If the pupil is trained to think of the position of any digit of a number with reference to the *units digit*, he will have no difficulty in answering such questions as the following: "By what power of 10 must the number $\cdot 0638$ be multiplied so as to bring the first significant figure into the units place?" "By what power of 10 must we divide $7382\cdot 7$ so as to bring the first significant figure into the units place?"; or in understanding that "the characteristic of the logarithm of a number whose first significant figure is k places to the left (or right) of the units digit is k (or $-k$)." In establishing this rule, we simply assume that the logarithm of a number whose first significant figure is in the units place (and which therefore lies between 1 and 10) lies between 0 and 1.

Assume for example that

$$7\cdot 382 = 10^{0\cdot 8682};$$

then

$$738\cdot 2 \quad = 10^{2\cdot 8682},$$

because the 7 is *two* places to the left of the units digit, and in moving it (and the other digits) 2 places to the left, we have multiplied by 10^2, and

$$\cdot 07382 = 10^{-2+\cdot 8682},$$

because the 7 is *two* places to the right of the units digit, and we have therefore divided $7\cdot 382$ by 10^2.

W. A. LINDSAY

The Possible Error in a Quotient.—Let the quotient be $A \div B$ where the possible errors in A and B respectively are a and b.

The correct value of the quotient lies between $\dfrac{A+a}{B-b}$ and $\dfrac{A-a}{B+b}$.

Now $$\frac{A+a}{B-b} = \frac{A}{B}\left(1+\frac{a}{A}\right)\left(\frac{1}{1-\dfrac{b}{B}}\right)$$

$$= \frac{A}{B}\left(1+\frac{a}{A}\right)\left\{1+\frac{b}{B}+\frac{b^2}{B^2}+\frac{\dfrac{b^3}{B^3}}{1-\dfrac{b}{B}}\right\} \qquad (1)$$

If we reject $\frac{ab}{AB}$, $\frac{b^2}{B^2}$ and quantities of a higher degree of smallness, we get

$$\frac{A+a}{B-b}=\frac{A}{B}\left(1+\frac{a}{A}+\frac{b}{B}\right) \tag{2}$$

$$=\frac{A}{B}+\frac{aB+bA}{B^2}.$$

Similarly

$$\frac{A-a}{B+b}=\frac{A}{B}-\frac{aB+bA}{B^2}. \tag{3}$$

The possible error in $\frac{A}{B}$ is thus $\frac{aB+bA}{B^2}$, or $\left(\frac{a}{A}+\frac{b}{B}\right)$ of $\frac{A}{B}$. (4)

It is easy to estimate the value of the terms rejected in (1) in any particular case.

Suppose, *e.g.* $\frac{a}{A}=\cdot01$, $\frac{b}{B}=\cdot01$; in most cases $\frac{a}{A}$ and $\frac{b}{B}$ will be less than ·01.

$$\text{Then } \frac{A+a}{B-b}=\frac{A}{B}\left\{1+\frac{a}{A}+\frac{b}{B}+\frac{b^2}{B^2}+\frac{ab}{AB}+\frac{ab^2}{AB^2}+\frac{\left(1+\frac{a}{A}\right)\frac{b^3}{B^3}}{1-\frac{b}{B}}\right\}$$

$$=\frac{A}{B}\left\{1+\frac{a}{A}+\frac{b}{B}+\cdot01^2\left(1+1+\cdot01+\frac{1\cdot01}{\cdot99}\times\cdot01\right)\right\}$$

$$=\frac{A}{B}\left\{1+\frac{a}{A}+\frac{b}{B}+\cdot01^2\left(2\cdot01+1\cdot0202\ldots\times\cdot01\right)\right\}$$

$$=\frac{A}{B}\left\{1+\frac{a}{A}+\frac{b}{B}+\cdot01^2\left(2\cdot020202\ldots\right)\right\}$$

$$=\frac{A}{B}\left\{1+\frac{a}{A}+\frac{b}{B}+k\right\},\text{ where } k<\cdot0003.$$

There are two interesting cases.

(i) Let $b=0$, *i.e.* let there be no error in B; the formula gives the possible error in $\frac{A}{B}$ as $\frac{a}{B}$, as we would expect.

If, for example, the possible error in A is ·01, and we are dividing A by 10, the possible error in the quotient is the 10th part of ·01.

(ii) Let $a = 0$, *i.e.* let there be no error in A.

The possible error in $\frac{A}{B}$ is $\frac{bA}{B^2}$, *i.e.* $\frac{b}{B}$ of $\frac{A}{B}$.

This result too may easily be obtained independently.

Example 1. Evaluate $\frac{1}{\pi}$. (A = 1.)

(i) Take $\pi = 3\cdot14$; the error is $\cdot00159\ldots$.

$$\text{The possible error in } \frac{1}{\pi} \text{ is } \frac{\cdot00159\ldots}{3\cdot14^2},$$
$$< \cdot00159\ldots/9,$$
$$< \cdot0002.$$

We need not therefore evaluate $\frac{1}{3\cdot14}$ to more than four places of decimals; the 4th figure cannot be correct.

$$\frac{1}{3\cdot14} = \cdot3184\ldots$$

This is $> \frac{1}{\pi}$, but the possible error is $< \cdot0002$.

Hence we may say $\frac{1}{\pi} = \cdot3184$ to within ·0002, or, remembering that ·3184 is $> \frac{1}{\pi}$, we may say $\frac{1}{\pi} = \cdot318$ (correct to 3 places).

(ii) Taking $\pi = 3\cdot1416$, the value obtained for $\frac{1}{\pi}$ will be too small by

$$< \frac{\cdot00001}{9}$$
$$i.e. \quad < \cdot0000011,$$
$$i.e. \quad < \frac{2}{10^6}.$$

Hence $$\frac{1}{\pi} = \cdot318309 \text{ to within } \cdot000002$$

or $$\frac{1}{\pi} = \cdot 31831 \quad \text{(correct to 5 places)}.$$

Example 2. The circumference of a circular disc is 23·93″, the diameter is 7·62″; find the ratio of the circumference to the diameter.

$$23\cdot 93 \div 7\cdot 63 = 3\cdot 136\ldots$$

The possible error in this quotient is

$$\frac{7\cdot 63 \times \cdot 005 + 23\cdot 93 \times \cdot 005}{7\cdot 63^2}$$

i.e. $$\frac{31\cdot 56 \times \cdot 005}{7\cdot 63^2}$$

i.e. $$\frac{\cdot 1578}{7\cdot 63^2}$$

i.e. $$< \frac{\cdot 1578}{50}, \text{ but} > \frac{\cdot 1578}{60},$$

i.e. $$< \cdot 0031\ldots \text{ but} > \cdot 0026.$$

We are not therefore justified in assuming that the 4th figure in 3·136 can be correct, or even that we can say 3·14 (correct to 2 places). The correct result lies between 3·139 (say) and 3·133, so that the first two figures alone can be depended on.

Equation (4) can be stated thus:—

$$\frac{\text{Possible error in }\dfrac{A}{B}}{\dfrac{A}{B}} = \frac{\text{Possible error in A}}{A} + \frac{\text{Possible error in B}}{B}$$

i.e. percentage error in $\frac{A}{B}$ = percentage error in A + percentage error in B.

Now if the percentage error in A is zero, or if it can be made as small as we please, then the percentage error in $\frac{A}{B}$ equals the percentage error in B; *i.e.* the percentage error in the quotient equals the percentage error in the divisor; which justifies the practice in contracted division of keeping only one significant figure *more* in the divisor than we wish in the quotient.

W. A. Lindsay