$$(21)\quad \max_{|z|=R>1} |p(z)| \geqq \begin{cases} R^{n/2} & \text{if } n \text{ is even} \\ R^{n/2}\psi(R) & \text{if } n \text{ is odd,} \end{cases}$$

or equivalently

$$(22)\quad \max_{|z|=\rho<1} |p(z)| \geqq \begin{cases} \rho^{n/2} & \text{if } n \text{ is even} \\ \rho^{n/2}\psi(1/\rho) & \text{if } n \text{ is odd.} \end{cases}$$

Inequalities (21) and (22) are sharp in case of even n as is shown by the example $z^{n/2}$.

It will be clear from the proof of Theorem 9 and that of Lemma 4 that $\psi(R) > 1$ for $R > 1$. Besides, from Lemma 1 we can easily deduce that

$$\psi(R) \geqq \tfrac{1}{2}(R^{1/2} + R^{-3/2}) \quad \text{for } R > 1.$$

Hence for odd n inequalities (21) and (22) may be replaced by

$$(21')\quad \max_{|z|=R>1} |p(z)| \geqq \max\{R^{n/2}, \tfrac{1}{2}(R^{(n+1)/2} + R^{(n-3)/2})\}$$

and

$$(22')\quad \max_{|z|=\rho<1} |p(z)| \geqq \max\{\rho^{n/2}, \tfrac{1}{2}(\rho^{(n-1)/2} + \rho^{(n+3)/2})\}$$

respectively. The example $\frac{1}{2}(z^{(n-1)/2} + z^{(n+1)/2})$ shows that the right-hand side of (21′) cannot be replaced by anything larger than $\frac{1}{2}(R^{(n+1)/2} + R^{(n-1)/2})$. The same example shows that the best we can possibly do as regards the right-hand side of (22′) is $\frac{1}{2}(\rho^{(n-1)/2} + \rho^{(n+1)/2})$.

Remark 2. We observe that the result of Dewan and Govil mentioned above can be deduced from Theorem 9 as well. To see this, let us suppose that (18) is false, i.e., there exists a polynomial p satisfying (11) such that for some $\alpha < 1$

$$\max_{|z|=1} |p'(z)| = \alpha M n/2$$

where $M := \max_{|z|=1} |p(z)|$. Then by a well known property of polynomials, mentioned below as Lemma 2, we have

$$\max_{|z|=t\geqq 1} |p'(z)| \leqq \alpha M \frac{n}{2} t^{n-1}$$

and so for $R > 1$ and $0 \leqq \theta < 2\pi$

$$|p(Re^{i\theta})| \leqq |p(e^{i\theta})| + \left|\int_1^R p'(te^{i\theta})e^{i\theta}dt\right| \leqq M + \alpha M \tfrac{1}{2}(R^n - 1)$$

which is less than $MR^{n/2}$ if $R < ((2-\alpha)/\alpha)^{2/n}$. This contradicts Theorem 9 and so (18) must hold.

2. Lemmas. We have either already used above or will need later the following auxiliary results.

LEMMA 1. *For* $0 < x < 1$, *we have*

$$\text{(23)} \qquad \frac{1 + 2\sum_{m=1}^{\infty} x^{m^2}}{\sum_{m=0}^{\infty} x^{m^2+m}} \geqq 1 + x.$$

Proof. It is easily seen that inequality (23) holds if and only if

$$1 \geqq \sum_{m=1}^{\infty} (x^{m^2+m-1} + x^{m^2+m}) + 2\sum_{m=2}^{\infty} x^{m^2-1}$$
$$= (1 - x)\sum_{m=1}^{\infty} (x^{m^2+m-1} + 2\sum_{k=1}^{m} x^{m^2+m+k-1}),$$

i.e.,

$$\sum_{m=0}^{\infty} x^m \geqq \sum_{m=1}^{\infty} \left(x^{m^2+m-1} + 2\sum_{k=1}^{m} x^{m^2+m+k-1} \right).$$

This latter inequality will be proved if we show that for $N = 1, 2, 3, \ldots$ and $0 < x < 1$

$$\sum_{k=N^2-1}^{N^2+2N-1} x^k \geqq x^{N^2+N-1} + 2\sum_{k=N^2+N}^{N^2+2N-1} x^k.$$

But clearly

$$\sum_{k=N^2-1}^{N^2+2N-1} x^k = \sum_{k=N^2-1}^{N^2+N-2} x^k + x^{N^2+N-1} + \sum_{k=N^2+N}^{N^2+2N-1} x^k$$
$$= x^{-N-1} \sum_{k=N^2+N}^{N^2+2N-1} x^k + x^{N^2+N-1} + \sum_{k=N^2+N}^{N^2+2N-1} x^k$$
$$> x^{N^2+N-1} + 2\sum_{k=N^2+N}^{N^2+2N-1} x^k \quad \text{if } 0 < x < 1.$$

LEMMA 2 [**9**, Part III, Chapter 6, Problem No. 269]. *If* p *is a polynomial of degree at most* n, *then*

$$\max_{|z|=R>1} |p(z)| \leqq R^n \max_{|z|=1} |p(z)|.$$

The following result of van der Corput and Visser [**4**, § 8] is crucial for our proof of Theorems 4′ and 5.

LEMMA 3. *If* $p(z) := \sum_{k=0}^{n} a_k z^k$ *is a polynomial of degree at most* n *such that* $\max_{|z|=1} |p(z)| \leqq 1$, *then*

$$\text{(24)} \qquad 2|a_0|\,|a_n| + \sum_{k=0}^{n} |a_k|^2 \leqq 1.$$

For the proof of Theorem 9 in the case of odd n we shall need

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LEMMA 4. *Let f be holomorphic in a domain containing $\mathscr{E}_R(R > 1)$ and its interior. If $f(0) = 0$ and $|f(z)| \leqq M$ for all z inside $\mathscr{E}_R$, then*

$$(25)\quad |f(t)| \leqq M \frac{2\sum_{l=0}^{\infty} R^{-(2l+1)^2}}{1 + 2\sum_{l=1}^{\infty} R^{-(2l)^2}} \quad \textit{for } -1 \leqq t \leqq 1.$$

Proof. The function

$$z = \psi(w) = \sqrt{k}\, sn\left(\frac{2K}{\pi}\text{arc sin } w\right),$$

where

$$K = \int_0^1 \frac{dt}{\sqrt{(1 - t^2)(1 - k^2t^2)}}$$

is the complete elliptic integral of the first kind with modulus k and the value of Jacobi's parameter q is R^{-4}, maps [**3**, see Example 5 on p. 414] the interior of the ellipse $\mathscr{E}_R$ conformally onto the open disk $|z| < 1$ such that $\psi(0) = 0$. Let $\varphi(z)$ be the inverse mapping. Then $G(z) := f(\varphi(z))$ is holomorphic in $|z| < 1$ with $|G(z)| \leqq M$ and $G(0) = 0$. Hence, by Schwarz's lemma

$$|G(z)| \leqq M|z| \quad \text{for } |z| < 1.$$

In particular (see § 14.7 of [**3**])

$$|f(1)| \leqq M|\psi(1)| = M\sqrt{k} = M\frac{2\sum_{l=0}^{\infty} R^{-(2l+1)^2}}{1 + 2\sum_{l=1}^{\infty} R^{-(2l)^2}},$$

which proves (25) for $t = 1$. In order to see that (25) holds for an arbitrary t in $[-1, 1]$ we may apply the above reasoning to the function $f(tz)$.

3. Proofs of the theorems.

Proof of Theorem 1. Let

$$p_m^*(z) := (Rz^m + iz^{m-1} + iz + R)/(2\sqrt{R^2 + 1}).$$

Then for all real θ

$$|p_m^*(e^{2i\theta})| = |R \cos m\theta + i \cos (m - 2)\theta|/\sqrt{R^2 + 1} \leqq 1 = |p_m^*(1)|,$$

and so

$$(26)\quad \max_{|z|=1} |p_m^*(z)| = 1.$$

Besides

(27) $$\max_{|z|=R>1} |p_m^*(z)| \geqq |p_m^*(iR)| = \frac{1}{2} R^{m-1}\sqrt{R^2+1} \geqq \frac{R^m + (\sqrt{2}-1)R^{m-2}}{2}.$$

Since $z^k + z^{-k}$ can be written as a polynomial of degree k in $(z + z^{-1})/2$ we see that

$$F(z) := z^{-n}p_{2n}^*(z) = (Rz^n + iz^{n-1} + iz^{-n+1} + Rz^{-n})/(2\sqrt{R^2+1})$$

is a polynomial $P_n^*((z + z^{-1})/2)$ of degree n in $(z + z^{-1})/2$. In view of (26) and (27)

$$\max_{-1\leqq x\leqq 1} |P_n^*(x)| = 1 \quad \text{and} \quad \max_{z\in\mathcal{E}_R} |P_n^*(z)| \geqq \frac{R^n + (\sqrt{2}-1)R^{n-2}}{2}.$$

Remark 3. It is clear that for values of R close to 1 the coefficient $(5 + \sqrt{17})/4$ of R^{n-2} in (12′) cannot be replaced by any number smaller than $\frac{1}{2}$ but while proving Theorem 1 we have shown that for no value of $R > 1$ it can be replaced by a number smaller than $(\sqrt{2} - 1)/2$. The same remark applies to the coefficient of ρ^2 in (12).

Proof of Theorem 4. As remarked earlier the stronger inequality

$$\max_{|z|=R>1} |p(z)| \leqq \tfrac{1}{2}(R^n + R^{n-2})$$

is known to be true if $n = 2$. So let $n \geqq 3$.

If $p(z) := \sum_{k=0}^{n} a_k z^k$ is a polynomial satisfying (11), then

(28) $$a_k = a_{n-k} \quad \text{for } 0 \leqq k \leqq n.$$

In particular, $|a_0| = |a_n| = \alpha$ (say) and $|a_1| = |a_{n-1}| = \beta$ (say). If

$$\max_{|z|=1} |p(z)| \leqq 1$$

then according to a result of Visser [**14**]

$$|a_0| + |a_n| \leqq 1$$

and so

(29) $$\alpha \leqq \tfrac{1}{2}.$$

Further, in view of (24), we have

(30) $$\beta \leqq \sqrt{\frac{1-4\alpha^2}{2}}.$$

Now let us write

$$p(z) = a_nz^n + a_{n-1}z^{n-1} + r(z).$$

It is easily checked that

$$(31)\quad |a_n z^n + a_{n-1}z^{n-1}| \leqq \alpha|z|^n + \sqrt{\frac{1-4\alpha^2}{2}}\,|z|^{n-1} \leqq \frac{1}{2}\,|z|^n + \frac{1+2\alpha}{4}\,|z|^{n-2}.$$

Since $r(z)$ is a polynomial of degree $n - 2$ such that

$$\max_{|z|=1} |r(z)| \leqq 1 + \alpha + \sqrt{\frac{1-4\alpha^2}{2}}$$

we may apply Lemma 2 to obtain

$$(32)\quad \max_{|z|=R>1} |r(z)| \leqq \left(1 + \alpha + \sqrt{\frac{1-4\alpha^2}{2}}\right)R^{n-2}.$$

Inequalities (31) and (32) imply that for $|z| = R > 1$

$$|p(z)| \leqq \frac{1}{2}\,R^n + \left(\frac{5}{4} + \frac{3\alpha}{2} + \sqrt{\frac{1-4\alpha^2}{2}}\right)R^{n-2}$$

from which inequality (12′) follows immediately.

Proof of Theorem 5. First let $n \geqq 7$ and write

$$p(z) := a_n z^n + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + a_{n-3}z^{n-3} + s(z)$$

where $s(z)$ is a polynomial of degree at most $n - 4$. If $|a_n| = \alpha$, $|a_{n-1}| = \beta$, $|a_{n-2}| = \gamma$, $|a_{n-3}| = \delta$, then in view of (24) and (28) we have

$$(33)\quad \delta \leqq \sqrt{\frac{1-4\alpha^2-2\beta^2-2\gamma^2}{2}}\,.$$

Using Lemma 2 to estimate $|s(z)|$ we obtain

$$\begin{aligned}(34)\quad \max_{|z|=R>1} |p(z)| &\leqq \alpha R^n + \beta R^{n-1} + \gamma R^{n-2} + \delta R^{n-3}\\ &\quad + (1+\alpha+\beta+\gamma+\delta)R^{n-4} \leqq (R^n + R^{n-4})\alpha\\ &\quad + (R^{n-1} + R^{n-4})\beta + (R^{n-2} + R^{n-4})\gamma\\ &\quad + (R^{n-3} + R^{n-4})\sqrt{\frac{1-4\alpha^2-2\beta^2-2\gamma^2}{2}} + R^{n-4}\\ &\leqq \tfrac{1}{2}\{(R^n + R^{n-4})^2 + 2(R^{n-1} + R^{n-4})^2 + 2(R^{n-2} + R^{n-4})^2\\ &\qquad + 2(R^{n-3} + R^{n-4})^2\}^{1/2} + R^{n-4}\end{aligned}$$

by Schwarz's inequality. Now it is a matter of simple verification that the right hand side of (34) is less than

$$\frac{1}{2}\,(R^n + R^{n-2}) + \frac{11}{4}\,R^{n-4}$$

for $R > 1$.

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Similar reasoning shows that in the case of $n = 5$ we have

$$\max_{|z|=R>1} |p(z)| \leqq \frac{1}{2}(R^5 + R^3) + \frac{9}{4}R.$$

Now let $n = 6$ and write

$$p(z) := a_6(z^6 + 1) + a_5(z^5 + z) + a_4(z^4 + z^2) + a_3z^3.$$

If we set $|a_6| = \alpha$, $|a_5| = \beta$, $|a_4| = \gamma$ and $|a_3| = \delta$, then because of (24) we must have

$$(33') \quad \delta \leqq \sqrt{1 - 4\alpha^2 - 2\beta^2 - 2\gamma^2}$$

and so again using Schwarz's inequality, we obtain

$$\max_{|z|=R>1} |p(z)| \leqq (R^6 + 1)\alpha + (R^5 + R)\beta + (R^4 + R^2)\gamma + R^3\sqrt{1 - 4\alpha^2 - 2\beta^2 - 2\gamma^2} \leqq \tfrac{1}{2}\{(R^6 + 1)^2 + 2(R^5 + R)^2 + 2(R^4 + R^2)^2 + 4R^6\}^{1/2} \leqq \frac{1}{2}(R^6 + R^4) + \frac{9}{4}R^2.$$

In the case of $n = 4$ we can similarly show that

$$\max_{|z|=R>1} |p(z)| \leqq \frac{1}{2}(R^4 + R^2) + \frac{9}{4}.$$

Proof of Theorem 6. For $n = 1$ the result is trivial, whereas for $n \geqq 2$ it can be proved in the same way as (12′); all we need to note is that if $p(z) := \sum_{k=0}^{n} a_k z^k$, then $|a_n| \leqq |a_0|$ and so

$$|a_n| \leqq \tfrac{1}{2} \quad \text{and} \quad |a_{n-1}| \leqq \sqrt{1 - 4|a_n|^2}.$$

Proof of Theorem 7. It is known (see for example [**10**, p. 8]) that if $q(z) := z^n\overline{p(1/\bar{z})}$, then for $R \geqq 1$ and $0 \leqq \theta < 2\pi$

$$(35) \quad |p'(Re^{i\theta})| + |q'(Re^{i\theta})| \leqq nR^{n-1} \max_{|z|=1} |p(z)|.$$

But

$$|q'(Re^{i\theta})| = R^{n-1}\left|\frac{d}{d\theta}\left\{e^{in\theta}\overline{p\left(\frac{1}{R}e^{i\theta}\right)}\right\}\right| = \frac{1}{R}\left|\frac{d}{d\theta}\left\{R^n e^{-in\theta}p\left(\frac{1}{R}e^{i\theta}\right)\right\}\right| = |p'(Re^{-i\theta})|$$

since $z^n p(1/z) \equiv p(z)$. Hence (35) can be written as (19).

Proof of Theorem 8. First assume ρ to be > 1. Then by inequality (5.3) (where there is an obvious misprint) of [**11**] for the special operator $B[p_n(z)] = p_n(z)$, we have

$$(36) \quad |p(\rho e^{i\theta})| + |q(\rho e^{i\theta})| \leqq (\rho^n + 1) \max_{|z|=1} |p(z)|.$$

But

$$|q(\rho e^{i\theta})| = \rho^n \left| p\left(\frac{1}{\rho} e^{i\theta}\right) \right| = |p(\rho e^{-i\theta})| \quad \text{since } z^n p(1/z) \equiv p(z).$$

Hence (36) can be written as (20).

That (20) holds also for $\rho < 1$ is easily seen by using the relationship $z^n p(1/z) \equiv p(z)$.

Proof of (20*). Note that if n is odd then a polynomial $p(z)$ satisfying (11) must vanish at -1. Therefore

$$p(-\rho) = \int_{-1}^{-\rho} p'(t)dt$$

and for $\rho > 1$, the desired result is a simple consequence of (19′). That (20*) holds for $\rho < 1$ as well follows from the fact that $z^n p(1/z) \equiv p(z)$.

Proof of Theorem 9. Let $p(z) := \sum_{k=0}^{n} a_k z^k$ and consider the function $g(z) := z^{-n} p(z^2)$. If $u_k(z) := z^k + z^{-k}$ then in view of (28) we can write $g(z)$ as a linear combination of $u_n(z), u_{n-2}(z), \ldots, u_0(z)$ if n is even and of $u_n(z), u_{n-2}(z), \ldots, u_1(z)$ if n is odd. Since $u_k(z)$ can be expressed as a polynomial $t_k((z + z^{-1})/2)$ of degree k in $(z + z^{-1})/2$ where $t_k(0) = 0$ for odd k we conclude that $g(z)$ is indeed a polynomial $P_n((z + z^{-1})/2)$ of degree n in $(z + z^{-1})/2$ and that $P_n(0) = 0$ if n is odd. The hypothesis $\max_{|z|=1} |p(z)| = 1$ implies that

(37) $\quad \max_{-1 \leqq w \leqq 1} |P_n(w)| = 1.$

Hence, if $E_{\sqrt{R}}$ is the ellipse in the w-plane with foci at -1, $+1$ and semi-axes $\frac{1}{2}(R^{1/2} + R^{-1/2})$, $\frac{1}{2}(R^{1/2} - R^{-1/2})$, then by the maximum modulus principle

(38) $\quad \max_{w \in E_{\sqrt{R}}} |P_n(w)| \geqq 1.$

But

$$\max_{w \in E_{\sqrt{R}}} |P_n(w)| = R^{-n/2} \max_{|z|=\sqrt{R}} |p(z^2)| = R^{-n/2} \max_{|z|=R} |p(z)|$$

and so

(39) $\quad \max_{|z|=R>1} |p(z)| \geqq R^{n/2}.$

For odd n we can do better since in that case $P_n(0) = 0$ and we may use (37) in conjunction with Lemma 4 to obtain the estimate given in (21).

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