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Some Questions about Semisimple Lie Groups Originating in Matrix Theory

Dragomir Ž. Đoković and Tin-Yau Tam

Abstract. We generalize the well-known result that a square traceless complex matrix is unitarily similar to a matrix with zero diagonal to arbitrary connected semisimple complex Lie groups G and their Lie algebras $\mathfrak{g}$ under the action of a maximal compact subgroup K of G. We also introduce a natural partial order on $\mathfrak{g}$: $x \le y$ if $f(K \cdot x) \subseteq f(K \cdot y)$ for all $f \in \mathfrak{g}^*$, the complex dual of $\mathfrak{g}$. This partial order is K-invariant and induces a partial order on the orbit space $\mathfrak{g}/K$. We prove that, under some restrictions on $\mathfrak{g}$, the set $f(K \cdot x)$ is star-shaped with respect to the origin.

1 Introduction

In this paper we consider some interesting well known facts from matrix theory and try to generalize them to arbitrary connected semisimple complex Lie groups. For instance, it is known that every n by n complex matrix x with zero trace is unitarily similar to a matrix with zero diagonal. We can view x as an element of the Lie algebra $\mathfrak{g} = \mathfrak{sl}(n, \mathbf{C})$ of the group $G = \mathrm{SL}(n, \mathbf{C})$ and the special unitary group $K = \mathrm{SU}(n)$ as a maximal compact subgroup of G. The diagonal matrices in $\mathfrak{g}$ form a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, and those with purely imaginary diagonal entries form a Cartan subalgebra $\mathfrak{t}$ of the Lie algebra $\mathfrak{k} = \mathfrak{su}(n)$ of K. The subspace of $\mathfrak{g}$ consisting of matrices with zero diagonal is just the sum of all root spaces of $(\mathfrak{g}, \mathfrak{h})$. Two matrices $x, y \in \mathfrak{g}$ are unitarily similar if and only if they belong to the same orbit of K under the restriction of the adjoint representation of G to K.

We show (see Theorem 3.4) that this matrix result continues to hold in general when G is an arbitrary connected semisimple complex Lie group, K a maximal compact subgroup of G, and $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$ which is obtained as the complexification of a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{k}$. Then it says that every K-orbit, say $K \cdot x$, in $\mathfrak{g}$ meets the sum of all root spaces of $(\mathfrak{g}, \mathfrak{h})$, which is also the orthogonal complement of $\mathfrak{h}$ in $\mathfrak{g}$ with respect to the Killing form of $\mathfrak{g}$.

As a real K-module, $\mathfrak{g}$ is just the direct sum of two copies of the adjoint module $\mathfrak{k}$ of K: $\mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{k}$. While the orbit space $\mathfrak{k}/K$ is homeomorphic to a closed Weyl chamber in $\mathfrak{t}$, we do not know in general the description of the orbit space $\mathfrak{g}/K$. For the special case $G = \mathrm{SL}(2, \mathbf{C})$ see Proposition 4.2.

We also introduce an interesting partial order "$\le$" in $\mathfrak{g}$: We say that $x \le y$ for $x, y \in \mathfrak{g}$ if $f(K \cdot x) \subseteq f(K \cdot y)$ for all complex linear functionals $f \in \mathfrak{g}^*$. This order is compatible with the action of K and so it induces a partial order on the orbit space

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$\mathfrak{g}/K$. One of us has conjectured that the image $f(K\cdot x)$ contains the origin and is star-shaped with respect to the origin (see Conjecture 3.8). We prove (see Theorem 3.11) that this conjecture is valid if $\mathfrak{g}$ is simply laced and has no components of type E_8.

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2 Preliminaries

Let K be a connected compact semisimple Lie group, G its complexification, and let $\mathfrak{k}$ and $\mathfrak{g}$ be their respective Lie algebras. Thus $\mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{k}$. We fix a maximal torus T of K and denote its Lie algebra by $\mathfrak{t}$. Then $\mathfrak{h} = \mathfrak{t} \oplus i\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}$.

Let l be the rank of G, *i.e.*, $l = \dim_{\mathbf{C}}(\mathfrak{h})$. By $\mathfrak{g}^*$ we denote the dual of $\mathfrak{g}$ (as a complex vector space).

Let R be the root system of $(\mathfrak{g}, \mathfrak{h})$ and Π a fixed base of R. The set of positive roots (with respect to Π) is denoted by R^+. As usual, $\mathfrak{g}^\alpha$ denotes the root space of a root α. We introduce the maximal nilpotent subalgebras $\mathfrak{n}$ and $\mathfrak{n}^-$ of $\mathfrak{g}$:

$$\mathfrak{n} = \sum_{\alpha\in R^+} \mathfrak{g}^\alpha, \quad \mathfrak{n}^- = \sum_{\alpha\in R^+} \mathfrak{g}^{-\alpha}.$$

Then $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}$ is a Borel subalgebra of $\mathfrak{g}$, and let B be the corresponding Borel subgroup of G. The coroot corresponding to a root α is denoted by H_α. Recall that $[\mathfrak{g}^\alpha, \mathfrak{g}^{-\alpha}]$ is a 1-dimensional subspace of $\mathfrak{h}$ and H_α is the unique element of $[\mathfrak{g}^\alpha, \mathfrak{g}^{-\alpha}]$ such that $\alpha(H_\alpha) = 2$. (For more details see *e.g.* [3, Chapitre 8, §2, Théorème 1].) The Weyl group of $(\mathfrak{g}, \mathfrak{h})$ will be denoted by W.

We denote by θ the Cartan involution of $\mathfrak{g}$ (when viewed as a real Lie algebra): it is identity on $\mathfrak{k}$ and negative identity on $i\mathfrak{k}$. It can be lifted to an anti-holomorphic involutorial automorphism of G, which we also denote by θ. Then $K = G^\theta$, *i.e.*, K is the set of θ-fixed points of G. We remark that $\theta(\mathfrak{h}) = \mathfrak{h}$ and $\theta(\mathfrak{g}^\alpha) = \mathfrak{g}^{-\alpha}$ for all $\alpha \in R$.

The Killing form of $\mathfrak{g}$ will be denoted by φ. Unless stated otherwise, the orthogonal complements will be taken with respect to φ. As φ is nondegenerate, it induces a vector space isomorphism $\mathfrak{g} \to \mathfrak{g}^*$ sending $x \to \varphi_x$ where $\varphi_x(y) = \varphi(x, y)$ for all $y \in \mathfrak{g}$.

Definition 2.1 An element $x \in \mathfrak{g}$ is *nilpotent* (resp. *semisimple*) if the linear operator $\mathrm{ad}(x)\colon \mathfrak{g} \to \mathfrak{g}$ has the same property. An element $f = \varphi_x \in \mathfrak{g}^*$ is *nilpotent* (resp. *semisimple*) if x has the same property.

We shall consider the adjoint action, Ad, of G on $\mathfrak{g}$ and its restriction to K. We write $a \cdot x$ instead of $\mathrm{Ad}(a)(x)$ for $a \in G$ and $x \in \mathfrak{g}$. The co-adjoint action of G on $\mathfrak{g}^*$ is defined by $a \cdot f = f \circ \mathrm{Ad}(a^{-1})$, where $a \in G$ and $f \in \mathfrak{g}^*$. Thus we have $(a \cdot f)(x) = f(a^{-1} \cdot x)$ for $a \in G$, $f \in \mathfrak{g}^*$, and $x \in \mathfrak{g}$.

Let us recall a few definitions that we will need. A connected Lie group is called *almost simple* if its Lie algebra is simple, and the quotient of a direct product of Lie groups by a discrete central subgroup is called an *almost direct product*. A root subsystem R_1 of a root system R is said to be *closed* if $\alpha, \beta \in R_1$ and $\alpha + \beta \in R$ imply that $\alpha + \beta \in R_1$.

We refer to the minimal ideals of $\mathfrak{g}$ as its *components*. A subset F of $\mathfrak{g}$ is *star-shaped* with respect to the origin if $x \in F$ and $t \in [0, 1)$ imply that $tx \in F$.

3 The Action of K on $\mathfrak{g}$

It is well known that $\mathfrak{b}$ meets every G-orbit in $\mathfrak{g}$ (see *e.g.* [7, Section 16]), which may be viewed as a generalization of the Jordan canonical form of an n by n complex matrix. This is also true for the K-orbits in $\mathfrak{g}$. It generalizes Schur triangularization theorem which asserts that each n by n complex matrix is unitarily similar to an upper triangular matrix.

Proposition 3.1 *$\mathfrak{b}$ meets every K-orbit in $\mathfrak{g}$.*

Proof This follows from the result just mentioned and the well known fact that $G = KB = BK$, which is a consequence of the global Iwasawa decomposition (see *e.g.* [5, Chapter VI, Theorem 6.3]). ■

We remark that Schur triangularization theorem also asserts that the diagonal elements in an upper triangular form of the n by n complex matrix x, *i.e.*, the eigenvalues of x, can be arranged in any order. Thus we ask Question (4) in Section 5.

In matrix theory, the following result is well known (see *e.g.* [6, Theorem 1.3.4]):

Proposition 3.2 *Every n by n complex matrix of trace 0 is unitarily similar to a matrix with zero diagonal.*

Let H denote the (algebraic) maximal torus of G with Lie algebra $\mathfrak{h}$. In order to extend the above result to complex semisimple Lie algebras, we need the following lemma.

Lemma 3.3 *There exists a closed connected θ-stable complex semisimple Lie subgroup S of G containing H and such that S is an almost direct product of θ-stable almost simple subgroups S_i ($i = 1, \dots, m$) of type A.*

Proof Without any loss of generality, we may assume that $\mathfrak{g}$ is simple, and not of type A_l. We remark that if R_1 is a closed root subsystem of R, then the corresponding semisimple subalgebra $\mathfrak{g}_1$ of $\mathfrak{g}$ is θ-stable.

If $-1 \in W$ then there exists a set $\{\beta_1, \dots, \beta_l\}$ of l strongly orthogonal roots in R [2, Chapitre VI, §1, Exercice 15]. We set $m = l$ and take $\mathfrak{s}_i = \mathfrak{g}^{\beta_i} + \mathfrak{g}^{-\beta_i} + \mathbf{C}H_{\beta_i}$ for $1 \le i \le l$. In this case each S_i is of type A_1.

Next assume that $\mathfrak{g}$ is of type D_l, with l odd. As R has a closed root subsystem of type A_l, we can take S to be the corresponding subgroup of type A_l.

Finally, if $\mathfrak{g}$ is of type E_6, then R has a closed root subsystem $\mathfrak{s}$ of type $3A_2$. The corresponding subgroup S is an almost direct product $S = S_1S_2S_3$, where each S_i is of type A_2.

We have exhausted all possibilities (see [2, Planches I–IX]). ■

Note that $\mathfrak{h}^{\perp} = \mathfrak{n}+\mathfrak{n}^{-}$. The following theorem generalizes the above matrix result to our setting.

Theorem 3.4 *$\mathfrak{h}^{\perp}$ meets every K-orbit in $\mathfrak{g}$.*

Proof Let $S = S_1S_2\cdots S_m$ be as in Lemma 3.3. Then $\mathfrak{h}_i = \mathfrak{s}_i\cap\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{s}_i$, and $\mathfrak{h}$ is a direct sum of the $\mathfrak{h}_i$. The algebra $\mathfrak{s}$ is a direct sum of its ideals $\mathfrak{s}_i$. Denote by $\mathfrak{q}$ the sum of the root spaces $\mathfrak{g}^{\alpha}$ that are not contained in $\mathfrak{s}$ and note that $\mathfrak{q} = \mathfrak{s}^{\perp}$. Then $\mathfrak{g} = \mathfrak{s}\oplus\mathfrak{q}$, $\mathfrak{q}\subseteq\mathfrak{h}^{\perp}$, and $\mathfrak{q}$ is S-stable. The subgroup $K_i = K\cap S_i$ is a maximal compact subgroup of S_i.

An arbitrary $x\in\mathfrak{g}$ can be decomposed uniquely as

$$x = \sum_{i=1}^{m} x_i + x'$$

where $x_i\in\mathfrak{s}_i$ and $x'\in\mathfrak{q}$. Since each S_i is of type A, Proposition 3.2 implies that there exists $a_i\in K_i$ such that $a_i\cdot x_i\in\mathfrak{h}^{\perp}\cap\mathfrak{s}_i$. If $a = a_1a_2\cdots a_m$, then $a\in K\cap S$ and

$$a\cdot x = \sum_{i=1}^{m} a_i\cdot x_i + a\cdot x' \in \sum_{i=1}^{m}\mathfrak{h}^{\perp}\cap\mathfrak{s}_i + \mathfrak{q}\subseteq\mathfrak{h}^{\perp}. \qquad\blacksquare$$

Let us illustrate this theorem by a concrete matrix example which does not seem to be known.

Example 3.5 Let x be an n by n skew-symmetric complex matrix. Then there exists a real orthogonal matrix a such that the matrix $y = axa^{-1}$ has the 2 by 2 diagonal blocks along the diagonal corresponding to the partition $\{1,2\},\{3,4\},\ldots$ all zero. This is obtained from Theorem 3.4 by taking $G = \mathrm{SO}(n,\mathbf{C})$, $K = \mathrm{SO}(n)$, $\mathfrak{g}$ to be the Lie algebra of all n by n skew-symmetric complex matrices, and $\mathfrak{h}$ to be the Cartan subalgebra [5, pp. 187–189] consisting of block-diagonal matrices with the diagonal blocks of size 2 corresponding to the above partition (except, when n is odd, the last block is of size 1).

Our next objective is to introduce a partial order on $\mathfrak{g}$ (which depends on our choice of K, a maximal compact subgroup of G).

Proposition 3.6 *For $x,y\in\mathfrak{g}$ the following are equivalent:*

(i) $f(K\cdot x)\subseteq f(K\cdot y)$, $\forall f\in\mathfrak{g}^*$,
(ii) $f(x)\in f(K\cdot y)$, $\forall f\in\mathfrak{g}^*$.

Proof It is obvious that (i) implies (ii). Assume that (ii) holds. Let $a\in K$ and $f\in\mathfrak{g}^*$ be arbitrary. As $f\circ\mathrm{Ad}(a)\in\mathfrak{g}^*$, the hypothesis gives:

$$f(a\cdot x) = f\circ\mathrm{Ad}(a)(x)\in f\circ\mathrm{Ad}(a)(K\cdot y) = f(aK\cdot y) = f(K\cdot y).$$

Hence (i) holds. $\blacksquare$

Definition 3.7 For $x, y \in \mathfrak{g}$ we write $x \leq y$ if the two conditions of the above proposition are satisfied.

Clearly, the relation "$\leq$" defines a partial order on $\mathfrak{g}$. This partial order is strongly K-invariant in the sense that $x \leq y$ implies that $a \cdot x \leq b \cdot y$ for $a, b \in K$, and so it induces a partial order on the orbit space $\mathfrak{g}/K$. A more transparent description of this important partial order is lacking. We shall take a closer look at the special case $\mathfrak{g} = \mathfrak{sl}_2$ in the next section.

Recently Cheung and Tsing [4] proved that if $\mathfrak{g}$ is of type A_l then for every $x \in \mathfrak{g}$ and every $f \in \mathfrak{g}^*$ the set $f(K \cdot x)$ is star-shaped with respect to the origin. In particular, $0 \in f(K \cdot x)$, *i.e.*, $\ker(f)$ meets $K \cdot x$. Then the second author conjectured that the result of Cheung and Tsing is valid in the general case.

Conjecture 3.8 (**[12]**) *For $x \in \mathfrak{g}$ and $f \in \mathfrak{g}^*$, the set $f(K \cdot x)$ is star-shaped with respect to the origin.*

This conjecture can be reformulated in terms of the partial order "$\leq$". The statement is independent of the choice of $\mathfrak{k}$ since the maximal compact subgroup K of G is unique up to an inner automorphism of G [5, p. 256].

Conjecture 3.9 (**[12]**) *For $x \in \mathfrak{g}$ and $t \in [0, 1]$, we have $tx \leq x$.*

It is not hard to reduce the proof of this conjecture to the case of simple Lie algebras. We do this in the next lemma.

Lemma 3.10 *Assume that $\mathfrak{k}$ is a direct sum of two nonzero ideals $\mathfrak{k} = \mathfrak{k}_1 \oplus \mathfrak{k}_2$ and let $K = K_1K_2$ be the corresponding (almost direct) decomposition of K. Then $\mathfrak{g}$ is a direct sum of the ideals $\mathfrak{g}_1 = \mathfrak{k}_1 + i\mathfrak{k}_1$ and $\mathfrak{g}_2 = \mathfrak{k}_2 + i\mathfrak{k}_2$. Let $x, y \in \mathfrak{g}$ be decomposed as $x = x_1 + x_2$, $y = y_1 + y_2$ with $x_1, y_1 \in \mathfrak{g}_1$ and $x_2, y_2 \in \mathfrak{g}_2$. Then $x_1 \leq y_1$ in $\mathfrak{g}_1$ and $x_2 \leq y_2$ in $\mathfrak{g}_2$ if and only if $x \leq y$ in $\mathfrak{g}$.*

Proof We have $K \cdot x = K_1 \cdot x_1 + K_2 \cdot x_2$ and $K \cdot y = K_1 \cdot y_1 + K_2 \cdot y_2$. Hence $K \cdot x \subseteq K \cdot y$ if and only if $K_1 \cdot x_1 \subseteq K_1 \cdot y_1$ and $K_2 \cdot x_2 \subseteq K_2 \cdot y_2$. ■

We say that $\mathfrak{g}$ is *simply laced* if the simple components of $\mathfrak{g}$ are of type A, D, or E. We can prove that the above conjecture is true if $\mathfrak{g}$ is simply laced and has no components of type E_8.

Theorem 3.11 *If $\mathfrak{g}$ is simply laced and has no components of type E_8, then Conjecture 3.8 is valid.*

Proof In view of Lemma 3.10 we may assume that $\mathfrak{g}$ is simple. If $\mathfrak{g}$ is of type A_l then the conjecture holds by the result of Cheung and Tsing [4].

Let $x \in \mathfrak{g}$ be arbitrary. We have to show that $tx \leq x$ for $t \in [0, 1)$. Since the partial order "$\leq$" is K-invariant, by Theorem 3.4 we may assume that $x \in \mathfrak{h}^\perp$, *i.e.*,

$$x = \sum_{\alpha \in R} x_\alpha, \quad x_\alpha \in \mathfrak{g}^\alpha. \tag{3.1}$$

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Assume first that $\mathfrak{g}$ is of type D_l. In this proof we assume that $\mathfrak{g}$ is realized as a Lie algebra of linear operators on a complex vector space V of dimension $2l$ as in [3, Chapitre VIII, §13, No. 4]. We shall make use of the basis $\{H_1, H_2, \dots, H_l\}$ of $\mathfrak{h}$ defined there, and its dual basis $\{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_l\}$ of $\mathfrak{h}^*$. We recall that $R = \{\pm\varepsilon_i \pm \varepsilon_j : 1 \leq i < j \leq l\}$ where all four sign combinations should be taken. Observe that each $H_m \in i\mathfrak{t}$ because all roots take real values on H_m.

Each $m \in \{1, 2, \dots, l\}$ determines a partition $R = R_m(-1) \cup R_m(0) \cup R_m(1)$, where

$$R_m(k) = \{\alpha \in R : \alpha(H_m) = k\}, \quad k \in \{0, \pm 1\}. \tag{3.2}$$

The subset $R_m(0)$ is a closed root subsystem of R of type D_{l-1}, and each of the subsets $R_m(\pm 1)$ has cardinality $2(l-1)$.

Let L_m be the linear operator on $\mathfrak{g}$ which fixes the elements of $\mathfrak{h}$ and those of the root spaces $\mathfrak{g}^\alpha$ for $\alpha \in R_m(0)$ and on the other root spaces acts as multiplication by the scalar $\sqrt{t}$. Thus we have

$$L_m(x) = \sum_{\alpha \in R_m(0)} x_\alpha + \sqrt{t} \sum_{\alpha \in R_m(\pm 1)} x_\alpha.$$

We claim that $L_m(x) \leq x$. To prove this claim, let $f \in \mathfrak{g}^*$ be arbitrary and we have to show that $f\big(L_m(x)\big) \in f(K \cdot x)$. Let s be a real parameter. Since $H_m \in i\mathfrak{t}$, we have $isH_m \in \mathfrak{t}$, and so $\exp(isH_m) \in K$. Moreover this element sends x_α to $e^{is\alpha(H_m)} x_\alpha$. Consequently

$$f\big(\exp(isH_m) \cdot x\big) = a + be^{is} + ce^{-is},$$

where

$$a = f\Big(\sum_{\alpha \in R_m(0)} x_\alpha\Big), \quad b = f\Big(\sum_{\alpha \in R_m(1)} x_\alpha\Big), \quad c = f\Big(\sum_{\alpha \in R_m(-1)} x_\alpha\Big).$$

We now make use of an argument from [4]. As s varies, the point $a + be^{is} + ce^{-is}$ traces an ellipse $\mathcal{E}$ in the complex plane, with a as its center. If $|b| = |c|$ the ellipse degenerates to a line segment or just a point. Since $t \in [0, 1)$, the point $f\big(L_m(x)\big) = a + \sqrt{t}(b + c)$ lies inside $\mathcal{E}$, or on $\mathcal{E}$ in the degenerate case. Clearly, we can dismiss the degenerate case.

Let $y \in \mathfrak{g}$ be such that $f = \varphi_y$. By Proposition 3.1 there exist $k_1, k_2 \in K$ such that $k_1 \cdot x, k_2 \cdot y \in \mathfrak{b}$. Choose continuous maps $u, v \colon [0, 1] \to K$ such that $u(1) = k_1$, $v(1) = k_2$, and $u(0) = v(0) = e$ (the identity element of K). Since

$$f\big(v(r)^{-1} \exp(isH_m) u(r) \cdot x\big) = \varphi\big(\exp(isH_m) u(r) \cdot x, v(r) \cdot y\big),$$

the point $f\big(v(r)^{-1} \exp(isH_m) u(r) \cdot x\big)$ for fixed r and variable s traces an ellipse $\mathcal{E}_r$ in the complex plane (which may be degenerate). Since $u(0) = v(0) = e$, we have $\mathcal{E}_0 = \mathcal{E}$. For $r = 1$ we have

$$f\big(v(1)^{-1} \exp(isH_m) u(1) \cdot x\big) = \varphi\big(\exp(isH_m) k_1 \cdot x, k_2 \cdot y\big).$$

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Since $k_1 \cdot x, k_2 \cdot y \in \mathfrak{b}$, the above expression is independent of s, *i.e.*, the "ellipse" $\mathcal{E}_1$ is just a point. Since $f\big(L_m(x)\big)$ is inside the ellipse $\mathcal{E}_0 = \mathcal{E}$, there exists $r_0 \in [0, 1)$ such that $f\big(L_m(x)\big)$ lies on $\mathcal{E}_{r_0}$. This proves our claim.

Since $x \in \mathfrak{h}^{\perp}$, we have $tx = L_1 L_2 \cdots L_l(x)$ and our claim implies that $tx \leq x$.

Now assume that $\mathfrak{g}$ is of type E_6. The proof in this case is similar to the one above but requires some modifications.

Denote by Σ the collection of closed root subsystems of R of type D_5. As $|W| = 2^7 \cdot 3^4 \cdot 5$ and the Weyl group of D_5 has order $2^4 \cdot 5! = 2^7 \cdot 3 \cdot 5$, it folllows that $|\Sigma| = 3^3$. For $\alpha \in R$ let m be the number of subsystems $S \in \Sigma$ not containing α. Clearly m does not depend on α, and so $m|R| = 2^5 |\Sigma|$ because $|R| - |S| = 2^5$ for $S \in \Sigma$. It follows that $m = 12$.

For a fixed $S \in \Sigma$ there exist exactly two elements $h \in \mathfrak{h}$ such that $\alpha(h) = 0$ for all $\alpha \in S$ and $\{\alpha(h) : \alpha \in R\} = \{0, \pm 1\}$. If h is one of these two elements, then $-h$ is the other one. We choose one of these two elements and denote it by h_S. Let $\Gamma = \{h_S : S \in \Sigma\}$ and for $h \in \Gamma$ let

$$R_h(k) = \{\alpha \in R : \alpha(h) = k\}, \quad k \in \{0, \pm 1\}.$$

Then $R_h(0) \in \Sigma$ and each of the subsets $R_h(\pm 1)$ has cardinality 16.

Let L_h be the linear operator on $\mathfrak{g}$ which fixes the elements of $\mathfrak{h}$ and those of the root spaces $\mathfrak{g}^{\alpha}$ for $\alpha \in R_h(0)$ and on the other root spaces acts as multiplication by the scalar $t^{1/12}$. Thus we have

$$L_h(x) = \sum_{\alpha \in R_h(0)} x_\alpha + t^{1/12} \sum_{\alpha \in R_h(\pm 1)} x_\alpha.$$

We claim that $L_h(x) \leq x$. The proof of this claim is the same as in the case of root systems of type D_l and we omit it. We just point out that H_m should be replaced by h $(\in \Gamma)$, L_h should play the role of L_m, and $\sqrt{t}$ has to be replaced by $t^{1/12}$.

Since $\{L_h : h \in \Gamma\}$ is a commuting set of operators and for each $\alpha \in R$ there are exactly 12 elements $h \in \Gamma$ such that $\alpha \notin R_h(0)$, we obtain that

$$\Big(\prod_{h \in \Gamma} L_h\Big)(x) = tx.$$

Hence our claim implies that $tx \leq x$.

If $\mathfrak{g}$ is of type E_7 the argument is similar and we omit the details. We mention only that one should take Σ to be the set of closed root subsystems of R of type E_6. ■

As Conjecture 3.9 is still open, it is of interest to ask whether or not $0 \leq x$ for all $x \in \mathfrak{g}$. We address this question in the following proposition.

Proposition 3.12 *The following three statements are equivalent to each other:*

(i) $\forall x \in \mathfrak{g}$, $0 \leq x$;
(ii) $\forall f \in \mathfrak{g}^*$, $\forall x \in \mathfrak{g}$, $\ker(f)$ *meets* $K \cdot x$;

(iii) $\forall f \in \mathfrak{g}^*$, $\mathfrak{g} = \bigcup_{a \in K} \ker(a \cdot f)$.

If $\mathfrak{g}$ has no components of type E_8, F_4, or G_2, then these statements hold.

Proof The equivalence of (i) and (ii) is immediate from the definition of the partial order "$\leq$". The equivalence of (ii) and (iii) follows from $(a \cdot f)(x) = f(a^{-1} \cdot x)$, where $a \in K$, $f \in \mathfrak{g}^*$, and $x \in \mathfrak{g}$. The first assertion is proved.

We now prove the second assertion. In view of Lemma 3.10 we may assume that $\mathfrak{g}$ is simple. By Theorem 3.11 we may exclude the cases A_l, D_l, E_6, and E_7. It remains to consider the cases B_l and C_l.

Let $\mathfrak{g}$ be of type B_l. Again we shall make use of notations from [3, Chapitre VIII, §13, No. 2]. In this case

$$R = \{\pm\varepsilon_i \pm \varepsilon_j : i \neq j,\ 1 \leq i, j \leq l\} \cup \{\pm\varepsilon_i : 1 \leq i \leq l\}.$$

The coroot of ε_i is the operator $2H_i$. Let Σ be the collection of all closed root subsystems S of type B_{l-1} (if $l = 2$ we require that S consists of two short roots). Define the subsets $R_m(k)$, for $1 \leq m \leq l$, $k = 0, \pm 1$, by (3.2). Fix $t \in [0, 1)$ and define the linear operators $L_m \colon \mathfrak{g} \to \mathfrak{g}$ as in the proof of Theorem 3.11. As in that proof, one can show that $L_m(x) \leq x$. The difference is that now, for x given by (3.1), we obtain that

$$L_1 L_2 \cdots L_l(x) = \sqrt{t} \sum_{\alpha \text{ short}} x_\alpha + t \sum_{\alpha \text{ long}} x_\alpha \leq x, \quad t \in [0, 1).$$

In the special case $t = 0$, we obtain that $0 \leq x$.

The proof when $\mathfrak{g}$ is of type C_l is similar to that for the type B_l, and we omit it. ■

Remark 3.13 Assume that $\mathfrak{g}$ is simple, and express the highest root as a linear combination of Π. All coefficients in this linear combination are positive integers. The exceptional cases E_8, F_4, and G_2 are characterized by the property that all these coefficients are ≥ 2.

An element $x \in \mathfrak{g}$ is said to be *normal* if $[x, \theta(x)] = 0$. When $\mathfrak{g} = \mathfrak{sl}(n, \mathbf{C})$ and $\mathfrak{k} = \mathfrak{su}(n)$, it reduces to the usual notion of normality of a matrix.

Lemma 3.14 *The element $x \in \mathfrak{g}$ is normal if and only if $K \cdot x$ meets $\mathfrak{h}$.*

Proof Let x be normal and write $x = y + iz$ where $y, z \in \mathfrak{k}$. Then $\theta(x) = y - iz$ and so $[y, z] = 0$. The assertion now follows from the fact that maximal abelian subalgebras of $\mathfrak{k}$ are its Cartan subalgebras and the latter are all K-conjugate. The converse is obvious. ■

Since the maximal tori of K are conjugate, the normality of $x \in \mathfrak{g}$ and Lemma 3.14 are independent of the choice of $\mathfrak{t}$ and thus of $\mathfrak{h}$. It is evident from the definition.

The following proposition is useful, although it is an immediate consequence of the definitions.

Proposition 3.15 *Let $\mathfrak{k}_1$ be a semisimple subalgebra of $\mathfrak{k}$ and $\mathfrak{g}_1 = \mathfrak{k}_1 + i\mathfrak{k}_1$ its complexification. If $x, y \in \mathfrak{g}_1$ and $x \le y$ in $\mathfrak{g}_1$, then also $x \le y$ in $\mathfrak{g}$.*

Proof If $f \in \mathfrak{g}^*$ then $f|_{\mathfrak{g}_1} \in \mathfrak{g}_1^*$ and so $f(x) \in f(K_1 \cdot y)$, where K_1 is the connected subgroup of K with Lie algebra $\mathfrak{k}_1$. Hence $f(x) \in f(K \cdot y)$ for all $f \in \mathfrak{g}^*$, *i.e.*, $x \le y$ is valid also in $\mathfrak{g}$. ■

We can now show that Conjecture 3.9 is true for normal elements.

Proposition 3.16 *If x is normal, then $tx \le x$ for all $t \in [0, 1]$.*

Proof Since x is normal, by Lemma 3.14, we may assume that $x \in \mathfrak{h}$. Let S be as in Lemma 3.3 and let $\mathfrak{s}$ be its Lie algebra. Since S is θ-stable, $\mathfrak{k}_1 = \mathfrak{s} \cap \mathfrak{k}$ is a compact real form of $\mathfrak{s}$. Since all simple components of $\mathfrak{s}$ are of type A and $x \in \mathfrak{h} \subseteq \mathfrak{s}$, Theorem 3.11 shows that $tx \le x$ in $\mathfrak{s}$ for all $t \in [0, 1]$. It remains to apply Proposition 3.15. ■

4 The Case $\mathfrak{g} = \mathfrak{sl}_2$

In this section it will be understood that $G = \mathrm{SL}(2, \mathbf{C})$, $K = \mathrm{SU}(2)$, $\mathfrak{g} = \mathfrak{sl}(2, \mathbf{C})$, and $\mathfrak{k} = \mathfrak{su}(2)$. We also set

$$x = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & -x_{11} \end{pmatrix} \in \mathfrak{g}.$$

Define the K-invariant polynomial functions $f_1 \colon \mathfrak{g} \to \mathbf{C}$ and $f_2 \colon \mathfrak{g} \to \mathbf{R}$ by

$$f_1(x) = \frac{1}{2}\operatorname{tr}(x^2), \quad f_2(x) = \frac{1}{2}\operatorname{tr}(xx^*).$$

Explicitly, we have

$$f_1(x) = x_{11}^2 + x_{12}x_{21}, \quad f_2(x) = |x_{11}|^2 + \frac{1}{2}(|x_{12}|^2 + |x_{21}|^2).$$

Since $|f_1(x)| \le f_2(x)$, the point $\big(f_1(x), f_2(x)\big)$ belongs to the closed convex cone

$$C = \{(z, t) \in \mathbf{C} \times \mathbf{R} : |z| \le t\}.$$

Define the continuous map $F \colon \mathfrak{g} \to C$ by $F(x) = \big(f_1(x), f_2(x)\big)$. It is well known that the invariants f_1 and f_2 separate the K-orbits in $\mathfrak{g}$, *i.e.*, two matrices $x, y \in \mathfrak{g}$ belong to the same K-orbit if and only if $F(x) = F(y)$. This is an old result of F. D. Murnaghan [9] (see also [11, Corollary 2.35]).

Let us define the closed subset S of $\mathfrak{g}$ by:

$$S = \left\{ w \begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix} : w \in \mathbf{C},\ \lambda \in [0, 1] \right\}.$$

Lemma 4.1 *$F(S) = F(\mathfrak{g}) = C$. Consequently, every K-orbit in $\mathfrak{g}$ meets S, i.e., $\mathfrak{g} = K \cdot S$.*

Proof In order to prove this assertion, it suffices to show that for a given point $(z, t) \in C$ there exist $w \in \mathbf{C}$ and $\lambda \in [0, 1]$ such that

$$\lambda w^2 = z, \quad (1 + \lambda^2)|w|^2 = 2t.$$

If $t = 0$ then also $z = 0$ and we can take $w = 0$ and $\lambda = 0$. If $t > 0$ and $z = 0$, we can take $\lambda = 0$ and $w = \sqrt{2t}$. Finally let $z \neq 0$. Then the second equation above can be replaced by

$$(1 + \lambda^2)|z| = 2\lambda t.$$

This equation has a unique solution for λ in the interval $(0, 1]$. After that we can solve the equation $\lambda w^2 = z$ for w. ■

We equip the orbit space $\mathfrak{g}/K$ with the quotient topology and denote by $\pi\colon \mathfrak{g} \to \mathfrak{g}/K$ the projection map. The map $F_0\colon \mathfrak{g}/K \to C$ induced by F is a continuous bijection. We shall prove that it is in fact a homeomorphism.

Proposition 4.2 *$F_0\colon \mathfrak{g}/K \to C$ is a homeomorphism.*

Proof It suffices to show that F is a proper map, *i.e.*, if $X \subseteq C$ is compact, then $F^{-1}(X)$ is also compact. Choose $t_0 \geq 0$ such that $t \leq t_0$ for all points $(z, t) \in X$. Then $f_2(x) \leq t_0$ for all matrices $x \in F^{-1}(X)$. Hence $F^{-1}(X)$ is a closed and bounded subset of $\mathfrak{g}$, and so it is compact. ■

The closed subset S fails to be a section of the map $F\colon \mathfrak{g} \to C$ since, for $x \in S \setminus \{0\}$, the intersection $S \cap K \cdot x$ is $\{\pm x\}$ if $x^2 \neq 0$, and $\{e^{i\theta} x : \theta \in \mathbf{R}\}$ if $x^2 = 0$. However, it will be convenient to use the elements of S as representatives of the K-orbits in $\mathfrak{g}$, taking into account the ambiguities just mentioned.

We now discuss the partial order "$\leq$". The following theorem is a special case of a result of Nakazato [10] (see [8] for another proof).

Theorem 4.3 *Let $\mathfrak{g} = \mathfrak{sl}(2, \mathbf{C})$, $K = \mathrm{SU}(2)$, and*

$$a = \begin{pmatrix} 0 & a_{12} \\ a_{21} & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & b_{12} \\ b_{21} & 0 \end{pmatrix},$$

with $a_{12} \geq a_{21} \geq 0$ and $b_{12} \geq b_{21} \geq 0$. If $f \in \mathfrak{g}^*$ is defined by $f(x) = \mathrm{tr}(ax)$, then $f(K \cdot b)$ is the elliptical disk in the complex plane in standard position with vertices $\pm(a_{12}b_{12} + a_{21}b_{21})$ and $\pm(a_{12}b_{12} - a_{21}b_{21})i$.

In order to make the order "$\leq$" useful, one needs a simple test for $x \leq y$ to hold. Unfortunately, we were not able to find such a test for arbitrary x and y. By using the above theorem, we can handle some particular cases. The proofs are straightforward and are omitted.

Lemma 4.4 *Let*

$$x = w\begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix}, \quad w \in \mathbf{C},\ \lambda \in [0,1].$$

If $b = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ *then*

$$x \le b \iff (1+\lambda)|w| \le 1,$$
$$b \le x \iff (1-\lambda)|w| \ge 1.$$

If $b = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ *then*

$$x \le b \iff \lambda = 1, \quad w \in \mathbf{R},\ |w| \le 1.$$

5 Some Open Questions

The readers may be interested in the following open questions:

(1) Are the assertions of Proposition 3.12 valid without any restrictions on $\mathfrak{g}$?
(2) Is Conjecture 3.8 true in general?
(3) Find a simple test for $x \le y$ to hold for arbitrary $x, y \in \mathfrak{g}$ when $\mathfrak{g} = \mathfrak{sl}(2, \mathbf{C})$, or (more ambitiously) for arbitrary semisimple $\mathfrak{g}$.
(4) If O is the orthogonal (with respect to the Killing form) projection of $\mathfrak{b} \cap K \cdot x$ to $\mathfrak{h}$, is it true that O is W-stable (or even a single W-orbit)?
(5) Describe the homeomorphism type of the orbit space $\mathfrak{g}/K$.

The following question was raised by the referee:

If P is the orthogonal (with respect to the Killing form) projection of $K \cdot x$ to $\mathfrak{h}$, is it true that P is convex?

Unfortunately, the answer to this question is negative even for $\mathfrak{g} = \mathfrak{sl}(n, \mathbf{C})$. This follows from a result of Au-Yeung and Sing [1].

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Department of Pure Mathematics
University of Waterloo
Waterloo, Ontario
N2L 3G1
email: djokovic@uwaterloo.ca

Department of Mathematics
Auburn University
AL 36849–5310
USA
email: tamtiny@auburn.edu