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ON THE CAYLEYNESS OF PRAEGER–XU GRAPHS

MARCO BARBIERI✉, VALENTINA GRAZIAN and PABLO SPIGA

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Abstract

We give a sufficient and necessary condition for a Praeger–Xu graph to be a Cayley graph.

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1. Scope of this note

The Praeger–Xu graphs, introduced by Praeger and Xu in [2], have exponentially large groups of automorphisms, with respect to the number of vertices. This fact causes various complications with regard to many natural questions.

In their recent work [1], Jajcay *et al.* gave a sufficient and necessary condition for a Praeger–Xu graph to be a Cayley graph. Explicitly, [1, Theorem 1.1] states that, for any positive integer $n \geq 3$, $n \neq 4$, and for any positive integer $k \leq n-1$, the Praeger–Xu graph $\mathrm{PX}(n,k)$ is a Cayley graph if and only if one of the following holds:

(i) the polynomial $t^n + 1$ has a divisor of degree $n-k$ in $\mathbb{Z}_2[t]$;
(ii) n is even, and there exist polynomials $f_1, f_2, g_1, g_2, u, v \in \mathbb{Z}_2[t]$ such that u, v are palindromic of degree $n-k$, and

$$t^n + 1 = f_1(t^2)u(t) + tg_1(t^2)v(t) = f_2(t^2)v(t) + tg_2(t^2)u(t). \tag{1.1}$$

Our aim here is to prove that (ii) implies (i), thus obtaining the following refinement. (It can be verified that $\mathrm{PX}(4,1)$, $\mathrm{PX}(4,2)$ and $\mathrm{PX}(4,3)$ are Cayley graphs.)

THEOREM 1.1. *For any positive integer $n \geq 3$ and for any positive integer $k \leq n-1$, the Praeger–Xu graph $\mathrm{PX}(n,k)$ is a Cayley graph if and only if the polynomial $t^n + 1$ has a divisor of degree $n-k$ in $\mathbb{Z}_2[t]$.*

Using the factorisation of $t^n + 1$ in $\mathbb{Z}_2[t]$, we give a purely arithmetic condition for the Cayleyness of $\mathrm{PX}(n,k)$. Let φ be the Euler φ-function and, for every positive integer d, let

$$\omega(d) := \min\{c \in \mathbb{N} \mid d \text{ divides } 2^c - 1\}$$

be the multiplicative order of 2 modulo d.

COROLLARY 1.2. *Let a be a nonnegative integer, let b be an odd positive integer, let $n := 2^a b$ with $n \geq 3$ and let k be a positive integer with $k \leq n-1$. The Praeger–Xu graph* $\mathrm{PX}(n,k)$ *is a Cayley graph if and only if k can be written as*

$$k = \sum_{d|b} \alpha_d \omega(d), \quad \textit{for some integers } \alpha_d \textit{ with } 0 \leq \alpha_d \leq \frac{2^a \varphi(d)}{\omega(d)}. \tag{1.2}$$

2. Proof of Theorem 1.1

Suppose (ii) holds. We aim to show that $t^n + 1$ is divisible by a polynomial of degree $n-k$ in $\mathbb{Z}_2[t]$, implying (i). Working in characteristic 2, (1.1) can be written as

$$t^n + 1 = f_1^2(t)u(t) + tg_1^2(t)v(t) = f_2^2(t)v(t) + tg_2^2(t)u(t),$$

in short,

$$t^n + 1 = f_1^2 u + tg_1^2 v = f_2^2 v + tg_2^2 u. \tag{2.1}$$

If $g_1 = 0$ or if $g_2 = 0$, then the result follows from (2.1), and the fact that u and v have degree $n-k$. Therefore, for the rest of the argument, we may suppose that $g_1, g_2 \neq 0$. Moreover, observe that $f_1, f_2 \neq 0$, because t does not divide $t^n + 1$.

We introduce four polynomials $u_e, u_o, v_e, v_o \in \mathbb{Z}_2[t]$ such that

$$u := u_e^2 + tu_o^2, \quad v := v_e^2 + tv_o^2.$$

Substituting these expansions for u and v in (2.1),

$$t^n + 1 = f_1^2 u_e^2 + t^2 g_1^2 v_o^2 + t(f_1^2 u_o^2 + g_1^2 v_e^2),$$

$$t^n + 1 = f_2^2 v_e^2 + t^2 g_2^2 u_o^2 + t(f_2^2 v_o^2 + g_2^2 u_e^2).$$

Recall that n is even. By splitting the equations into even and odd degree terms, we obtain

$$t^n + 1 = f_1^2 u_e^2 + t^2 g_1^2 v_o^2, \quad 0 = t(f_1^2 u_o^2 + g_1^2 v_e^2),$$

$$t^n + 1 = f_2^2 v_e^2 + t^2 g_2^2 u_o^2, \quad 0 = t(f_2^2 v_o^2 + g_2^2 u_e^2).$$

Set $m := n/2$. Since we are working in characteristic 2,

$$t^m + 1 = f_1 u_e + tg_1 v_o, \quad t^m + 1 = f_2 v_e + tg_2 u_o, \tag{2.2}$$

$$f_1 u_o = g_1 v_e, \qquad f_2 v_o = g_2 u_e. \tag{2.3}$$

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Since u and v are palindromic by assumption, we get $1 = u(0) = u_e(0)$ and $1 = v(0) = v_e(0)$. In particular, both u_e and v_e are not zero. From (2.2) and (2.3),

$$f_1 = \frac{t^m + 1}{u_e v_e + t u_o v_o} v_e, \quad g_1 = \frac{t^m + 1}{u_e v_e + t u_o v_o} u_o,$$
$$f_2 = \frac{t^m + 1}{u_e v_e + t u_o v_o} u_e, \quad g_2 = \frac{t^m + 1}{u_e v_e + t u_o v_o} v_o. \tag{2.4}$$

Our candidate for the desired divisor of $t^n + 1$ is $s := u_e v_e + t u_o v_o$. Let us show first that $\deg(s) = n - k$. Since $u_e v_e$ and $u_o v_o$ have even degree, we deduce

$$\deg(s) = \max\{\deg(u_e v_e), \deg(t u_o v_o)\}.$$

Recall $u = u_e^2 + t u_o^2$ and $v = v_e^2 + t v_o^2$. If $n - k$ is even, then

$$\deg(u_e) = \deg(v_e) = \frac{n-k}{2} \quad \text{and} \quad \deg(u_o), \deg(v_o) < \frac{n-k-1}{2}.$$

However, if $n - k$ is odd, then

$$\deg(u_e), \deg(v_e) < \frac{n-k}{2} \quad \text{and} \quad \deg(u_o) = \deg(v_o) = \frac{n-k-1}{2}.$$

Therefore, in both cases, $\deg(s) = n - k$.

It remains to prove that s divides $t^n + 1$. Since f_1, g_1, f_2, g_2 are polynomials, by (2.4), s divides

$$\gcd((t^m + 1)v_e, (t^m + 1)v_o, (t^m + 1)u_e, (t^m + 1)u_o) = (t^m + 1)\gcd(v_e, v_o, u_e, u_o).$$

Observe that $\gcd(v_e, v_o, u_e, u_o)$ divides $f_1 u_e + t g_1 v_o$, and hence, in view of the first equation in (2.2), $\gcd(v_e, v_o, u_e, u_o)$ divides $t^m + 1$. Therefore, s divides $(t^m + 1)^2 = t^n + 1$.

3. Proof of Corollary 1.2

By Theorem 1.1, deciding if a Praeger–Xu graph $\mathrm{PX}(n, k)$ is a Cayley graph is tantamount to deciding if $t^n + 1$ admits a divisor of order k in $\mathbb{Z}_2[t]$. An immediate way to proceed is to study how $t^n + 1$ can be factorised in irreducible polynomials.

Let $n = 2^a b$, with $\gcd(2, b) = 1$. Since we are in characteristic 2,

$$t^n + 1 = t^{2^a b} + 1 = (t^b + 1)^{2^a}.$$

Furthermore, if $\lambda_d(t) \in \mathbb{Z}[t]$ denotes the dth cyclotomic polynomial, then

$$t^b + 1 = \prod_{d \mid b} \lambda_d(t)$$

is the factorisation of $t^b + 1$ in irreducible polynomials over $\mathbb{Q}[t]$, by Gauss' theorem. Since the Galois group of any field extension of $\mathbb{Z}_2$ is a cyclic group generated by the Frobenius automorphism, the degree of an irreducible factor of $\lambda_d(t)$ in $\mathbb{Z}_2[t]$ is the smallest c such that a dth primitive root ζ raised to the power 2^c is ζ, that is, $\omega(d)$.

Hence, $\lambda_d(t)$ in $\mathbb{Z}_2[t]$ factorises into $\varphi(d)/\omega(d)$ irreducible polynomials, each having degree $\omega(d)$.

Therefore, $t^n + 1 \in \mathbb{Z}_2[t]$ has a divisor of degree k if and only if k can be written as the sum of some $\omega(d)$ terms, each summand repeated at most $2^a\varphi(d)/\omega(d)$ times, which is exactly (1.2).

References

[1] R. Jajcay, P. Potočnik and S. Wilson, 'On the Cayleyness of Praeger–Xu graphs', *J. Combin. Theory Ser. B* **152** (2022), 55–79.

[2] C. E. Praeger and M.-Y. Xu, 'A characterization of a class of symmetric graphs of twice prime valency', *European J. Combin.* **10** (1989), 91–102.

MARCO BARBIERI, Dipartimento di Matematica 'Felice Casorati',
University of Pavia, Via Ferrata 5, 27100 Pavia, Italy
e-mail: marco.barbieri07@universitadipavia.it

VALENTINA GRAZIAN, Dipartimento di Matematica e Applicazioni,
University of Milano-Bicocca, Via Cozzi 55, 20125 Milano, Italy
e-mail: valentina.grazian@unimib.it

PABLO SPIGA, Dipartimento di Matematica e Applicazioni,
University of Milano-Bicocca, Via Cozzi 55, 20125 Milano, Italy
e-mail: pablo.spiga@unimib.it