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SCHUR'S COLOURING THEOREM FOR NONCOMMUTING PAIRS

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Abstract

For G a finite non-Abelian group we write $c(G)$ for the probability that two randomly chosen elements commute and $k(G)$ for the largest integer such that any $k(G)$-colouring of G is guaranteed to contain a monochromatic quadruple (x, y, xy, yx) with $xy \neq yx$. We show that $c(G) \to 0$ if and only if $k(G) \to \infty$.

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1. Introduction

Our starting point is Schur's theorem [18, Hilfsatz], the proof of which adapts to give the following result.

THEOREM 1.1. *Suppose that G is a finite group and C is a cover of G of size k. Then there is a set $A \in C$ with at least $c_k|G|^2$ triples $(x, y, xy) \in A^3$ where c_k is a constant depending only on k.*

The proof is a routine adaptation, but we shall not give it as the result as stated also follows from our next theorem.

If G is non-Abelian then we might like to ask for quadruples $(x, y, xy, yx) \in A^4$ instead of triples. Establishing the following result (which we do in Section 2) is the main purpose of the paper.

THEOREM 1.2. *Suppose that G is a finite group and C is a cover of G of size k. Then there is a set $A \in C$ with $c_k|G|^2$ quadruples $(x, y, xy, yx) \in A^4$ where c_k is a constant depending only on k.*

When G is non-Abelian we should like to ensure that at least one of the quadruples found in Theorem 1.2 has $xy \neq yx$, and to this end we define the *commuting probability* of a finite group G to be

$$c(G) := \frac{1}{|G|^2} \sum_{x,y\in G} 1_{[xy=yx]};$$

in words, it is the probability that a pair $(x, y) \in G^2$ chosen uniformly at random has $xy = yx$. There are many nice results about the commuting probability (see the introduction to [10] for details) and it is an instructive exercise (see [9]) to check that if $c(G) < 1$ then $c(G) \leq \frac{5}{8}$, so that if a group is non-Abelian there are 'many' pairs that do not commute. Despite this we prove the following result in Section 3.

PROPOSITION 1.3. *Suppose that G is a finite group and $c(G) \geq \epsilon$. Then there is a cover C of G of size $\exp((2 + o_{\epsilon\to 0}(1))\epsilon^{-1}\log\epsilon^{-1})$ such that if $A \in C$ and $(x, y, xy, yx) \in A^4$ then $xy = yx$.*

If G is non-Abelian we write $k(G)$ for the *noncommuting Schur number* of G, that is, the largest natural number such that for any cover C of G of size $k(G)$ there is some $A \in C$ and $(x, y, xy, yx) \in A^4$ with $xy \neq yx$. (Note that since G is assumed non-Abelian we certainly have $k(G) \geq 1$.)

The number $k(G)$ has been studied for a range of specific groups by McCutcheon in [12] and we direct the interested reader there for examples and further questions.

THEOREM 1.4. *Let $(G_n)_n$ be a sequence of non-Abelian groups. Then $c(G_n) \to 0$ if and only if $k(G_n) \to \infty$.*

PROOF. The right to left implication follows immediately from Proposition 1.3. We can assume that c_k is monotonically decreasing. Suppose that $c(G_n) \to 0$ and there is a k_0 and an infinite set S of natural numbers such that $k(G_n) < k_0$ for all $n \in S$. Let $n \in S$ be such that $c(G_n) < c_{k_0}$ which can be done since $c(G_n) \to 0$ and $c_{k_0} > 0$.

Since $k(G_n) < k_0$ there is a cover C of G_n of size k_0 such that if $A \in C$ and $(x, y, xy, yx) \in A^4$ then $xy = yx$. By Theorem 1.2 there is an $A \in C$ such that $(x, y, xy, yx) \in A^4$ for at least $c_{k_0}|G_n|^2$ quadruples. But then by design $xy = yx$ for all these pairs and so $c(G_n) \geq c_{k_0}$, a contradiction which proves the result. □

Before closing this section we need to acknowledge our debt to previous work. In [13] McCutcheon proves that $k(S_n) \to \infty$ as $n \to \infty$. A short calculation shows that $c(S_n) \to 0$ as $n \to \infty$, and the possibility of showing that $k(G_n) \to \infty$ as $c(G_n) \to 0$ is identified by Bergelson and Tao in the remarks after [5, Theorem 11]. Earlier, in [5, Footnote 4], they also observe the significance of Neumann's work [14] which is the main idea behind the proof of Proposition 1.3.

Write $D(G)$ for the smallest dimension of a nontrivial unitary representation of G. (This is called the quasirandomness of G in [5, Definition 1] following the work of Gowers [8].) In [5, Corollary 8] the authors show that $k(G_n) \to \infty$ as $D(G_n) \to \infty$, and in fact go further proving a density result. For general finite groups there can be no density result; we refer the reader to the discussion after [5, Theorem 11] for more details.

2. Proof of Theorem 1.2

The proof of Theorem 1.2 is inspired by an attempt to translate the proof of [3, Theorem 3.4] into a combinatorial setting. There the authors use a recurrence

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theorem [4, Theorem 5.2]; in its place we use a version of the Ajtai–Szemerédi corners theorem [1] for finite groups. This was proved by Solymosi [22, Theorem 2.1] using the triangle removal lemma.

Theorem 2.1. *There is a function* $f_\Delta : (0,1] \to (0,1]$ *such that if* G *is a finite group and* $\mathcal{A} \subset G^2$ *has size at least* $\alpha|G|^2$ *then*

$$S(\mathcal{A}) := \frac{1}{|G|^3} \sum_{x,y,z \in G} 1_{\mathcal{A}}(x,y)1_{\mathcal{A}}(zx,y)1_{\mathcal{A}}(x,yz) \geq f_\Delta(\alpha).$$

Proof. Following the proof of [22, Theorem 2.1], form a tripartite graph with three copies of G as the vertex sets (call them V_1, V_2, V_3) and joining $(x,y) \in V_1 \times V_2$ if and only if $(x,y) \in \mathcal{A}$; $(y,w) \in V_2 \times V_3$ if and only if $(y^{-1}w, y) \in \mathcal{A}$; and $(x,w) \in V_1 \times V_3$ if and only if $(x, wx^{-1}) \in \mathcal{A}$. The map $G^3 \to G^3; (x,y,w) \mapsto (x,y,y^{-1}wx^{-1})$ is a bijection and (x,y,w) is a triangle in this graph if and only if $(x,y),(zx,y),(x,yz) \in \mathcal{A}$ where $z = y^{-1}wx^{-1}$.

It follows from [23, Theorem 1.1] that one can remove at most

$$3 \cdot o_{S(\mathcal{A}) \to 0}(|G|^2) = o_{S(\mathcal{A}) \to 0}(|G|^2)$$

elements from $\mathcal{A}$ to make the graph triangle-free. On the other hand if $(x,y) \in \mathcal{A}$ then (x,y,xy) is a triangle in the above graph and hence we must have removed all elements from $\mathcal{A}$ and $\alpha|G|^2 \leq o_{S(\mathcal{A}) \to 0}(|G|^2)$ from which the result follows. □

There are a number of subtleties around the extent to which one can replace, say, (zx,y) with (xz,y), and we refer the reader to the papers of Solymosi [22] and Austin [2] for some discussion.

We take the convention, as we can, that the function f_Δ is monotonically increasing and $f_\Delta(x) \leq x$ for all $x \in (0,1]$. Even with Fox's work [7], in general we only have $f_\Delta(\alpha)^{-1} \leq T(O(\log \alpha^{-1}))$. However, when G is Abelian much better bounds are known as a result of the beautiful arguments of Shkredov [19–21]. It seems likely that these could be adapted to give a bound with a tower of bounded height if the Fourier analysis is adapted to the non-Abelian setting in the same way as it is for Roth's theorem in [17]. Doing so would give a quantitative version of [5, Theorem 10] (see [5, Remark 44]), but the improvement to Theorem 1.2 would only be to replace a wowzer-type function with a tower as we shall see shortly.

We shall prove the following proposition from which Theorem 1.2 follows immediately on inserting the bound for f_Δ given by Theorem 2.1.

Proposition 2.2. *Suppose* G *is a finite group and* C *is a cover of* G *of size* k. *Then there is a set* $A \in C$ *with* $(g^{(k+1)}(1))^2|G|^2$ *quadruples* $(x,y,xy,yx) \in A^4$, *where* $g^{(k+1)}$ *is the* $(k+1)$*-fold composition of* g *with itself and* $g : (0,1] \to (0,1]; \alpha \mapsto (3k)^{-1} f_\Delta(\alpha^k)$.

Proof. Write $A_1, \dots, A_k$ for the sets in C ordered so that their respective densities are $\alpha_1 \geq \cdots \geq \alpha_k$; since C is a cover we have $\alpha_1 \geq 1/k$. Let $r \in \{1, \dots, k\}$ be minimal such that

$$\tfrac{1}{3} f_\Delta(\alpha_1 \cdots \alpha_r) \geq \alpha_{r+1} + \cdots + \alpha_k, \tag{2.1}$$

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which is possible since the sum on the right is empty and so 0 when $r = k$. From minimality and the order of the α_is,

$$\alpha_{i+1} > \frac{1}{3k} f_\Delta(\alpha_1 \cdots \alpha_i) \quad \text{for all } 1 \leq i \leq r-1.$$

The function f_Δ is monotonically increasing and $f_\Delta(x) \leq x$ for all $x \in (0, 1]$ so it follows from the above that $\alpha_r \geq g^{(r)}(1) \geq g^{(k)}(1)$.

Now, suppose that $s_1, \ldots, s_r \in G$ and write

$$\mathcal{A}_i := \{(x, y) \in G^2 : xs_iy \in A_i\} \quad \text{for } 1 \leq i \leq r.$$

Then

$$\mathbb{E}_{s_i \in G} 1_{\mathcal{A}_i}(x, y) = \alpha_i \quad \text{for all } x, y \in G \text{ and } 1 \leq i \leq r,$$

and so

$$\mathbb{E}_{s \in G^r} \Big| \bigcap_{i=1}^{r} \mathcal{A}_i \Big| = \sum_{x,y \in G} \mathbb{E}_{s \in G^r} \prod_{i=1}^{r} 1_{\mathcal{A}_i}(x, y) = \alpha_1 \cdots \alpha_r |G|^2.$$

By averaging we can pick some $s \in G^r$ such that $\mathcal{A} := \bigcap_{i=1}^{r} \mathcal{A}_i$ has $|\mathcal{A}| \geq \alpha_1 \cdots \alpha_r |G|^2$.

By the definition of f_Δ (from Theorem 2.1),

$$\mathbb{E}_{x,y,z \in G} 1_{\mathcal{A}}(x, y) 1_{\mathcal{A}}(zx, y) 1_{\mathcal{A}}(x, yz) = S(\mathcal{A}) \geq f_\Delta(\alpha_1 \cdots \alpha_r);$$

write

$$Z := \{z \in G : \mathbb{E}_{x,y \in G} 1_{\mathcal{A}}(x, y) 1_{\mathcal{A}}(zx, y) 1_{\mathcal{A}}(x, yz) \geq \tfrac{1}{3} f(\alpha_1 \cdots \alpha_r)\}.$$

Then

$$\begin{aligned} \mathbb{P}(Z) + \tfrac{1}{3} f_\Delta(\alpha_1 \cdots \alpha_r) &\geq \mathbb{E}_{x,y,z \in G} 1_{Z \sqcup (G \setminus Z)}(z) 1_{\mathcal{A}}(x, y) 1_{\mathcal{A}}(zx, y) 1_{\mathcal{A}}(x, yz) \\ &= S(\mathcal{A}) \geq f_\Delta(\alpha_1 \cdots \alpha_r), \end{aligned}$$

and hence $\mathbb{P}(Z) \geq \frac{2}{3} f_\Delta(\alpha_1 \cdots \alpha_r)$. But then

$$\begin{aligned} \mathbb{P}(Z \setminus (A_{r+1} \cup \cdots \cup A_k)) &\geq \tfrac{2}{3} f_\Delta(\alpha_1 \cdots \alpha_r) - (\alpha_{r+1} + \cdots + \alpha_k) \\ &\geq \tfrac{1}{3} f_\Delta(\alpha_1 \cdots \alpha_r) \end{aligned}$$

by (2.1). Since $\bigcup_{i=1}^{k} A_i = G$, we conclude that there is some i with $1 \leq i \leq k$ such that

$$\mathbb{P}((Z \setminus (A_{r+1} \cup \cdots \cup A_k)) \cap A_i) \geq \frac{1}{3r} f_\Delta(\alpha_1 \cdots \alpha_r).$$

Of course $(Z \setminus (A_{r+1} \cup \cdots \cup A_k)) \cap A_j = \emptyset$ for $r < j \leq k$ and so we may assume $i \leq r$.

Write $Z' := (Z \setminus (A_{r+1} \cup \cdots \cup A_k)) \cap A_i$. Since $Z' \subset Z$,

$$\mathbb{E}_{x,y} 1_{\mathcal{A}_i}(x, y) 1_{\mathcal{A}_i}(zx, y) 1_{\mathcal{A}_i}(x, yz) \geq \mathbb{E}_{x,y} 1_{\mathcal{A}}(x, y) 1_{\mathcal{A}}(zx, y) 1_{\mathcal{A}}(x, yz) \geq \tfrac{1}{3} f_\Delta(\alpha_1 \cdots \alpha_r)$$

for all $z \in Z'$. On the other hand, every $z \in Z'$ has $z \in A_i$ and so we conclude that there are at least

$$\frac{1}{3} f_\Delta(\alpha_1 \cdots \alpha_r) |G|^2 \cdot \frac{1}{3r} f_\Delta(\alpha_1 \cdots \alpha_r) |G|$$

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triples $(x, y, z) \in G^3$ such that

$$z \in A_i, \quad xs_iy \in A_i, \quad zxs_iy \in A_i \quad \text{and} \quad xs_iyz \in A_i.$$

The map $(x, y, z) \mapsto (xs_iy, z)$ has all fibres of size $|G|$ and so there are at least

$$\frac{1}{9r} f_\Delta(\alpha_1 \cdots \alpha_r)^2 |G|^2 \geq (g(\alpha_r))^2 |G|^2$$

pairs $(a, b) \in G^2$ such that $a, b, ab, ba \in A_i$. This gives the result. □

3. Proof of Proposition 1.3

The key idea comes from Neumann's theorem [14, Theorem 1] which is already identified in [5, Footnote 4]. Neumann's theorem describes the structure of groups G for which $c(G) \geq \epsilon$; they are the groups containing normal subgroups $K \leq H \leq G$ such that K and G/H have size $O_\epsilon(1)$ and H/K is Abelian. Neumann's theorem was further developed in [6, Theorem 2.4], but both arguments provide a more detailed structure than we require.

We have made some effort to control the exponent; results such as [6, Lemma 2.1] or [15, Theorem 2.2] could be used in place of Kemperman's theorem in what follows at the possible expense of the 2 becoming slightly larger. Moving the $2 + o_{\epsilon\to 0}(1)$ below 1 would require a slightly different approach as we normalise a subgroup of index around ϵ^{-1} at a certain point which costs us a term of size $\epsilon^{-1}!$.

Proposition (Proposition 1.3). *Suppose that G is a finite group and $c(G) \geq \epsilon$. Then there is a cover C of G of size $\exp((2 + o_{\epsilon\to 0}(1))\epsilon^{-1} \log \epsilon^{-1})$ such that if $A \in C$ and $(x, y, xy, yx) \in A^4$ then $xy = yx$.*

Proof. We work with the conjugation action of G on itself (that is, $(g, x) \mapsto g^{-1}xg$) and write x^G for the conjugacy class of x (the orbit of x under this action) and $C_G(x)$ for the centre of x in G (the stabiliser of x under this action).

Let $\eta, \nu \in (0, 1]$ be parameters (we shall take $\nu = \frac{1}{2}$ and $\eta = \epsilon/\log \epsilon^{-1}$) to be optimised later and put

$$X := \{x \in G : |x^G| \leq \eta^{-1}\}.$$

Then

$$\epsilon|G|^2 \leq |G|^2 \mathbb{P}(xy = yx) = \sum_x |C_G(x)| = |G| \sum_x \frac{1}{|x^G|} \leq \sum_{x\in X} |G| + \sum_{x\notin X} \eta|G|.$$

Writing $\kappa := |X|/|G|$ we can rearrange the above to see that $\kappa \geq (\epsilon - \eta)/(1 - \eta)$.

Suppose that $s \in \mathbb{N}$ is maximal such that

$$|\overbrace{X \cdots X}^{s \text{ times}}| \geq (1 + (1 - \nu)(s - 1))|X|.$$

There is some $s \in \mathbb{N}$ since the inequality certainly holds for $s = 1$, and there is a maximal such s with $s \leq (\kappa^{-1} - \nu)/(1 - \nu)$ since $|X| \geq \kappa|G|$.

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Since $1_G^G = \{1_G\}$ we have $1_G \in X$ and $1_G \in X \cdots X$ for any s-fold product. By Kemperman's theorem [11, Theorem 5] (also recorded on [16, page 111], and which despite the additive notation does not assume commutativity) it follows that there is some $H \leq G$ such that

$$|\overbrace{X \cdots X}^{s+1 \text{ times}}| \geq |\overbrace{X \cdots X}^{s \text{ times}}| + |X| - |H| \quad \text{and} \quad H \subset \overbrace{X \cdots X}^{s+1 \text{ times}}.$$

By the maximality of s,

$$(1 + (1 - \nu)s)|X| > |\overbrace{X \cdots X}^{s+1 \text{ times}}| \geq (1 + (1 - \nu)(s - 1))|X| + |X| - |H|.$$

Consequently $|H| > \nu|X|$ and so $|G/H| < \nu^{-1}\kappa^{-1}$.

Let K be the kernel of the action of left multiplication by G on G/H, that is, $K := \{x \in G : xgH = gH \text{ for all } g \in G\}$. The action induces a homomorphism from G to $\mathrm{Sym}(G/H)$ so that by the First Isomorphism Theorem

$$K \lhd G \quad \text{and} \quad |G/K| \leq |\mathrm{Sym}(G/H)| \leq |G/H|!.$$

Each $x \in H$ (and hence each $x \in K$ since $xH = H$ for such x) can be written as a product of $s + 1$ elements of X. Moreover, the function $x \mapsto |x^G|$ is submultiplicative, that is $|(xy)^G| \leq |x^G||y^G|$, and so it follows that

$$|x^G| \leq \eta^{(s+1)} \leq R := \lfloor \eta^{-(\kappa^{-1}+1-2\nu)/(1-\nu)} \rfloor$$

for all $x \in X^{s+1}$ and in particular for all $x \in K$. Thus for each $x \in K$ there is an injection $\phi_{x^G} : x^G \to \{1, \ldots, R\}$. With this notation we can define our covering; let

$$\mathcal{S} := \{\{x \in K : \phi_{x^G}(x) = i\} : 1 \leq i \leq R\} \quad \text{and} \quad C := ((G/K) \setminus \{K\}) \cup \mathcal{S},$$

so that $\mathcal{S}$ is a cover of K and C is a cover of G. Now

$$\begin{aligned} |C| &\leq |G/K| - 1 + |\mathcal{S}| \leq \lfloor \nu^{-1}\kappa^{-1} \rfloor! - 1 + R \\ &\leq \exp\Big(\max\Big\{\nu^{-1}\kappa^{-1} \log \nu^{-1}\kappa^{-1}, \frac{\kappa^{-1} + 1 - 2\nu}{1 - \nu} \log \eta^{-1}\Big\} + O(1)\Big). \end{aligned}$$

Optimise this by taking $\nu = \frac{1}{2}$ and $\eta = \epsilon / \log \epsilon^{-1}$ as mentioned before so that $\kappa \geq \epsilon(1 - o_{\epsilon \to 0}(1))$ and $\log \eta^{-1} = (1 + o_{\epsilon \to 0}(1)) \log \epsilon^{-1}$.

Suppose that $A \in C$ and $x, y, xy, yx \in A$. If $A \in (G/K) \setminus \{K\}$ then $xK = yK = xyK = yxK = A$. Since $K \lhd G$ we have $xK = xyK = (xK)(yK)$ and so $yK = K$ which is a contradiction. It follows that $A \in \mathcal{S}$ and hence $x, y, xy, yx \in K$. We conclude that $\phi_{(xy)^G}(xy) = \phi_{(yx)^G}(yx)$ but $xy = y^{-1}(yx)y$ and so $(xy)^G = (yx)^G$. Since $\phi_{(xy)^G}$ is an injection, $xy = yx$ as required.

The result is proved. □

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