ON EXCEPTIONAL VALUES OF MEROMORPHIC FUNCTIONS WITH THE SET OF SINGULARITIES OF CAPACITY ZERO[1]

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1. Let E be a compact set in the z-plane and let Ω be its complement with respect to the extended z-plane. Suppose that E is of capacity zero. Then Ω is a domain and we shall consider a single-valued meromorphic function $w = f(z)$ on Ω which has an essential singularity at each point of E. We shall say that a value w is exceptional for $f(z)$ at a point $\zeta \in E$ if there exists a neighborhood of ζ where the function $f(z)$ does not take this value w.

In our previous paper [7], we showed that the set of all exceptional values of $f(z)$ at a point ζ of E may be non-countable. In fact, we proved the following:

For every K_σ-set K[2] of capacity zero in the w-plane, there exist a compact set E of capacity zero in the z-plane and a single-valued meromorphic function $f(z)$ on its complementary domain Ω such that $f(z)$ has an essential singularity at each point of E and such that the set of exceptional values at each singularity coincides with K.

In the opposite direction, we do not know, except for countable sets, any characterization of sets E for which all functions have very few exceptional values. Here we raise the following question: Is there any perfect set E in the z-plane such that any function, which is single-valued and meromorphic in the complementary domain Ω of E and has an essential singularity at each point ζ of E, has "at most two" or "at most a countable number of" exceptional values at each $\zeta \in E$?

The purpose of this paper is to give a sufficient condition for sets E for which every function $f(z)$ has at most a finite number of exceptional values. We shall show the existence of such a perfect set E by means of a Cantor set.

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[1] In this paper, capacity is always logarithmic.

[2] By a K_σ-set we mean the union of an at most countable number of compact sets.

2. Let $\{\Omega_n\}_{n=0,1,2,\ldots}$ be an exhaustion of Ω with the following conditions:

1°) $\Omega_n \supset \bar{\Omega}_{n-1}$ for every n,

2°) for each n, the boundary $\partial\Omega_n$ of Ω_n consists of a finite number of closed analytic curves,

3°) each component of the open set $\mathscr{C}\bar{\Omega}_n$[3] contains points of E,

4°) each component of the open set $\Omega_n - \bar{\Omega}_{n-1}$ is doubly-connected.

We shall use in the sequel the graph associated with $\{\Omega_n\}$ which is defined as follows[4]: The open set $\Omega_n - \bar{\Omega}_{n-1}$ $(n \geqq 1)$ consists of a finite number of doubly-connected domains $R_{n,k}$ $(k = 1, 2, \ldots, N(n))$. The boundary of $R_{n,k}$ consists of closed curves contained in $\partial\Omega_{n-1} \cup \partial\Omega_n$. Denote by $\alpha_{n-1,k}$ the part of the boundary of $R_{n,k}$ on $\partial\Omega_{n-1}$ and $\beta_{n,k}$ that on $\partial\Omega_n$. Let $u_{n,k}(z)$ be the harmonic function in $R_{n,k}$ which vanishes on $\alpha_{n-1,k}$ and is equal to a constant $\mu_{n,k}$ on $\beta_{n,k}$ and whose conjugate function $v_{n,k}(z)$ satisfies

$$\int_{\beta_{n,k}} dv_{n,k} = 2\pi,$$

where the integral is taken in the positive sense with respect to $R_{n,k}$. The quantity $\mu_{n,k}$ is called the harmonic modulus of $R_{n,k}$. Now we define the harmonic modulus σ_n of the open set $\Omega_n - \bar{\Omega}_{n-1}$. Let $u_n(z)$ be the harmonic function in $\Omega_n - \bar{\Omega}_{n-1}$ which is equal to zero on $\partial\Omega_{n-1}$ and to σ_n on $\partial\Omega_n$ and whose conjugate function $v_n(z)$ has the variation 2π, i.e.,

$$\int_{\partial\Omega_{n-1}} dv_n = 2\pi.$$

This quantity σ_n is called the harmonic modulus of $\Omega_n - \bar{\Omega}_{n-1}$. If we choose an additive constant of $v_n(z)$ suitably, the regular function $u_n(z) + iv_n(z)$ maps $R_{n,k}$ $(k = 1, 2, \ldots, N(n))$ with one suitable slit onto a rectangle $0 < u_n < \sigma_n$, $b_k < v_n < a_k + b_k$ one-to-one conformally, where $a_k (k = 1, 2, \ldots, N(n))$ and b_k $(k = 1, 2, \ldots, N(n))$ are constants satisfying the relations that

$$a_k = 2\pi \frac{\sigma_n}{\mu_{n,k}}, \quad \sum_{k=1}^{N(n)} a_k = 2\pi$$

and

$$b_1 = 0, \; b_k = \sum_{i=1}^{k-1} a_i \qquad (1 < k \leqq N(n)).$$

[3] We denote the complement of a set A with respect to the extended complex plane by $\mathscr{C}A$.

[4] See Kuroda [6].

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Consequently, the function $u_n(z)+iv_n(z)$ maps $\Omega_n - \bar{\Omega}_{n-1}$ with $N(n)$ suitable slits onto a slit-rectangle $0<u_n<\sigma_n$, $0<v_n<2\pi$ one-to-one conformally. We define the function $u(z)+iv(z)$ by $u_n(z)+iv_n(z)+\sum_{j=1}^{n-1}\sigma_j$ on each $\Omega_n - \bar{\Omega}_{n-1}$ $(n\geqq 1)$. Then this function $u(z)+iv(z)$ maps $\Omega - \bar{\Omega}_0$ with at most a countable number of suitable slits onto a strip domain $0<u<R$, $0<v<2\pi$ with a countable number of slits one-to-one conformally, where

$$R=\sum_{j=1}^{\infty}\sigma_j \leqq +\infty.$$

This strip domain is the graph of Ω associated with the exhaustion $\{\Omega_n\}$ in the sense of Noshiro [8]. The number R is called the length of this graph. By the theorems of Sario [11] and Noshiro [8], Ω is the complementary domain of a compact set of capacity zero in the z-plane if and only if there exists a graph of Ω whose length R is infinite.

3. Let γ_r be the niveau curve $u(z)=r$ $(0<r<R)$ on Ω. The niveau curve γ_r consists of a finite number of simple closed curves $\gamma_{r,k}$ $(k=1,2,\ldots,n(r))$. If $\sum_{j=1}^{n-1}\sigma_j<r<\sum_{j=1}^{n}\sigma_j$, then each $\gamma_{r,k}$ $(k=1,2,\ldots,n(r)=N(n))$ is a simple closed analytic curve in $R_{n,k}$ which separates $\alpha_{n-1,k}$ from $\beta_{n,k}$. If $r=\sum_{j=1}^{n-1}\sigma_j$, then each $\gamma_{r,k}$ $(k=1,2,\ldots,n(r)=N(n))$ coincides with $\alpha_{n-1,k}$. We shall call each component of the open set $\Omega_n - \bar{\Omega}_m$ $(n>m)$ an R-chain. For every $\gamma_{r,k}$ $(0<r<R,\ 1\leqq k\leqq n(r))$ we consider the longest doubly-connected R-chain $R(\gamma_{r,k})$ such that $\gamma_{r,k}$ is contained in $R(\gamma_{r,k})$ or is the one of the two boundary components of $R(\gamma_{r,k})$, and denote by $\mu(\gamma_{r,k})$ the harmonic modulus of this R-chain. We set

$$\mu(r)=\min_{1\leqq k\leqq n(r)}\mu(\gamma_{r,k}).$$

Here we note that if $\sum_{j=1}^{n-1}\sigma_j\leqq r<\sum_{j=1}^{n}\sigma_j$, then $R(\gamma_{r,k})\supset R_{n,k}$ because of the condition 4°) of $\{\Omega_n\}$.

Generally $R_{n,k}$ may branch off into a finite number of $R_{n+1,m}$'s. If every $R_{n,k}$ $(n=1,2,\ldots;\ k=1,2,\ldots,N(n))$ branches off into at most ρ number of $R_{n+1,m}$'s, we say that the exhaustion $\{\Omega_n\}$ branches off at most ρ-times everywhere. Then we obtain the following

THEOREM 1. *Let E be a compact set of capacity zero in the z-plane and let*

Ω be its complementary domain. If there exists an exhaustion $\{\Omega_n\}$ of Ω which satisfies the conditions 1°), 2°), 3°) *and* 4°) *stated in* § 2, *branhces off at most ρ_0-times everywhere and has the graph with infinite length satisfying the conditions that*

$$\lim_{r\to\infty}\mu(r) = +\infty \quad \textit{and} \quad \lim_{r\to\infty}\frac{n(r)}{r} = 0,$$

then every function, which is single-valued and meromorphic in Ω and has an essential singularity at each point ζ of E, has at most ρ_0+1 exceptional values at each singularity.

If we replace the last condition of the above by the condition

$$\overline{\lim_{r\to\infty}}\frac{n(r)}{r} < +\infty,$$

then the functions have at most a finite number of exceptional values at each singularity.

4. Before proving the theorem, we give two lemmas. Let C_1 and C_2 be two disjoint closed discs in the extended w-plane and let $\{\Lambda\}$ be the class of rectifiable curves[5] which lie outside C_1 and C_2 except for their end points and join C_1 and C_2. For a subclass $\{\Lambda'\}$ of $\{\Lambda\}$, we can consider the extremal length $\lambda\{\Lambda'\}$, which is defined as follows: Let $\{\rho\}$ be the collection of functions ρ which are non-negative and lower semi-continuous in the extended w-plane. The quantity

$$\lambda\{\Lambda'\} = \sup_{\rho}\frac{\inf_{\Lambda'}\int_{\Lambda'}\rho|dw|}{\iint\rho^2\,dudv} \qquad (w = u+iv)$$

is called the extremal length of $\{\Lambda'\}$, where we understand that $0/0 = \infty/\infty = 0$ (Ahlfors and Beurling [1], Ahlfors and Sario [2]).[6] We have

$$0 < \lambda\{\Lambda\} < +\infty.$$

If we consider a set c consisting of a finite number of continua in the closure of the ring domain (C_1, C_2) and set

5) This means that curves are rectifiable with respect to the spherical distance.

6) For properties of extremal lengths, see, e.g., Ahlfors and Sario [2], Hersch [5], Ohtsuka [9].

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$$\{\Lambda'\}_c = \{\Lambda' \in \{\Lambda\};\ \Lambda' \cap c = \phi\},$$

then it holds that

$$+\infty \geqq \lambda\{\Lambda'\}_c \geqq \lambda\{\Lambda\}.$$

Given a positve number τ, we shall denote by C_τ the class of sets c with the property that

$$\sum_\nu d(\kappa_\nu) < \tau,$$

where $\{\kappa_\nu\}$ are the components of c and $d(\kappa_\nu)$ means the spherical diameter of κ_ν.

LEMMA 1. *There is a positive number τ such that*

$$\sup_{c \in C_\tau} \lambda\{\Lambda'\}_c < +\infty.$$

Proof. By means of linear transformations, which correspond to rotations of sphere around the center and hence do not change spherical distance, we may assume that C_2 is a disc $|w| \geqq R$. If we denote by $d_e(\kappa_\nu)$ the diameter of κ_ν with respect to the euclidean metric, then we have that

$$\sum_\nu d_e(\kappa_\nu) \leqq (1+R^2) \sum_\nu d(\kappa_\nu).$$

We map the ring domain (C_1, C_2) conformally onto the annulus $1 < |\zeta| < \mu$ by $\zeta(w)$, where $\mu = e^{2\pi\lambda(\Lambda)}$. With an interior point α of C_1, we can represent the function $\zeta(w)$ by

$$\zeta(w) = e^{i\theta} \mu R \frac{w-\alpha}{R^2 - \bar{\alpha} w}$$

and hence we see that

$$M = \sup_{w_1, w_2 \in (C_1, C_2)} \frac{|\zeta(w_1) - \zeta(w_2)|}{|w_1 - w_2|} \leqq \frac{\mu(R + |\alpha|)}{R(R - |\alpha|)} < +\infty.$$

Therefore we have that

$$\sum_\nu d_e(\zeta(\kappa_\nu)) \leqq M \sum_\nu d_e(\kappa_\nu) \leqq M(1+R^2) \sum_\nu d(\kappa_\nu).$$

The number

$$\tau = \frac{\pi}{M(1+R^2)}$$

is one of the wanted. In fact, if we delete from the annulus $1<|\zeta|<\mu$ all segments s_θ: $\arg\zeta=\theta(0\leqq\theta<2\pi)$, $1<|\zeta|<\mu$, which intersect $\bigcup_\nu \zeta(\kappa_\nu)$, then we have a finite number of domains D_i: $\theta_{2i-1}<\arg\zeta<\theta_{2i}$, $1<|\zeta|<\mu$ $(i=1, 2, \ldots, N)$ such that they are disjoint from each other and

$$\sum_{i=1}^{N}(\theta_{2i}-\theta_{2i-1})\geqq\pi.$$

Let $\{\gamma\}$ be the class of all curves in the annulus $1<|\zeta|<\mu$ which join two boundary circles of the annulus and do not touch $\bigcup_\nu \zeta(\kappa_\nu)$ and let $\{s_i\}$ be the class of segments s_θ: $\theta_{2i-1}<\theta<\theta_{2i}$. Then

$$\lambda\{s_i\}=\frac{1}{\theta_{2i}-\theta_{2i-1}}\log\mu=\frac{2\pi\lambda\{A\}}{\theta_{2i}-\theta_{2i-1}} \text{ [6)]}$$

and since domains D_i are disjoint from each other

$$\lambda\{A'\}_c=\lambda\{\gamma\}\leqq\lambda(\bigcup_{i=1}^{N}\{s_i\})=\frac{1}{\sum_{i=1}^{N}\frac{1}{\lambda\{s_i\}}}=\frac{2\pi\lambda\{A\}}{\sum_{i=1}^{N}(\theta_{2i}-\theta_{2i-1})}\leqq 2\lambda\{A\}.\text{[6)]}$$

Thus our proof is complete.

We shall consider distinct $n(\geqq 3)$ points $w_1, w_2, \ldots, w_n$ in the extended w-plane and denote by $\zeta=T^i_{j,m}(w)$ $(i\neq j, m$ and $j\neq m)$ the linear transformation which transforms w_i, w_j and w_m to the point at infinity, the origin and the point $\zeta=1$ respectively. Now we prove the following lemma which is a consequence of Bohr-Landau's theorem [3]:

If $g(z)$ is regular in $|z|<1$ and $g(z)\neq 0, 1$ there, then

$$\max_{|z|=r}|g(z)|\leqq\exp\left(\frac{A\log(|g(0)|+2)}{1-r}\right)\qquad\text{for all } r<1,$$

where A is a positive constant (a precise form of Schottky's theorem).

LEMMA 2. *Let R be an annulus $a<|z|<b$ in the z-plane and let c and d be positive numbers such that*

$$a<c<d<b \text{ and } \log\frac{c}{a},\ \log\frac{b}{d}\geqq\sigma(\sigma>0).$$

Then there is a positive constant δ with the following properties:

1) *the spherical closed discs C_i $(i=1, 2, \ldots, n)$ with the centers at w_i and with the spherical radius δ are mutually disjoint and*

$$C_i \subset (T^m_{i,j})^{-1}(U)^{7)} \qquad (i \neq j, m \text{ and } j \neq m),$$

where U is the unit disc $|\zeta|<1$,

2) *if, for all r $(c \leqq r \leqq d)$, a single-valued meromorphic function $f(z)$ in R omitting the values $w_1, w_2, \ldots, w_n$ takes on $|z|=r$ a value contained in $(T^i_{j,m})^{-1}(U)$, then $f(z)$ takes no value in C_i in the annulus $c<|z|<d$.*

Here δ depends only on σ and does not depend on R and $f(z)$.

Proof. From Bohr-Landau's theorem we can see easily that if $g(z)$ is a regular function in R such that

$$g(z) \neq 0, 1 \text{ and } \min_{|z|=r} |g(z)|<1 \text{ for all } r\colon\ c \leqq r \leqq d,$$

then there is a positive constant K depending only on σ and satisfying

$$|g(z)| \leqq K \qquad \text{for every } z\colon\ c \leqq |z| \leqq d.$$

Therefore, if $T^i_{j,m}(f(z))$ has the same properties as $g(z)$, it holds that

$$|T^i_{j,m}(f(z))| \leqq K \qquad \text{for every } z\colon\ c \leqq |z| \leqq d.$$

Hence the image of the outside V of $|\zeta| \leqq K$ by $(T^i_{j,m})^{-1}$ is an open disc which contains w_i and has the following property: If, for all r $(c \leqq r \leqq d)$, $f(z)$ takes on $|z|=r$ a value contained in $(T^i_{j,m})^{-1}(U)$, $f(z)$ takes no value in $(T^i_{j,m})^{-1}(V)$ in the annulus $c<|z|<d$. Set

$$U(w_i) = \bigcap_{\substack{j \neq m \\ j,\, m \neq i}} ((T^i_{j,m})^{-1}(V) \cap (T^m_{i,j})^{-1}(U)).$$

Since $(T^i_{j,m})^{-1}(V)$ and $(T^m_{i,j})^{-1}(U)$ are open discs containing w_i, each term in the right side is a non-empty open set containing w_i and hence $U(w_i)$ is also a non-empty open set containing w_i. Therefore

$$0<\delta_i = \min_{w \in \partial U(w_i)} \frac{|w-w_i|}{\sqrt{(1+|w|^2)(1+|w_i|^2)}},$$

and hence

$$\delta' = \min_{1 \leqq i \leqq n} \delta_i > 0.$$

If we choose a positive number $\delta \leqq \delta'$ so that the spherical closed disc C_i with

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[7] Note that $T^m_{i,j} \neq T^i_{j,m}$; that is, $T^m_{i,j}$ transforms w_i, w_j and w_m to the origin, the point $w=1$ and the point at infinity respectively and $T^i_{j,m}$ transforms w_i, w_j and w_m to the point at infinity, the origin and the point $w=1$ respectively.

the centers at w_i and with the spherical radius δ are mutually disjoint, then discs C_i satisfy all conditions of the lemma.

5. *Proof of the theorem.* In the case where $\rho_0 = 1$, E consists of just one point and hence our assertion is true from Picard's theorem.

Let ρ_0 be greater than 1. Contrary to our assertion, let us suppose that there exists a function $f(z)$ which is single-valued and meromorphic in Ω, has an essential singularity at each point of E and has more than $\rho_0 + 1$ exceptional values at a singularity $\zeta \in E$. We denote by $U(\zeta)$ a neighborhood of ζ where $f(z)$ does not take distinct $\rho_0 + 2$ values $w_1, w_2, \ldots, w_{\rho_0+2}$. Then we can find an n and a k such that the domain $R_{n,k}$ is contained in $U(\zeta)$ and separates the boundary of $U(\zeta)$ from ζ. Consider the component Ω', containing ζ, of the complement of Ω_{n-1} with respect to the extended z-plane. The complement of the closed set $E \cap \Omega'$ with respect to the extended z-plane is a domain and if we take $\mathscr{C}\bar{\Omega}$, as the first domain of an exhaustion $\{\Omega'_m\}$ of $\mathscr{C}(E \cap \Omega')$ and $\mathscr{C}\Omega' \cup (\Omega' \cap \Omega_{n+p-1})$ as the $(p+1)$-th $(p \geqq 1)$, the graph associated with this exhaustion satisfies our conditions too. In the below we shall use the notation Ω instead of $\mathscr{C}(E \cap \Omega')$ and the notation $\{\Omega_n\}$ instead of $\{\Omega'_m\}$. We consider the graph associated with this exhaustion and denote by $u(z) + iv(z)$ the function which maps one-to-one conformally $\Omega - \bar{\Omega}_0$ with at most a countable number of suitable slits onto our graph.

First we shall show that there exist a positive number τ and an r_0 such that for all $r \geqq r_0$, the spherical length of the image of the niveau curve γ_r: $u(z) = r$ is not less than τ, i.e., for all $r \geqq r_0$,

$$L(r) = \int_{\gamma_r} \frac{|f'(z)|}{1 + |f(z)|^2} |dz| \geqq \tau > 0.$$

Applying Lemma 2 to the set of points $w_1, w_2, \ldots, w_{\rho_0+2}$, we can find a positive constant δ such that the spherical closed discs C_i $(i = 1, 2, \ldots, \rho_0 + 2)$ with the centers at w_i and with the spherical radius δ satisfy the conditions of the lemma.

Let $\{\Lambda_{i,j}\}$ $(i \neq j)$ be the class of rectifiable curves in the extended w-plane which lie outside C_i and C_j except for their end points and join C_i and C_j. From Lemma 1 we can find a positive constant $\tau_{i,j}$ such that

$$\mu'_{i,j} = 2\pi \sup_{c \in C_{\tau_{i,j}}} \lambda\{\Lambda'_{i,j}\}_c < +\infty,$$

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where the definition of $C_{\tau_{i,j}}$ was given just before Lemma 1.

Set

$$\tau' = \min_{i \neq j} \tau_{i,j}, \qquad \tau = \min\left\{\frac{\tau'}{2}, \frac{\delta}{2}\right\}$$

and

$$\mu = \max_{i \neq j} \mu'_{i,j}.$$

Suppose that there is an increasing sequence of positive numbers $\{r_n\}$ such that

$$r_1 < r_2 < \cdots < r_n < \cdots \to +\infty$$

and for each n,

$$L(r_n) < \tau.$$

We may assume from the assumption of our theorem that

$$\mu(r) > \mu + 2\sigma$$

for all $r \geqq r_1$, where σ is a positive constant. Further we may assume that $f(z)$ takes in a component $\Omega(r_1, r_2)$ of the open set $r_1 < u(z) < r_2$ two values w_0' and w_0'' such that they lie outside $\bigcup_{i=1}^{p_0+2} C_i$ and the spherical distance between them is greater than 2τ, because E is of capacity zero and hence $f(z)$ takes all values w infinitely often with possible exception of a set of capacity zero in any neighborhood of each point of E. Let n and p be positive integers with the property that

$$\sum_{j=1}^{n-p-1} \sigma_j \leqq r_1 < \sum_{j=1}^{n-p} \sigma_j \text{ and } \sum_{j=1}^{n-1} \sigma_j \leqq r_2 < \sum_{j=1}^{n} \sigma_j.$$

The boundary $\partial\Omega(r_1, r_2)$ of $\Omega(r_1, r_2)$ consists of one of $\{\gamma_{r_1,k}\}_{k=1,2,\ldots,n(r_1)}$, say $\gamma_{r_1,1}$, and some of $\{\gamma_{r_2,k}\}_{k=1,2,\ldots,n(r_2)}$, say $\{\gamma_{r_2,k}\}_{k=1,2,\ldots,m}$ $(m \leqq n(r_2))$. We shall say that $\gamma_{r_2,i}$ and $\gamma_{r_2,j}$ are of ν-th kin if a component $R(\gamma_{r_2,i}, \gamma_{r_2,j})$ of $\Omega_n - \Omega_{n-\nu-1}$ is the smallest R-chain which contains $\gamma_{r_2,i} \cup \gamma_{r_2,j}$. Since

$$d(f(\gamma_{r_1,1})) \leqq \frac{L(r_1)}{2} < \frac{\tau}{2} \text{ and } \sum_{k=1}^{m} d(f(\gamma_{r_2,k})) \leqq \frac{L(r_2)}{2} < \frac{\tau}{2},$$

we can cover $\bigcup_{k=1}^{m} f(\gamma_{r_2,k})$ by a finite number of mutually disjoint spherical closed discs S_q $(q = 1, 2, \ldots, m';\ m' \leqq m)$ with the property that

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$$\sum_{q=1}^{m'} d(S_q) < \tau,$$

and $\bigcup_{k=1}^{m} f(\gamma_{r_2,k}) \cup f(\gamma_{r_1,1})$ by a finite number of mutually disjoint spherical closed discs S'_q ($q=1, 2, \ldots, m''$; $m'' \leqq m'+1$) satisfying that

$$\bigcup_{q=1}^{m'} S_q \subset \bigcup_{q=1}^{m''} S'_q \text{ and } \sum_{q=1}^{m''} d(S'_q) < 2\tau.$$

Let z'_0 and z''_0 be the points of $\Omega(r_1, r_2)$ satisfying that $f(z'_0) = w'_0$ and $f(z''_0) = w''_0$ and let γ be an arbitrary curve in $\Omega(r_1, r_2)$ joining z'_0 and z''_0. Since the image $f(\gamma)$ of γ joins w'_0 and w''_0 and the spherical distance between w'_0 and w''_0 is greater than 2τ, we can find a point $w_0 \in f(\gamma)$ such that for all i there are curves Λ_i which join w_0 and $\tilde{C}_i$ and do not touch $\bigcup_{q=1}^{m''} S'_q$. Here we denote by $\tilde{C}_i$ the concentric spherical closed disc of C_i with the spherical diameter δ. Let $z_0 \in \gamma$ be a point satisfying that $f(z_0) = w_0$. Since $f(z)$ does not take values $\{w_i\}_{i=1,2,\ldots,\rho_0+2}$ on $\bar{\Omega}(r_1, r_2)$, all curves in $\tilde{C}_i$ joining the end point of Λ_i on $\tilde{C}_i$ and w_i intersect the image of $\partial\Omega(r_1, r_2)$. In fact, if there is a curve $\tilde{\Lambda}$ not intersecting this image, the element $e(w; z_0)$ of the inverse function f^{-1} corresponding to z_0 can be continued analytically in the wider sense along $\Lambda_i \cup \tilde{\Lambda}$ up to a point arbitrarily near w_i so that the path corresponding to this continuation is contained in $\Omega(r_1, r_2)$. This is a contradiction. Observing that

$$d(f(\gamma_{r_2,k})) \leqq \frac{L(r_2)}{2} \leqq \frac{\tau}{2} \leqq \frac{\delta}{4} \qquad (k=1, 2, \ldots, m),$$

we see that the inside of each C_i contains the image of at least one $\gamma_{r_2,k} \subset \partial\Omega(r_1, r_2)$ possibly except for one C_i which may contain the image of $\gamma_{r_1,1}$. Let $(\gamma_{r_2,h}, \gamma_{r_2,h'})$ be one of the nearest of kin among all pairs $(\gamma_{r_2,k}, \gamma_{r_2,k'})$ whose images are contained in distinct discs, let C_i and $C_{i'}$ $(i \neq i')$ be the discs containing their images respectively, let them be of ν-th kin and let $R_{n-\nu,t}$ be the domain which determines their kinship. Since our exhaustion branches off at most ρ_0-times everywhere, we can find at least two discs, say C_j and $C_{j'}$ $(j \neq j')$, which do not contain the image of any $\gamma_{r_2,k}$ of ν-th or nearer than ν-th kin to $\gamma_{r_2,h}$ or $\gamma_{r_2,h'}$.

Let R be the longest doubly-connected R-chain containing $R_{n-\nu,t}$. Then from our assumption the harmonic modulus of R is greater than $\mu+2\sigma$. Further $R \neq R(\gamma_{r_1,1})$ and hence $R \subset \Omega(r_1, r_2)$, for if $R = R(\gamma_{r_1,1})$ all $\gamma_{r_2,k}$ $(k=1, 2, \ldots, m)$

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are of ν-th or nearer than ν-th kin to $\gamma_{r_2,h}$ or $\gamma_{r_2,h'}$ and C_j and $C_{j'}$ can not contain the image of any $\gamma_{r_2,k}$ $(k=1, 2, \ldots, m)$. We may consider R as an annulus $a<|z|<b$ $(\log b/a>\mu+2\sigma)$ and denote ae^{σ} and $be^{-\sigma}$ by c and d respectively. We observe that for each s $(a<s<b)$ the image of the circle K_s: $|z|=s$ by $f(z)$ intersects $(T^i_{i',j'})^{-1}(U)$ and $(T^j_{i',j'})^{-1}(U)$. In fact, we know that $C_{i'}\subset (T^i_{i',j'})^{-1}(U)\cap(T^j_{i',j'})^{-1}(U)$. Suppose that $f(K_s)\cap C_{i'}=\phi$ and denote by $\gamma(h')$ a curve in $\Omega(r_1, r_2)$ joining z_0 and a point of $\gamma_{r_2,h'}$. The images of $\gamma_{r_2,k}$ of ν-th or nearer than ν-th kin to $\gamma_{r_2,h}$ or $\gamma_{r_2,h'}$ are covered by some of $\{S_q\}(1\leqq q\leqq m')$, say $\{S_q\}(1\leqq q\leqq m_0';\ m_0'<m')$. Since

$$\sum_{q=1}^{m_0'} d(S_q)<\tau\leqq\frac{\delta}{2}$$

and since $(T^i_{i',j'})^{-1}(U)$ is spherical disc containing $C_{i'}$, there are point $w'\in f(\gamma(h'))\cap C_{i'}$ outside $\bigcup_{q=1}^{m_0'} S_q$ and a curve Λ in $(T^i_{i',j'})^{-1}(U)$ which joins w' and $w_{j'}$ and does not touch $\bigcup_{q=1}^{m_0'} S_q$. Let z' be the point of $\gamma(h')$ such that $f(z')=w'$ and let $e(w;z')$ be the element of f^{-1} corresponding to z'. Continue $e(w;z')$ along Λ. If Λ does not intersect $f(K_s)$, $f(z)$ must take the value $w_{j'}$ in $\Omega(r_1, r_2)$; this is a contradiction. By the same reasoning we see that $f(K_s)\cap(T^j_{i',j'})^{-1}(U)\neq\emptyset$. It now follows by Lemma 2 that $f(z)$ does not take any value of $C_i\cap C_j$ in $c<|z|<d$. Consequently if we consider the class of all rectifiable curves $\{\Lambda_{i,j}\}$ which lie outside $C_i\cup C_j$ except for their end points, join C_i and C_j and do not intersect $\bigcup_{q=1}^{m''} S_q'$, then by the same reasoning as above, we can see easily that each $\Lambda_{i,j}$ contains a curve which is the image of a curve Γ' in the annulus $c<|z|<d$ joining its two boundary circles. Let $\{\Gamma\}$ be the class of all rectifiable curves in $c<|z|<d$ joining its two boundary circles and let $\{\Gamma'\}$ be the subclass of $\{\Gamma\}$ such that for each Γ' there is a $\Lambda_{i,j}$ containing its image $f(\Gamma')$. Then we have that

$$\lambda\{\Gamma\}\leqq\lambda\{\Gamma'\}\leqq\lambda\{f(\Gamma')\}\leqq\lambda\{\Lambda_{i,j}\}.$$ [6)]

Since

$$\sum_{q=1}^{m''} d(S_q')<2\tau\leqq\tau',$$

we see from the definitions of τ' and μ that

$$2\pi\lambda\{\Lambda_{i,j}\}\leqq\mu$$

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and hence

$$2\pi\lambda\{\Gamma\} \leqq \mu.$$

But on the other hand we have

$$2\pi\lambda\{\Gamma\} = \text{mod (the annulus } c < |z| < d) > \mu^{8)\,6)}.$$

Thus we are led to a contradiction and we can conclude that there is an r_0 such that

$$L(r) \geqq \tau$$

for all $r \geqq r_0$.

Let Ω_r denote the subdomain of Ω bounded by the niveau curve γ_r: $u(z) = r$, let Φ_r denote the Riemannian image of Ω_r and let $A(r)$ denote the spherical area of Φ_r. Then

$$A(r) = \int_0^r \int_0^{2\pi} \frac{|f'(z)|^2}{(1+|f(z)|^2)^2} |\varphi'(u+iv)|^2 \, dv du,$$

where we denote by φ the invease function of $u(z) + iv(z)$. Set

$$S(r) = \frac{A(r)}{\pi} \quad \text{and} \quad \xi = \overline{\lim_{r\to\infty}} \frac{n(r)}{S(r)}.$$

Then the following holds:

If ξ is finite, then $f(z)$ takes every value in the extended w-plane infinitely often with possibie $2+[\xi]$ exceptions, where $[\xi]$ denotes the greatest integer not exceeding ξ (Hällström [4], Tsuji [12], [13]).

Hence our theorem is obtained immediately. In fact, we showed in the above that for all $r \geqq r_0$

$$\tau \leqq L(r) = \int_0^{2\pi} \frac{|f'(z)|}{1+|f(z)|^2} |\varphi'(u+iv)| \, dv.$$

By the Schwarz inequality, we have

$$\tau^2 \leqq 2\pi \int_0^{2\pi} \frac{|f'(z)|^2}{(1+|f(z)|^2)^2} |\varphi'(u+iv)|^2 \, dv,$$

and hence

$$\frac{\tau^2}{2\pi}(r - r_0) \leqq \int_0^r \int_0^{2\pi} \frac{|f'(z)|^2}{(1+|f(z)|^2)^2} |\varphi'(u+iv)|^2 \, dv du = A(r).$$

8) We denote by mod R the harmonic modulus of a doubly-connected domain R.

From our condition, it follows that

$$0 \leqq \varliminf_{r\to\infty} \frac{n(r)}{S(r)} \leqq \frac{2\pi^2}{\tau^2} \lim_{r\to\infty} \frac{n(r)}{(r-r_0)} = \frac{2\pi^2}{\tau^2} \lim_{r\to\infty} \frac{n(r)}{r} = 0.$$

This contradicts the assumption that $f(z)$ has more than two exceptional values.

By the same arguments, we can see that $f(z)$ has at most a finite number of exceptional values if

$$\varlimsup_{r\to\infty} \frac{n(r)}{r} < +\infty.$$

Thus our theorem is established.

Remark. Let D be a domain containing E completely. From our proof we can see that every function, which is single-valued and meromorphic in the domain $D-E$ and has an essential singularity at each point of E, has at most ρ_0+1 exceptional values at each singularity.

6. Let E be the boundary of a domain D in the z-plane and let ζ be a point of E. If ζ has a neighborhood $U(\zeta)$ whose boundary consists of one closed analytic curve not touching E, $\Omega_\zeta = D \cap U(\zeta)$ is a domain. An exhaustion of Ω_ζ whose first domain is $\mathscr{C}\overline{U(\zeta)}$ is called a local exhaustion of D at ζ and the graph associated with this is called a local graph at ζ.

THEOREM 2. *Let E be the boundary of a domain D in the z-plane and let ζ be a point of E. If there is a local exhaustion at ζ which satisfies the conditions* 1°), 2°), 3°) *and* 4°) *stated in* § **2**, *branches off at most ρ_0-times everywhere and has the local graph with infinite length satisfying the conditions that*

$$\lim_{r\to\infty} \mu(r) = +\infty \quad \textit{and} \quad \lim_{r\to\infty} \frac{n(r)}{r} = 0,$$

or if there exists a sequence of points $\{\zeta_n\}$ of E converging to ζ, at each point of which the local exhaustion with the same properties as above is found, then every function, which is single-valued and meromorphic in D and has an essential singularity at each point of E, has at most ρ_0+1 exceptional values at ζ.

If at each ζ_n, the local exhaustion branches off at most ρ_n-times everywhere and has the local graph with infinite length satisfying the conditions that

$$\lim_{r\to\infty} \mu(r) = +\infty \quad \textit{and} \quad \lim_{r\to\infty} \frac{n(r)}{r} < +\infty,$$

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then the functions have at most a countable number of exceptional values at ζ. *(We remark that integers* ρ_n *depend on n and need not be bounded.)*

For in the case where there is a local exhaustion at ζ satisfying our conditions, the assertion is true obviously from Theorem 1. If ζ is the limiting point of $\{\zeta_n\}$, each neighborhood contains points of $\{\zeta_n\}$ and hence $f(z)$ has at most ρ_0+1 exceptional values at ζ.

If we replace the condition $\lim_{r\to\infty} n(r)/r=0$ *by* $\varliminf_{r\to\infty} n(r)/r<+\infty$ and if $f(z)$ has non-countable number of exceptional values, then we can find a neighborhood where $f(z)$ does not take an infinite number of values; this contradicts the fact that this neighborhood contains points of $\{\zeta_n\}$ where $f(z)$ has at most a finite number of exceptional values.

7. In this section we shall show the existence of general Cantor sets in whose complement the functions have a finite number of exceptional values.

First we state the definition of general Cantor sets.[9] Let $k_1, k_2, \ldots$ be integers greater than 1 and let $p_1, p_2, \ldots$ be finite numbers also greater than 1. We set $h_q=1/(k_q p_q)$. Let I be a closed interval with the length $d>0$. We delete (k_q-1) intervals of equal length from I so that there remain k_q intervals of equal length $h_q d$. We call this operation the q-operation applied to I. We begin by applying the 1-operation to [0, 1], next apply the 2-operation to each of the remaining intervals $I_{1\nu}(1\leqq\nu\leqq k_1)$, further apply the 3-operation to each of the remaining intervals $I_{2\nu}(1\leqq\nu\leqq k_1k_2)$ and so on. We call the limiting set of the union of $I_{n\nu}$'s$(1\leqq\nu\leqq\prod_{q=1}^{n}k_q)$ a general Cantor set and denote by $F(k_q, p_q)$.

Now we prove the following

THEOREM 3. *If*

$$k_q\leqq\rho_0(q\geqq 1)\quad \textit{and}\quad \lim_{q\to\infty}p_q=+\infty,$$

and if

$$\varlimsup_{n\to\infty}\frac{\prod_{q=1}^{n-1}k_q}{\sum_{j=2}^{n-1}\frac{1}{\prod_{q=1}^{j-1}k_q}\log\frac{1+\frac{\rho_0}{\rho_0-1}(P_{j-1}-1)}{2}}=0\qquad(<+\infty,\ \textit{resp.}),$$

[9] See Ohtsuka [10].

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then every function, which is single-valued and meromorpic in the complementary domain Ω of $F(k_q, p_q)$[10] *and has an essential singularity at each point of $F(k_q, p_q)$, has at most ρ_0+1 (a finite number of resp.) exceptional values at each singularity.*

Proof. It is sufficient for us to prove that under the conditions of the theorem, $F(k_q, p_q)$ has the complement satisfying the conditions of Theorem 1.

Since $p_q \to \infty$ as $q \to \infty$ and since it suffices to prove locally, we may assume that $p_q \geqq 2$ for all q. We define an exhaustion $\{\Omega_n\}$ of Ω as follows: First we take the outside of the disc $|z-\frac{1}{2}| \leqq 1$ as the first domain Ω_0. Let $C_{1\nu} (1 \leqq \nu \leqq k_1)$ be the circles with the centers at the middle points of $I_{1\nu} (1 \leqq \nu \leqq k_1)$ and with the same radius

$$\frac{1}{2}\left(h_1+\frac{1}{k_1-1}\left(1-\frac{1}{p_1}\right)\right).$$

Then for each $\nu (1 \leqq \nu < k_1)$ $C_{1\nu}$ touches $C_{1(\nu+1)}$. The domain bounded by all of $C_{1\nu}$ is taken the second domain Ω_1. $\Omega_1-\bar{\Omega}_0$ is a doubly-connected domain with the harmonic modulus

$$\mu_{1,1}=\sigma_1>\log\frac{2}{1+\frac{1}{k_1-1}\left(1-\frac{1}{p_1}\right)}>0,$$

because it contains the annulus $1>|z-\frac{1}{2}|>\frac{1}{2}\left(1+\frac{1}{k_1-1}\left(1-\frac{1}{p_1}\right)\right)$. Next we draw the circles $C_{2\nu}$ $(1 \leqq \nu \leqq k_1 k_2)$ with the centers at the middle points of $I_{2\nu}$ $(1 \leqq \nu \leqq k_1 k_2)$ and with the equal radius

$$\frac{1}{2}h_1\left(h_2+\frac{1}{k_2-1}\left(1-\frac{1}{p_2}\right)\right).$$

Then for each ν $((m-1)k_2+1 \leqq \nu < mk_2;\ m=1, 2, \ldots, k_1)$ $C_{2\nu}$ touches $C_{2(\nu+1)}$. We take as the third domain Ω_2 the domain bounded by all of $C_{2\nu}$ and see that the open set $\Omega_2-\bar{\Omega}_1$ consists of k_1 doubly-connected domains $R_{2,1}, R_{2,2}, \ldots, R_{2,k_1}$ which are congruent and hence have the equal harmonic modulus

$$\mu_{2,k}=k_1\sigma_2>\log\frac{h_1+\frac{1}{k_1-1}\left(1-\frac{1}{p_1}\right)}{h_1\left(1+\frac{1}{k_2-1}\left(1-\frac{1}{p_2}\right)\right)}\geqq\log\frac{1+\frac{\rho_0}{\rho_0-1}(p_1-1)}{2}>0$$

$$(k=1, 2, \ldots, k_1),$$

[10] See the remark of Theorem 1.

because they contain the annulus bounded by the concentric circles with radii $\frac{1}{2}\left(h_1+\frac{1}{k_1-1}\left(1-\frac{1}{p_1}\right)\right)$ and $\frac{1}{2}h_1\left(1+\frac{1}{k_2-1}\left(1-\frac{1}{p_2}\right)\right)$. Generally, let $C_{n\nu}$ $(1\leqq \nu\leqq \prod_{q=1}^{n} k_q)$ be the circles with the centers at the middle points of $I_{n\nu}(1\leqq\nu\leqq\prod_{q=1}^{n}k_q)$ and with the equal radius

$$\frac{1}{2}\prod_{q=1}^{n-1}h_q\left(h_n+\frac{1}{k_n-1}\left(1-\frac{1}{p_n}\right)\right).$$

Take the domain bounded by these circles as the $(n+1)$-th domain Ω_n. Then, since for each ν $((m-1)k_n+1\leqq\nu< mk_n;\ m=1, 2, \ldots, \prod_{q=1}^{n-1}k_q)$ $C_{n\nu}$ touches $C_{n(\nu+1)}$, the open set $\Omega_n-\bar{\Omega}_{n-1}$ consists of $\prod_{q=1}^{n-1}k_q$ congruent doubly-connected domains $R_{n,k}$ $(1\leqq k\leqq\prod_{q=1}^{n-1}k_q)$ with the equal harmonic modulus

$$\mu_{n,k}=(\prod_{q=1}^{n-1}k_q)\sigma_n>\log\frac{h_{n-1}+\frac{1}{k_{n-1}-1}\left(1-\frac{1}{p_{n-1}}\right)}{h_{n-1}\left(1+\frac{1}{k_n-1}\left(1-\frac{1}{p_n}\right)\right)}\geqq\log\frac{1+\frac{\rho_0}{\rho_0-1}(p_{n-1}-1)}{2}>0$$

$$(1\leqq k\leqq\prod_{q=1}^{n-1}k_q).$$

For they contain the annulus bounded by the concentric circles with radii $\frac{1}{2}\left(\prod_{q=1}^{n-2}h_q\right)\left(h_{n-1}+\frac{1}{k_{n-1}-1}\left(1-\frac{1}{p_{n-1}}\right)\right)$ and $\frac{1}{2}\left(\prod_{q=1}^{n-1}h_q\right)\left(1+\frac{1}{k_n-1}\left(1-\frac{1}{p_n}\right)\right)$. The domains Ω_n form obviously an exhaustion of Ω which satisfies 1°), 2°), 3°) and 4°) in §2 and branches off at most ρ_0-times everywhere.

Now we consider the graph associated with this exhaustion. The open sets $\Omega_n-\bar{\Omega}_{n-1}$ $(n\geqq1)$ have harmonic moduli σ_n such that

$$\sigma_1>0\quad\text{and}\quad\sigma_n=\frac{\mu_{n,k}}{\prod_{q=1}^{n-1}k_q}>\frac{1}{\prod_{q=1}^{n-1}k_q}\log\frac{1+\frac{\rho_0}{\rho_0-1}(p_{n-1}-1)}{2}\qquad(n\geqq2)$$

and hence we see from our assumption that

$$R=\sum_{n=1}^{\infty}\sigma_n>\sum_{n=2}^{\infty}\frac{1}{\prod_{q=1}^{n-1}k_q}\log\frac{1+\frac{\rho_0}{\rho_0-1}(p_{n-1}-1)}{2}=+\infty,$$

that is, the length of the graph is infinite. We shall show that

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$$\lim_{r\to\infty}\mu(r)=+\infty \quad \text{and} \quad \lim_{r\to\infty}\frac{n(r)}{r}=0 \qquad (<+\infty, \text{ resp.}).$$

Since

$$\mu(r)=\mu_{n,k}>\log\frac{1+\frac{\rho_0}{\rho_0-1}(p_{n-1}-1)}{2} \qquad \left(\sum_{j=1}^{n-1}\sigma_j\leqq r<\sum_{j=1}^{n}\sigma_j\right)$$

and since $p_n\to+\infty$ as $n\to\infty$, the first relation holds. From our condition and from the facts that

$$n(r)=\prod_{q=1}^{n-1}k_q\left(\sum_{j=1}^{n-1}\sigma_j\leqq r<\sum_{j=1}^{n}\sigma_j\right) \text{ and } \sum_{j=1}^{n-1}\sigma_j>\sum_{j=2}^{n-1}\frac{1}{\prod_{q=1}^{j-1}k_q}\log\frac{1+\frac{\rho_0}{\rho_0-1}(p_{n-1}-1)}{2},$$

we have

$$\varlimsup_{r\to\infty}\frac{n(r)}{r}\leqq\varlimsup_{n\to\infty}\frac{\prod_{q=1}^{n-1}k_q}{\sum_{j=2}^{n-1}\frac{1}{\prod_{q=1}^{j-1}k_q}\log\frac{1+\frac{\rho_0}{\rho_0-1}(p_{j-1}-1)}{2}}=0$$

$$(<+\infty, \text{ resp.}).$$

Thus we see that all conditions of Theorem 1 are satisfied. The proof is now complete.

For instance, the general Cantor set $F(k_q, p_q)$ such that

$$k_q=\rho_0(q\geqq1) \quad \text{and} \quad p_q=2\exp\rho_0^{\alpha q} \qquad (\alpha>2)$$

satisfies the conditions of Theorem 3. In fact, we have that

$$\varlimsup_{n\to\infty}\frac{\rho_0^{n-1}}{\sum_{j=2}^{n-1}\frac{1}{\rho_0^{j-1}}\log\frac{1+\frac{\rho_0}{\rho_0-1}(2e^{\rho_0^{\alpha(j-1)}}-1)}{2}}\leqq\varlimsup_{n\to\infty}\frac{\rho_0^{n-1}}{\sum_{j=2}^{n-1}\rho_0^{(\alpha-1)(j-1)}}=0.$$

8. In this last section, we shall show by an example that the conditions of Theorem 1 for $\rho_0=2$ are not sufficient in order that the number of exceptional values is not greater than two, that is, there exist a perfect set E satisfying the conditions of Theorem 1 for $\rho_0=2$ and a function $f(z)$ which is single valued and meromorphic in the complement of E, has an essential singularity at each point of E and has three exceptional values at each singularity.

EXAMPLE. We delete from the w-plane the origin and the point $w=1$ and denote by R the resulting domain. By induction we shall construct convering surfaces $\hat{R}^n$ of the w-plane and define an exhaustion $\{\hat{R}_k\}_{k=0,1,2,\ldots}$ of their limiting surface $\hat{R}$ in the below.

Let A, B, C, . . . denote simple closed analytic curves in R. Consider three points: the point at infinity, the origin and the point $w=1$. We shall denote by $\{A, B; C, D, E\}_\infty$ ($\{A, B; C, D, E\}_0$, $\{A, B; C, D, E\}_1$, resp.) a set of five curves such that A and B separate the point at infinity (the origin, the point $w=1$, resp.) from the other two points and touch each other, such that C separates A from the point at infinity (the origin, the point $w=1$, resp.) and such that D and E surround the origin and the point $w=1$ respectively (the point $w=1$ and the point at infinity respectively, the point at infinity and the origin respectively, resp.), touch each other and form with B the boundary of a doubly-connected domain $(B, D\cup E)$[11]. Further we shall denote by $\{F, G; H, I\}_\infty$($\{F, G; H, I\}_0$, $\{F, G; H, I\}_1$, resp.) a set of four curves such that F separates the point at infinity (the origin, the point $w=1$, resp.) from the others, G is homotopic to zero with respect to R and they touch each other and that H and I separate F and G, respectively, from the point at infinity (the origin, the point $w=1$, resp.).

First we take a replica $\hat{R}^1$ of R. We can determine there $\{\alpha_{1,1}, \alpha_{1,2}; \alpha_{2,1}, \alpha_{2,2}, \alpha_{2,3}\}_\infty$ so that the harmonic moduli of doubly connected domains $(\alpha_{1,1}, \alpha_{2,1})$ and $(\alpha_{1,2}, \alpha_{2,2}\cup\alpha_{2,3})$ are not less than 8. In fact, first we determine curves $\alpha_{2,2}$ and $\alpha_{2,3}$, next determine $\alpha_{1,2}$ so that

$$\text{mod}\,(\alpha_{1,2}, \alpha_{2,2}\cup\alpha_{2,3})\geqq 8$$

and last determine $\alpha_{1,1}$ and $\alpha_{2,1}$ so that

$$\text{mod}\,(\alpha_{1,1}, \alpha_{2,1})\geqq 8.$$

The domain bounded by $\alpha_{1,1}\cup\alpha_{1,2}$ is taken as $\hat{R}_1$ and the domain bounded by $\alpha_{2,1}\cup\alpha_{2,2}\cup\alpha_{2,3}$ is taken as $\hat{R}_2$. We determine $\hat{R}_0$ so that $\overline{\hat{R}_0}\subset\hat{R}_1$ and $\hat{R}_1-\overline{\hat{R}_0}$ is a doubly-connected domain with the harmonic modulus not less than 2. Denoting by σ_j the harmonic moduli of the open sets $\hat{R}_j-\overline{\hat{R}_{j-1}}$, we observe that

$$\sigma_1\geqq 2,\ \sigma_2\geqq 4 \text{ and } n(r)\leqq 2 \text{ for all } r:\ 0\leqq r<\sigma_1+\sigma_2.$$

[11] We denote by (C_1, C_2) a doubly-connected domain, if C_1 is one of its boundary components and C_2 is the other.

Next we take three replicas $\{R_i\}_{i=1,2,3}$ of R. We draw $\{\alpha_{3,1}, \alpha_{3,2}; \alpha_{4,1}, \alpha_{5,1}\}_\infty$ in $\hat{R}^1$ and $\{\alpha_{4,2}, \alpha_{4,3}; \alpha_{5,2}, \alpha_{5,3}, \alpha_{5,4}\}_\infty$ in R_1 as follows: First we determine $\alpha_{4,3}$, $\alpha_{5,3}$ and $\alpha_{5,4}$ in R_1 so that

$$\bmod(\alpha_{4,3}, \alpha_{5,3} \cup \alpha_{5,4}) \geqq 9 \cdot 2^5,$$

and next determine $\{\alpha_{3,1}, \alpha_{3,2}; \alpha_{4,1}, \alpha_{5,1}\}_\infty$ in $\hat{R}^1$ so that $\alpha_{3,1} \cup \alpha_{3,2}$ is contained in the end part of $\hat{R}^1$ bounded by $\alpha_{2,1}$ and does not intersect the same curve as $\alpha_{4,3}$ drawn in $\hat{R}^1$, and that

$$\bmod(\alpha_{2,1}, \alpha_{3,1} \cup \alpha_{3,2}) \geqq 3 \cdot 2^3, \ \bmod(\alpha_{3,1}, \alpha_{4,1}) \geqq 6 \cdot 2^4 \text{ and } \bmod(\alpha_{4,1}, \alpha_{5,1}) \geqq 9 \cdot 2^5.$$

Last we determine $\alpha_{4,2}$ and $\alpha_{5,2}$ so that the domain bounded by $\alpha_{4,2}$ and $\alpha_{4,3}$ contains the same curve as $\alpha_{3,2}$ drawn in R_1 and

$$\bmod(\alpha_{4,2}, \alpha_{5,2}) \geqq 9 \cdot 2^5.$$

We connect R_1 with $\hat{R}^1$ crosswise across a slit in the domain bounded by $\alpha_{3,2}$. If we choose this slit sufficiently small, we have

$$\bmod(\alpha_{3,2}, \alpha_{4,2} \cup \alpha_{4,3}) \geqq 6 \cdot 2^4.$$

In the similar manner, we draw $\{\alpha_{3,3}, \alpha_{3,4}; \alpha_{4,4}, \alpha_{5,5}\}_0$ and $\{\alpha_{3,5}, \alpha_{3,6}; \alpha_{4,7}, \alpha_{5,9}\}_1$ in $\hat{R}^1$, $\{\alpha_{4,5}, \alpha_{4,6}; \alpha_{5,6}, \alpha_{5,7}, \alpha_{5,8}\}_0$ in R_2 and $\{\alpha_{4,8}, \alpha_{4,9}; \alpha_{5,10}, \alpha_{5,11}, \alpha_{5,12}\}_1$ in R_3 and connect R_2 and R_3 with $\hat{R}^1$ across suitable slits in domains bounded by $\alpha_{3,4}$ and $\alpha_{3,6}$. The resulting surface is denoted by $\hat{R}^2$. We take as $\hat{R}_3$ the domain of $\hat{R}^2$ bounded by $\bigcup_{i=1}^{6} \alpha_{3,i}$, as $\hat{R}_4$ one bounded by $\bigcup_{i=1}^{9} \alpha_{4,i}$ and as $\hat{R}_5$ one bounded by $\bigcup_{i=1}^{12} \alpha_{5,i}$. Then we see that

$$\sigma_j \geqq 2^j \ (1 \leqq j \leqq 5) \text{ and } n(r) \leqq 9 \text{ for all } r: \sum_{j=1}^{2} \sigma_j \leqq r < \sum_{j=1}^{5} \sigma_j.$$

Suppose that $\hat{R}^n$ and $\hat{R}_k$ $(0 \leqq k \leqq 3n-1)$ are obtained so that $\hat{R}^n$ has 4^{n-1} sheets and $\partial \hat{R}_{3n-1}$ consists of $3 \cdot 4^{n-1}$ simple closed analytic curves $\alpha_{3n-1,i}$ $(1 \leqq i \leqq 3 \cdot 4^{n-1})$, each of which separates one of the three points from the other two, and that

$$\sigma_j \geqq 2^j (1 \leqq j \leqq 3n-1) \text{ and } n(r) \leqq 9 \cdot 4^{p-2} \text{ for all } r: \sum_{j=1}^{3p-4} \sigma_j \leqq r < \sum_{j=1}^{3p-1} \sigma_j \quad (2 \leqq p \leqq n).$$

Then we take $3 \cdot 4^{n-1}$ replicas R_i $(1 \leqq i \leqq 3 \cdot 4^{n-1})$ of R and connect each R_i with $\hat{R}^n$ crosswise across a suitable slit in the end part of $\hat{R}^n$ divided by $\alpha_{3n-1,i}$ as follows: We consider only the case where $\alpha_{3n-1,i}$ surrounds the point at infinity. (In the other cases, it is sufficient for us to replace ∞ by 0 or 1 in the below.) In a similar way as above we determine $\{\alpha_{3n,2i-1}, \alpha_{3n,2i}; \alpha_{3n+1,3i-2}, \alpha_{3n+2,4i-3}\}_\infty$ in the end part of $\hat{R}^n$ divided by $\alpha_{3n-1,i}$ and $\{\alpha_{3n+1,3i-1}, \alpha_{3n+1,3i}; \alpha_{3n+2,4i-2}, \alpha_{3n+2,4i-1}, \alpha_{3n+2,4i}\}_\infty$ in R_i so that the harmonic moduli of the doubly-connected domains $(\alpha_{3n+1,3i-2}, \alpha_{3n+2,4i-3})$, $(\alpha_{3n+1,3i-1}, \alpha_{3n+2,4i-2})$ and $(\alpha_{3n+1,3i}, \alpha_{3n+2,4i-1} \cup \alpha_{3n+2,4i})$ are not less than 2^{3^n+2}, and that

$$\text{mod}\,(\alpha_{3n,2i-1}, \alpha_{3n+1,3i-2}) \geqq 2^{3n+1} \text{ and } \text{mod}\,(\alpha_{3n-1,i}, \alpha_{3n,2i-1}) \geqq 2^{3n}.$$

Then we connect R_i with $\hat{R}^n$ crosswise across a slit in the domain bounded by $\alpha_{3n,2i}$, where we choose it so small that

$$\text{mod}\,(\alpha_{3n,2i}, \alpha_{3n+1,3i-1} \cup \alpha_{3n+1,3i}) \geqq 2^{3n+1}.$$

In the surface $\hat{R}^{n+1}$ thus obtained we determine $\hat{R}_{3n}$, $\hat{R}_{3n+1}$ and $\hat{R}_{3n+2}$ as the domains bounded by $\bigcup_i \alpha_{3n,i}$, $\bigcup_i \alpha_{3n+1,i}$ and $\bigcup_i \alpha_{3n+2,i}$ respectively. It is easily seen that $\hat{R}^{n+1}$ and $\hat{R}_k$ $(0 \leqq k \leqq 3n+2)$ satisfy the all conditions added on $\hat{R}^n$ and $\hat{R}_k$ $(0 \leqq k \leqq 3n-1)$ for $n+1$. The limiting surface $\hat{R}$ is a covering surface of the w-plane, has a null boundery and is of planar character.

We map $\hat{R}$ one-to-one conformally onto a domain Ω in the z-plane which is the complement of a compact set E of capacity zero and denote this mapping function by $\hat{f}$. By the same arguments used in [7] we see that $f(z) = \varphi \circ \hat{f}^{-1}(z)$ is single-valued and meromorphic in Ω, has an essential singularity at each point of E and has at each singularity three exceptional values: values 0, 1 and infinity, where we denote by φ the projection of $\hat{R}$ into the w-plane. But E satisfies the conditions of Theorem 1. In fact, if we take as an exhaustion of Ω $\{\Omega_k = \hat{f}^{-1}(\hat{R}_k)\}_{k=0,1,2,\ldots}$, it satisfies obviously the conditions 1°), 2°), 3°) and 4°) in §2 and branches off at most 2-times everywhere. Furthermore since the harmonic moduli of the open sets $\Omega_k - \Omega_{k-1}$ $(k \geqq 1)$ are equal to $\sigma_k \geqq 2^k$ and since

$$n(r) \leqq 9 \cdot 4^{p-2} \text{ for all } r: \sum_{j=1}^{3p-4} \sigma_j \leqq r < \sum_{j=1}^{3p-1} \sigma_j \qquad (p \geqq 2),$$

we have that

$$\lim_{r\to\infty} \mu(r) \geqq \lim_{k\to\infty} 2^k = +\infty \text{ and } \lim_{r\to\infty} \frac{n(r)}{r} \leqq \lim_{p\to\infty} \frac{9 \cdot 4^{p-1}}{\sum_{j=1}^{3p-1} 2^j} = \frac{9}{4} \lim_{p\to\infty} \frac{1}{2^p(1-2^{-3p})} = 0.$$

Remark. It is still open whether there is a perfect set E for which every function has at most two exceptional values at each singularity.

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Added in proofs: During the proofs of this paper, the author found that Carleson gave an important theorem, which is closely related to ours, in his recent paper: A remark on Picard's theorem, Bull. Amer. Math. Soc. **67** (1961), pp. 142-144.