UNITARY REPRESENTATIONS OF SOME LINEAR GROUPS

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§0. Introduction. Recently I. Gelfand and M. Neumark [2] have determined the types of irreducible unitary representations of the group $\mathbf{G}_1$ of linear transformations of the straight line. The analogous result is obtained for the group $\mathbf{G}_2$ of transformations $z \to az + b$ in the complex-number plane $\mathfrak{C}$, where a and b run over all complex numbers with the exception of $a = 0$, which may be considered as the group of all sense-preserving similar transformations in the two-dimensional euclidean space E^2. In this paper, we shall determine the types of cyclic[1] unitary representations and irreducible unitary representations of the group $\mathbf{G}$ of all sense-preserving congruent transformations in E^2, which may be realized as the group of all transformations in $\mathfrak{C}$ of the form $z \to az + b$; $a, b \in \mathfrak{C}$ and $|a| = 1$, The method is due to the same idea as Gelfand-Neumark's one [2], but we need Lemma 2 (§2) which is not necessary in the case of $\mathbf{G}_1$ and of $\mathbf{G}_2$. Our method may be applied to the group $\mathbf{G}'$ of all transformations $q \to aq + b$ in the field $\mathbf{Q}$ of quaternions, where $a, b \in \mathbf{Q}$ and $\|a\| = 1$.[2]

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§1. Main results. Let $\mathbf{G}$ be the group of all transformations $z \to az + b$ in the complex-number plane $\mathfrak{C}$ where $a, b \in \mathfrak{C}$ and $|a| = 1$. Then the group $\mathbf{U}$ of all rotations $z \to az$, $|a| = 1$, is a subgroup of $\mathbf{G}$ and the group $\mathbf{V}$ of all translations $z \to z + b$ is a commutative normal subgroup of $\mathbf{G}$, and it holds that

$$(1.1) \qquad \begin{cases} \mathbf{G} = \mathbf{U} \cdot \mathbf{V}, \quad \mathbf{U} \cap \mathbf{V} = \{e\} \qquad (e = \text{the identity of } \mathbf{G}), \\ \mathbf{G}/\mathbf{V} \cong \mathbf{U}. \end{cases}$$

Hereafter we shall denote by u_a and v_b the elements of $\mathbf{U}$ and $\mathbf{V}$ corresponding to the complex number a ($|a| = 1$) and b respectively. Then we have $u_1 = v_0 = e$ and

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[1] It is called "simple" in [3].

[2] The group $\mathbf{G}'$ is different from the group of all sense-preserving congruent transformations in E^4. It seems to be more complicated to determine the types of unitary representations of the group of all sense-preserving congruent transformations in E^n for $n \geqq 3$; — see §4.

(1.2) $$u_a v_b = v_{ab} u_a.$$

Let X be the character group of $\mathbf{V}$ and χ_0 be the identity character. Then X is isomorphic to the two-dimensional vector group as well as $\mathbf{V}$ and consequently every element χ of X may be considered as a complex number $r\exp(i\theta)(r \geqq 0)$. Hereafter we shall denote every $\chi \equiv r\exp(i\theta) \in X$ by the couple $\langle s, r\rangle$ where $s = \exp(i\theta)$; such a couple is unique for $\chi \neq \chi_0 \equiv 0$, and $\tilde{X} = X - \{\chi_0\}$ is the topological product space of the unit circle S in the complex-number plane and $R = (0, \infty)$. Thus we may consider the transformations $\chi \to a\chi$ in X and $s \to as$ $(|a| = 1)$ in S as the multiplication of complex numbers.

We shall here state the main theorems.

THEOREM 1. *Let $\sigma(\Gamma)$ $(\Gamma \subset S)$ be the measure on S invariant under rotations;—*

i) *Fix an arbitrary element $r_0 \in R$, and define the unitary operator $U(g)$ $(g \in \mathbf{G})$ in the Hilbert space $\mathfrak{H} = L^2(S, \sigma)$ as follows: $U_a\psi(s) = \psi(a^{-1}s)$, $V_b\psi(s) = (b, \langle s, r_0\rangle)\psi(s)$*[3] *$(\psi(s) \in L^2(S, \sigma))$ and $U(g) = U_a V_b$ for $g = u_a v_b$.*[4] *Then $\{\mathfrak{H}, U(g)\}$ is an irreducible unitary representation of $\mathbf{G}$, and for any fixed $\psi_0(s) \in L^2(S, \sigma)$ such that $\|\psi_0\| = 1$ the function*

$$\Phi(g) \equiv \Phi(u_a v_b) = \int_S (b, \langle a^{-1}s, r_0\rangle)\, \psi_0(a^{-1}s)\overline{\psi_0(s)}d\sigma(s) \quad (g = u_a v_b) \tag{1.3}$$

is the normal elementary[5] *p. d.*[6] *function on $\mathbf{G}$ corresponding to the above irreducible unitary representation.*

ii) *If $r_1, r_2 \in R$ and $r_1 \neq r_2$, then the unitary representation as stated in* i) *corresponding to r_1 is not unitary equivalent to that corresponding to r_2.*

iii) *Let $\mathfrak{H}$ be the one-dimensional unitary space and l be any fixed integer $(\gtreqless 0)$, and define the unitary operator $U(g)$ by $U_a\psi = a^l\psi$, $V_b\psi = \psi$ $(\psi \in \mathfrak{H})$ and $U(g) = U_a V_b$ for $g = u_a v_b$. Then $\{\mathfrak{H}, U(g)\}$ is an irreducible unitary representation of $\mathbf{G}$, and*

$$\Phi(g) \equiv \Phi(u_a v_b) = a^l \equiv \exp(il\theta) \quad (\textit{for } a = \exp(i\theta)) \tag{1.4}$$

is the corresponding normal elementary p. d. function on $\mathbf{G}$.

iv) *Every irreducible unitary representation of $\mathbf{G}$ is unitary equivalent to one of the above stated types. Consequently every normal elementary p. d. function on $\mathbf{G}$ is expressible in the form* (1.3) *or* (1.4).

THEOREM 2. *Let $\sigma(\Gamma)$ be as stated in Theorem* 1, *and $\rho_j(\Delta)$ $(\Delta \subset R)$, $j = 1$,*

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3) (b, χ) denotes the value of character χ $(\in X)$ at the element $v_b \in \mathbf{V}$.

4) Any element $g \in \mathbf{G}$ is uniquely expressible in this form by virtue of (1.1) and (1.2).

5) See [3] § 15.

6) Abbreviated for *positive definite*.

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2, . . . , n ($\leqq \infty$), *be measures on* R *such that* $\rho_j(R) < \infty$;—

i) *In every Hilbert space* $\mathfrak{M}_j = L^2(\tilde{X}, \sigma \otimes \rho_j)$,[7] *we define the unitary operator* $U(g)$ $(g \in \mathrm{G})$ *as follows:* $U_a\psi(s, r) = \psi(a^{-1}s, r)$ $V_b\psi(s, r) = (b, \langle s, r\rangle)\psi(s, r)$ $(\psi(s, r) \in L^2(\tilde{X}, \sigma \otimes \rho_j))$ *and* $U(g) = U_a V_b$ *for* $g = u_a v_b$; *and let* $f_j(s, r)$, $j = 1, 2, \ldots, n (\leqq \infty)$, *be functions as follows:*

1°) $f_j(s, r) \in L^2(\tilde{X}, \sigma \otimes \rho_j)$ *for every* j,

2°) $\int_S |f_j(s, r)|^2 \, d\sigma(s) = 1$ *for* ρ_j-*almost all* r,

3°) $f_j(s, r)/f_k(s, r)$ *is not constant essentially* (σ) *as a function of* s *for* ρ_j- *or* ρ_k-*almost all* r.

Let $\{\mathfrak{N}_l, U_l(g)\}$ *be the irreducible unitary representation of* G *as stated in Theorem* 1 ii) *corresponding to the integer* l, f'_l *be an arbitrarily fixed element of* $\mathfrak{N}_l$, *and* $\{l_1, l_2, \ldots, l_N\}$ $(N \leqq \infty)$ *be a sequence of integers such that* $k \neq j$ *implies* $l_k \neq l_j$. *Then any of* $\{\mathfrak{M}_j, U(g), f_j\}$ $(j = 1, 2, \ldots, n)$ *and* $\{\mathfrak{H}, U(g), f^\circ\}$ *defined by*

$$\{\mathfrak{H}, U(g)\} = [\bigoplus_{j=1}^{n}\{\mathfrak{M}_j, U(g)\}] \oplus [\bigoplus_{k=1}^{N}\{\mathfrak{N}_{l_k}, U_{l_k}(g)\}]^{8)}$$

and

$$f^\circ = \sum_{j=1}^{n} \alpha_j f_j + \sum_{k=1}^{N} \beta_k f'_{l_k}{}^{9)} \qquad \begin{cases} \sum_{j=1}^{n} |\alpha_j|^2 < \infty \; (\text{if } n = \infty) \\ \sum_{k=1}^{N} |\beta_k|^2 < \infty \; (\text{if } N = \infty) \end{cases}$$

are cyclic unitary representations of G. *The p. d. function* $\Psi(g)$ *corresponding to the unitary representation* $\{\mathfrak{H}, U(g), f^\circ\}$ *is as follows:*

$$\begin{aligned} \Psi(g) &\equiv \Psi(u_a v_b) \\ &= \sum_{j=1}^{n} A_j \int_R d\rho_j(r) \int_S (b, \langle a^{-1}s, r\rangle) f_j(a^{-1}s, r)\overline{f_j(s, r)} d\sigma(s) \\ &\quad + \sum_{k=1}^{N} B_k \exp(i l_k \theta) \qquad \text{for } g = u_a v_b,\ a = e^{i\theta}. \\ &\qquad (A_j = |\alpha_j|^2, \quad B_k = |\beta_k|^2). \end{aligned} \tag{1.5}$$

ii) *Every cyclic unitary representation of* G *is unitary equivalent to that of above stated type, and any p. d. function on* G *is expressible in the form* (1.5), *where* $0 \leqq n \leqq \infty$ *and* $0 \leqq N \leqq \infty$. *The functions*

$$\begin{aligned} \Phi_j(g; r) &\equiv \Phi_j(u_a v_b, r) \\ &= \int_S (b, \langle a^{-1}s, r\rangle) f_j(a^{-1}s, r)\overline{f_j(s, r)} d\sigma(s) \\ &\qquad (r \in R;\ j = 1, 2, \ldots) \end{aligned}$$

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[7] $\sigma \otimes \rho_j$ denotes the product measure of σ and ρ_j.

[8] See [3] § 5 as for the direct sum of unitary representations.

[9] The right-hand side means the summation as elements of the Hilbert space $\mathfrak{H}$.

and

$$\chi_l(g) \equiv \chi_l(u_a v_b) = \exp(il\theta) \quad for \quad a = e^{i\theta}$$
$$(l = \ldots, -2, -1, 0, 1, 2, \ldots)$$

are normal elementary p. d. functions on **G** *and any p. d. function* $\Psi(g)$ *is expressible in the form*

$$\Psi(g) = \sum_{j=1}^{\infty} A_j \int_R \Phi_j(g; r) d\rho_j(r) + \sum_{l=-\infty}^{\infty} B_l \chi_l(g), \tag{1.6}$$

where $A,\ B \geqq 0$, $\sum_{j=1}^{\infty} A_j \rho_j(R) < \infty$ *and* $\sum_{l=-\infty}^{\infty} B_l < \infty$. (Cf. Bochner-Raikov's theorem for p. d. functions on commutative groups.)

As for the group **G′** of all transformations $q \to aq + b$, $\|a\| = 1$, in the field **Q** of quaternions, any irreducible unitary representation and any cyclic unitary representation of **G′** may be obtained by the same methods as stated in Theorems 1 and 2, where the irreducible unitary representation stated in Theorem 1 iii) must be replaced by an irreducible unitary representation of the compact group of all transformations $q \to aq$ ($\|a\| = 1$) in Q; such modifications are necessary for cyclic unitary representations.

After some preliminaries in § 2, we shall prove Theorem 1 in § 3 and Theorem 2 in § 4. Some supplementary remarks will be also given in § 4.

§ 2. Preliminary lemmas.

LEMMA 1. *Let* $\{\mathfrak{M}, U(x)\}$ *be a unitary representation (not necessarily cyclic) of the n-dimensional vector group* **X**, *where* $\mathfrak{M}$ *is a separable Hilbert space. Then there exists a resolution of the identity* $\{E(\Lambda)\}$ *in* $\mathfrak{M}$ *on the character group* X *of the group* **X** *such that*

$$U(x) = \int_X (x, \chi) dE(\chi).$$

Further the space $\mathfrak{M}$ *can be realized as an at most countable direct sum of spaces* $\mathfrak{M}_j$ $(j = 1, 2, \ldots)$ *of the function* $f_j(\chi)$ *such that*

$$\|f_j\| = \int_X |f_j(\chi)|^2 dF_j(\chi) < \infty$$

where $F_j(\Lambda)$ *is a measure on* X *such that* $F_j(X) = 1$ *and every* $F_j(\Lambda)$ *is absolutely continuous with respect to* $F_{j-1}(\Lambda)$ $(j > 1)$; *furthermore, if* $f \in \mathfrak{M}$ *is realized by* $\{f_j(\chi) \ / \ j = 1, 2, \ldots\}$, *then* $U(x)f$ *by* $\{(x, \chi) f_j(\chi) \ / \ j = 1, 2, \ldots\}$.

This lemma is well known as Stone's theorem and Hahn-Hellinger's theory[10] in the case $n = 1$, and may be proved in our general case by the same idea.

LEMMA 2. *Let* $\tilde{X}$, R *and* S *be as stated in* § 1 *and* $F(\Lambda)$ $(\Lambda \subset \tilde{X} \equiv S \times R)$ *be a measure on* $\tilde{X}$ *such that* $F(\tilde{X}) < \infty$, *and assume that there exists a non-nega-*

10) See [5] Chapter VII.

tive function $u(a; \chi)$ *on* $S\times\tilde{X}$ $(a\in S, \chi\in\tilde{X})$, *B-measurable in* $\langle a, \chi\rangle$ *and summable on* $\tilde{X}$ *with respect to the measure* $F(\Lambda)$ *for every* $a\in S$, *such that*

$$F(a^{-1}\Lambda)=\int_{\Lambda}u(a;\chi)dF(\chi)\text{ [11]} \tag{2.1}$$

for any $\Lambda\subset\tilde{X}$ *and any* $a\in S$. *Then there exist a non-negative B-measurable function* $\omega(s, r)$ *on* $\tilde{X}=S\times R$ *and a measure* $\rho(\varDelta)$ *on* R, $\rho(R)<\infty$, *such that* $F(\Lambda)$ *is given by*

$$F(\Lambda)=\int_{\Lambda}\omega(s, r)d\sigma(s)d\rho(r) \tag{2.2}$$

where $\sigma(\Gamma)$ *is the measure on* S *invariant under rotations.*

Proof. For any fixed $\varDelta\subset R$, $F_{\Delta}(\Gamma)=F(\Gamma\times\varDelta)$ $(\Gamma\subset S)$ is a measure on S and it follows from the assumption (2.1) that $F_{\Delta}(a\Gamma)$ is absolutely continuous with respect to $F_{\Delta}(\Gamma)$ for every $a\in S$. Hence $F_{\Delta}(\Gamma)$ is absolutely continuous with respect to the invariant measure $\sigma(\Gamma)$.[12] And hence there exists a function $\mu(s, \varDelta)$ of a point $s\in S$ and a set $\varDelta\subset R$ such that

i) for any fixed $s\in S$, $\mu(s, \varDelta)$ is a regular measure on R and $\mu(s, R)<\infty$,

ii) for any fixed $\varDelta\subset R$, $\mu(s, \varDelta)$ is B-measurable in s, and

iii) for any $\Gamma\subset S$ and $\varDelta\subset R$, $F(\Gamma\times\varDelta)=\int_{\Gamma}\mu(s, \varDelta)d\sigma(s)$; this fact is proved by J. L. Doob [1] as the existence- and uniqueness-theorem of the conditional probability law. Consequently for any $\varphi(\chi)\equiv\varphi(s, r)\in L^1(\tilde{X}, F)$, we have

$$\int_{\tilde{X}}\varphi(s, r)dF(\chi)=\int_{S}d\sigma(s)\int_{R}\varphi(s, r)\mu(s, dr); \tag{2.3}$$

the iterated integral in the right-hand side is well defined by i) and ii), and this equals the left-hand side by iii). From (2.1) and (2.3), we get

$$\begin{aligned}\int_{\Gamma}\mu(as, \varDelta)d\sigma(s)=F(a^{-1}\Gamma\times\varDelta)&=\int_{\Gamma\times\Delta}u(a;\chi)dF(\chi)\\&=\int_{\Gamma}d\sigma(s)\int_{\Delta}u(a; s, r)\mu(s, dr)\end{aligned}$$

for any $\Gamma\subset S$, $\varDelta\subset R$ and any $a\in S$, where $u(a; s, r)=u(a;\chi)$ for $\chi=\langle s, r\rangle$. And hence, for any $\varDelta$, we have

$$\mu(as, \varDelta)=\int_{\Delta}u(a; s, r)\mu(s, dr) \qquad \text{for } \sigma\text{-almost all } s\in S. \tag{2.4}$$

By Fubini's theorem, (2.4) is true for σ-almost all a for σ-almost all s. Since the space R has countable open bases and since $\mu(s, \varDelta)$ is a regular measure

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[11] $a^{-1}\Lambda=\{a^{-1}\chi\ /\ \chi\in\Lambda\}$; — see § 1.

[12] This fact is well known as D. Raikov's lemma.

on R for every s, there exists a point $s_0 \in S$, independent of Δ, such that

$$\mu(as_0, \Delta) = \int_\Delta u(a; s_0, r)\mu(s_0, dr) \qquad \text{for } \sigma\text{-almost all } a \in S.$$

Since the transformation $a \to as_0^{-1}$ is measure-preserving, we obtain by putting $a = ss_0^{-1}$ that

$$(2.5) \qquad \mu(s, \Delta) = \int_\Delta u(ss_0^{-1}; s_0, r)\mu(s_0, dr) \qquad \text{for } \sigma\text{-almost all } s \in S.$$

If we put $\omega(s, r) = u(ss_0^{-1}; s_0, r)$ and $\rho(\Delta) = \mu(s_0, \Delta)$, then $\omega(s, r)$ is B-measurable in $\langle s, r\rangle$ and, by (2.3), (2.4) and Fubini's theorem, we have

$$\int_{\tilde{X}} \varphi(s, r)dF(\chi) = \int_S d\sigma(s)\int_R \varphi(s, r)\omega(s, r)d\rho(r)$$
$$= \int_{\tilde{X}} \varphi(s, r)\omega(s, r)d\sigma(s)d\rho(r)$$

for any $\varphi \in L^1(\tilde{X}, F)$; this implies (2.2), *q.e.d.*

LEMMA 3. *Let* **U**, **V** *and* $\tilde{X}$ *etc. be as in Theorem* 2, $f_1(s, r)$ *be a function* $\in L^2 \equiv L^2(\tilde{X}, \sigma\otimes\rho_1)$ *such that* $\sigma(\{s \;/\; f_1(s, r) \neq 0\}) > 0$ *for* ρ_1*-almost all* $r \in R$, *and* L *be the totality of linear combinations of the functions of the form* $(b, \langle s, r\rangle)f_1(a^{-1}s, r)$, $|a| = 1$. *Then* L *is dense in* L^2 *with respect to the norm in* L^2.

Proof (*outline*). For any set $A \subset \tilde{X}$ and any $r \in R$, A_r denotes the set $\{s \;/\; \langle s, r\rangle \in A\}$ by definition. Let Δ be any fixed subset of R. If $\sigma(A_r) > 0$ for ρ_1-almost all $r \in \Delta$ and $A' \subset S \times \Delta$, then there exist $u_{a_1}, \ldots, u_{a_n} \in \mathbf{U}$ for any $\varepsilon > 0$ such that $\sigma\otimes\rho_1(A' - [a_1A \cup \ldots \cup a_nA]) < \varepsilon$. On the other hand, any continuous function on $\tilde{X}$ is approximated uniformly on any compact subset of $\tilde{X}$ by means of linear combinations of characters. By making use of these facts, we may prove that any continuous function on $\tilde{X}$ which vanishes outside of a compact set is approximated in L^2 by means of functions $\in L$. Lemma 3 follows from this result at once.

§ **3. Proof of Theorem 1.** Let **G**, **U** and **V** etc. be as stated in Theorem 1 and $\{\mathfrak{H}, U(g), f^\circ\}$ be a cyclic unitary representation of **G**, and put $U_a = U(u_a)$ for $u_a \in \mathbf{U}$ and $V_b = U(v_b)$ for $v_b \in \mathbf{V}$. Then it follows from (1.2) that

$$(3.1) \qquad U_a V_b = V_{ab} U_a.$$

Since **G** satisfies the second countability axiom and since the representation is cyclic, the Hilbert space $\mathfrak{H}$ is separable. Put

$$\mathfrak{N} = \{f \in \mathfrak{H} \;/\; V_b f = f \quad \text{for all } v_b \in \mathbf{V}\}.$$

Then, since **V** is a normal subgroup of **G**, $f \in \mathfrak{N}$ implies that $V_b U(g)f = U(g)U(g^{-1}v_b g)f = U(g)f$ for any $g \in \mathbf{G}$ and $v_b \in \mathbf{V}$. Therefore $\mathfrak{N}$ and consequently $\mathfrak{M} = \mathfrak{H} \ominus \mathfrak{N}$ are $U(g)$-invariant subspaces of $\mathfrak{H}$. The representation, considered in $\mathfrak{N}$, yields a representation of the group $\mathbf{U}$ $(\cong \mathbf{G}/\mathbf{V})$.

Consider the representation in $\mathfrak{M}$; $\mathfrak{M}$ is separable as well as $\mathfrak{H}$. By Lemma 1, there exists a resolution of the identity $\{E(\Lambda)\}$ in $\mathfrak{M}$ on X such that

$$V_b = \int_X (b, \chi) dE(\chi);$$

and the space $\mathfrak{M}$ may be realized as an at most countable direct sum of the spaces $\mathfrak{M}_j$ of functions:

$$\mathfrak{M}_j = \{f_j(\chi) \,/\, \|f_j\|^2 = \int_X |f_j(\chi)|^2 dF_j(\chi) < \infty\},$$

where $F_j(\Lambda)$ is a measure on X such that $F_j(X) = 1$ and every $F_j(\Lambda)$ $(j > 1)$ is absolutely continuous with respect to $F_{j-1}(\Lambda)$. When $f \in \mathfrak{M}$ is realized by $\{f_j(\chi)\}$, we write $f \sim \{f_j(\chi)\}$; then

$$V_b f \sim \{(b, \chi) f_j(\chi)\} \qquad \text{for any } v_b \in \mathbf{V}. \tag{3.2}$$

Since 0 is the only one element of $\mathfrak{M}$ that fulfills $V_b f = f$ for all $v_b \in \mathbf{V}$ we obtain $F_j(\{\chi_0\}) = 0$, $j = 1, 2, \ldots$. Thus we may consider $F_j(\Lambda)$, $j = 1, 2, \ldots$, as measures on $\tilde{X} = X - \{\chi_0\}$.

The operator U_a is expressible as a matrix $(U_{jk}(a))$ where $U_{jk}(a)$ is a bounded operator from $\mathfrak{M}_k$ into $\mathfrak{M}_j$ such that

$$U_a f \sim \{\sum_k U_{jk}(a) f_k(\chi)\}_{j=1,2,\ldots} \qquad \text{for } f \sim \{f_j(\chi)\}.$$

Since U_a is unitary, we have

$$\sum_j \|f_j\|^2 = \sum_j \|\sum_k U_{jk}(a) f_k\|^2. \tag{3.3}$$

Next, if we put $U_{jk}(a)\cdot 1 = u_{jk}^{\circ}(a\,;\chi)$, then

$$\|u_{jk}^{\circ}(a\,;\chi) - u_{jk}^{\circ}(b\,;\chi)\|^2 \leqq \|U_a f^k - U_b f^k\|_{\mathfrak{H}}^2 \quad (|a| = |b| = 1),$$

where $f^k \sim \{f_j(\chi)\}$ such that $f_k(\chi) \equiv 1$ and $f_j(\chi) \equiv 0$ $(j \neq k)$, and $\|\cdot\|_{\mathfrak{H}}$ denotes the norm in $\mathfrak{H}$; moreover $\mathbf{U}$ satisfies the second axiom of countability. Hence we may construct a function $u_{jk}(a\,;\chi)$ B-measurable in $\langle a, \chi\rangle$ and such that $u_{jk}(a\,;\chi) = u_{jk}^{\circ}(a\,;\chi)$ for F_j-almost all χ for every a.[13] Thus we may consider that $U_{jk}(a)\cdot 1 = u_{jk}(a\,;\chi)$. Then we get

$$U_{jk}(a) f_k(\chi) = u_{jk}(a\,;\chi) f_k(a^{-1}\chi). \tag{3.4}$$

At first we can prove this equality for functions of the form $f_k(\chi) = (b, \chi)$ (for any fixed b) by making use of (3.1), (3.2) and the fact that $(ab, \chi) = (b, a^{-1}\chi)$

[13] Such $u_{jk}(a\,;\chi)$ may be obtained by the same way as constructing the "measurable kernel" of a stochastic process. See [4].

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($|a|=1$). Since the totality of linear combinations of "characters" (b, χ) is dense in $L^2(\tilde{X}, F_k)$, (3.4) is true for all $f_k \in L^2(\tilde{X}, F_k)$. Hence (3.3) becomes as follows:

$$\sum_j \int_{\tilde{X}} |f_j(\chi)|^2 \, dF_j(\chi) = \sum_j \int_{\tilde{X}} |\sum_k u_{jk}(a;\chi) f_k(a^{-1}\chi)|^2 dF_j(\chi). \tag{3.5}$$

Let $\varphi(\chi)$ be the characteristic function of $\Lambda \subset \tilde{X} = S \times R$ and put in (3.5) $f_1(\chi) = \varphi(a\chi)$ and $f_j(\chi) \equiv 0$ *for* $j \neq 1$. Then we obtain

$$\begin{aligned} F_1(a^{-1}\Lambda) &= \int_{\tilde{X}} \varphi(a\chi) dF_1(\chi) = \sum_j \int_{\tilde{X}} |u_{j1}(a;\chi)\varphi(\chi)|^2 dF_j(\chi) \\ &= \sum_j \int_\Lambda |u_{j1}(a;\chi)|^2 dF_j(\chi). \end{aligned} \tag{3.6}$$

Since all $F_j(\Lambda)$ are absolutely continuous with respect to $F_1(\Lambda)$ (by Lemma 1), we may write

$$F_j(\Lambda) = \int_\Lambda \Phi_j(\chi) dF_1(\chi)$$

where every $\Phi_j(\chi)$ is non-negative, B-measurable in χ and summable on $\tilde{X}$ with respect to F_1. Then the function

$$u(a;\chi) = \sum_j |u_{j1}(a;\chi)|^2 \Phi_j(\chi) \quad (\geqq 0)$$

is B-measurable in $\langle a;\chi\rangle$ and summable on $\tilde{X}$ with respect to F_1 for any a, and it follows from (3.6) and by Lebesgue's convergence theorem that

$$F_1(a^{-1}\Lambda) = \int_\Lambda u(a;\chi) dF_1(\chi). \tag{3.7}$$

Hence, by Lemma 2, there exist a non-negative B-measurable function $\omega(s, r)$ on $\tilde{X}$ and a measure $\rho(\Delta)$ on R such that $\rho(R) = 1$ and $F_1(\Lambda)$ is given by

$$F_1(\Lambda) = \int_\Lambda \omega(s, r) d\sigma(s) d\rho(r),$$

and consequently there exist non-negative B-measurable functions $\omega_j(s, r)$, $j = 1, 2, \ldots$, on $\tilde{X} = S \times R$ such that

$$F_j(\Lambda) = \int_\Lambda \omega_j(s, r) d\sigma(s) d\rho(r). \tag{3.8}$$

Now put $\Lambda_j = \{\langle s, r\rangle \,/\, \omega_j(s, r) = 0\}$. Evidently $\Lambda_1 \subset \Lambda_2 \subset \ldots$. Put $\varphi_j(s, r) = \omega_j(s, r) f_j(s, r)$ for every $f \sim \{f_j(s, r)\}$ and define the norm of φ_j by

$$\|\varphi_j\|^2 = \int_{\tilde{X}} |\varphi_j(s, r)|^2 d\sigma(s) d\rho(r).$$

Then we have $\|\varphi_j\|^2 = \|f_j\|^2$, and hence the mapping $f_j \to \varphi_j$ is an isometric mapping from $\mathfrak{M}_j$ onto

$$\mathfrak{L}_j = \{\varphi_j(s, r) \ / \ \|\varphi_j\|^2 < \infty, \ \varphi_j(s, r) = 0 \text{ on } \Lambda_j\}.$$

So we can realize $\mathfrak{M}$ as a direct sum of $\mathfrak{L}_j$. The mapping $f_j \to \varphi_j$ carries $U_{jk}(a)$ into operators on $\{\varphi_j(s, r)\}$; we denote them by $U_{jk}(a)$ again. Define

$$u'_{jk}(a\,;\, s,\, r) = \begin{cases} \omega_j(s, r)u_{jk}(a\,;\, s,\, r)\omega_k(a^{-1}s,\, r)^{-1} & \text{if } \langle a^{-1}s,\, r\rangle \notin \Lambda_k, \\ 0 & \text{if } \langle a^{-1}s,\, r\rangle \in \Lambda_k \end{cases}$$

$\left(u_{jk}(a\,;\, s,\, r) \equiv u_{jk}(a\,;\, \chi) \text{ for } \chi = \langle s,\, r\rangle\right)$. Then it follows from (3.4) and by the definition of $\varphi_j(s, r)$ that

$$U_{jk}(a)\varphi_k(s,\, r) = u'_{jk}(a\,;\, s,\, r)\varphi_k(a^{-1}s,\, r), \tag{3.9}$$

and unitary condition (3.5) becomes

$$\begin{aligned} \sum_j \int_{\tilde{X}} |\varphi_j(s,\, r)|^2 d\sigma(s) d\rho(r) &= \sum_j \int_{\tilde{X}} |\sum_k u'_{jk}(a\,;\, s,\, r)\varphi_k(a^{-1}s,\, r)|^2 d\sigma(s) d\rho(r) \\ &= \sum_j \int_{\tilde{X}} |\sum_k u'_{jk}(a\,;\, as,\, r)\varphi_k(s,\, r)|^2 d\sigma(s) d\rho(r). \end{aligned} \tag{3.10}$$

Denote by n ($\leqq \infty$) the number of $\mathfrak{M}_j$ and by $\mathfrak{H}_0$ the unitary space of all sequences $\xi = \{\xi_j\} \equiv \{\xi_1, \ldots, \xi_n\}$ of complex numbers such that $\|\xi\|^2 = \sum_{j=1}^{n} |\xi_j|^2 < \infty$ (if $n = \infty$) and by $\mathfrak{H}_k$ $(k = 1, 2, \ldots)$ the finite-dimensional subspace of $\mathfrak{H}_0$ defined by the condition $\xi_k = \xi_{k+1} = \ldots = 0$. $f \sim \varphi(\chi) = \{\varphi_j(s,\, r)\}$ means that $f \in \mathfrak{M}$ is realized as a vector function $\varphi(\chi)$ such that $\varphi(\chi) \in \mathfrak{H}_0$ for $\chi \notin \bigcup_{k=1}^{n} \Lambda_k$ and $\varphi(\chi) \in \mathfrak{H}_k$ for $\chi \in \Lambda_k$. Denote the matrix $(u'_{jk}(a\,;\, s,\, r))$ by $M(a\,;\, s,\, r)$ for every $\langle a\,;\, s,\, r\rangle$. Then $f \sim \varphi(\chi) \equiv \varphi(s,\, r)$ implies that

$$\begin{cases} \|f\|^2_{\mathfrak{H}} = \int_{\tilde{X}} \|\varphi(s, r)\|^2 d\sigma(s) d\rho(r) \qquad (\|\varphi(s,\, r)\|^2 = \sum_j |\varphi_j(s,\, r)|^2), \\ U_a f \sim M(a\,;\, s,\, r)\varphi(a^{-1}s,\, r), \\ V_b f \sim (b,\, \langle s,\, r\rangle)\varphi(s,\, r) \end{cases} \tag{3.11}$$

by (3.2), (3.9) and the definition of $\varphi_j(s,\, r)$.

(3.10) is now written as follows:

$$\int_{\tilde{X}} \|\varphi(s,\, r)\|^2 d\sigma(s) d\rho(r) = \int_{\tilde{X}} \|M(a\,;\, as,\, r)\varphi(s,\, r)\|^2 d\sigma(s) d\rho(r).$$

If we put in this equality $\varphi(s,\, r) = \{\xi_j \varphi_\Lambda(s,\, r)\}$ where $\xi = \{\xi_j\} \in \mathfrak{H}_k$ and $\varphi_\Lambda(s,\, r)$ is the characteristic function of any assigned Borel set $\Lambda \subset \Lambda_k - \Lambda_{k-1}$, then

$$\int_\Lambda \|\xi\|^2 d\sigma(s) d\rho(r) = \int_\Lambda \|M(a\,;\, as,\, r)\xi\|^2 d\sigma(s) d\rho(r).$$

This implies that, for any $u_a \in \mathbf{U}$, $M(a\,;\, s,\, r)$ considered on $\mathfrak{H}_k$ is an isometric operator for almost all[14] $\langle s,\, r\rangle \in a(\Lambda_k - \Lambda_{k-1})$. Further, by the definition of

[14] Here we mean "for almost all $\langle s,\, r\rangle$ with respect to the product measure $\sigma \otimes \rho$."

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$u'_{jk}(a; s, r)$, the range of $M(a; s, r)$ is $\mathfrak{H}_k$ for almost all $\langle s, r\rangle \in (\Lambda_k - \Lambda_{k-1})$ $(k \geqq 2)$. Since $\Lambda_1 \subset \Lambda_2 \subset \ldots$, it follows that for almost all $\langle s, r\rangle \in [a(\Lambda_k - \Lambda_{k-1}) - (\Lambda_k - \Lambda_{k-1})]$ the operator $M(a; s, r)$ maps $\mathfrak{H}_k$ isometrically onto $\mathfrak{H}_j$ for some $j \neq k$. Hence every $(\Lambda_k - \Lambda_{k-1})$ $(k \geqq 2)$ must be of the form $S \times \Delta_k$ $(\Delta_k \subset R)$ (with the exception of the set of measure zero). On the other hand, Λ_1 is of the form $S \times \Delta$ $(\Delta \subset R)$ from (3.7) and the definition of Λ_1. Hence the same is true for every Λ_k $(k = 1, 2, \ldots)$.

Hereafter we shall say that a matrix $M_1(a; s, r) = (u^1_{jk}(a; s, r))$ is equal to another matrix $M_2(a; s, r) = (u^2_{jk}(a; s, r))$ for a. a. (= almost all) $\langle s, r\rangle$ if and only if $u^1_{jk}(a; s, r) = u^2_{jk}(a; s, r)$ for $\sigma \otimes \rho$-almost all $\langle s, r\rangle \notin \Lambda_k$ for $j = 1, 2, \ldots, n$; this condition is equivalent to the following one: $M_1(a; s, r) = M_2(a; s, r)$ as operetors stated in (3.11). By the above obtained result concerning the form of Λ_k, if $M_1(a; s, r) = M_2(a; s, r)$ for a. a. $\langle s, r\rangle$ then, for any b $(|b| = 1)$, $M_1(a; bs, r) = M_2(a; bs, r)$ for a. a. $\langle s, r\rangle$.

It follows from (3.11) that for any a, b $(|a| = |b| = 1)$ and any $\varphi(s, r) = \{\varphi_j(s, r)\}$ $(\varphi_j \in \mathfrak{L}_j)$

$$(3.12) \qquad M(a; s, r)\varphi(s, r) = M(b; s, r)M(b^{-1}a; b^{-1}s, r)\varphi(s, r)$$

as elements of $\mathfrak{M}$. We fix an arbitrary element $u_a \in \mathbf{U}$. From (3.12) and by Fubini's theorem, we have

$$(3.13) \qquad M(a; s, r) = M(b; s, r)M(b^{-1}a; b^{-1}s, r) \text{ for a. a. } \langle b, s, r\rangle.$$

Since the transformation $\langle b, s, r\rangle \to \langle sb, s, r\rangle$ is measure-preserving, (3.13) implies that

$$M(a; s, r) = M(sb; s, r)M(b^{-1}s^{-1}a; b^{-1}, r) \quad \text{for a. a. } \langle b, s, r\rangle;$$

this holds for any fixed $u_a \in \mathbf{U}$. Since $\mathbf{U}$ is separable, there exists a countable set $\mathbf{U}_0 \subset \mathbf{U}$ which is dense in $\mathbf{U}$ and contains the identity e of $\mathbf{G}$. Hence we may take an element $b_0 \in S$ such that

$$M(a; s, r) = M(sb_0; s, r)M(b_0^{-1}(a^{-1}s)^{-1}; b_0^{-1}, r) \quad \text{for a. a. } \langle s, r\rangle$$

for all $u_a \in \mathbf{U}_0$, and that $N_1(s, r) = M(sb_0; s, r)$ and $N_2(s, r) = M(b_0^{-1}s^{-1}, b_0^{-1}, r)$ are isometric operator for a. a. $\langle s, r\rangle$. Thus we obtain

$$(3.14) \qquad M(a; s, r) = N_1(s, r)N_2(a^{-1}s, r) \quad \text{for a. a. } \langle s, r\rangle$$

for all $u_a \in \mathbf{U}_0$. Putting $u_a = e (\in \mathbf{U}_0)$, we get

$$(3.15) \qquad N_1(s, r)N_2(s, r) = I \quad \text{for a. a. } \langle s, r\rangle.$$

Now put $\psi(s, r) = N_2(s, r)\varphi(s, r)$; then $\|\psi(s, r)\| = \|\varphi(s, r)\|$ and $\varphi(s, r) = N_1(s, r)\psi(s, r)$ (by (3.15)) for a. a. $\langle s, r\rangle$. And hence, by (3.14) and (3.11), $f \sim \varphi(s, r) \sim \psi(s, r)$ implies

$$\begin{cases} \|f\|_{\mathfrak{H}}^2 = \int_{\tilde{X}} \|\varphi(s, r)\|^2 d\sigma(s) d\rho(r)\,; \\ U_a f \sim \psi(a^{-1}s, r) \quad \text{for any} \quad u_a \in \mathbf{U}_0\,; \\ V_b f \sim (b, \langle s, r \rangle) \psi(s, r) \quad \text{for any} \quad v_b \in \mathbf{V}. \end{cases}$$

By the definition of $\mathfrak{H}_0$, $\psi(s, r) = \{\psi_1(s, r), \psi_2(s, r), \ldots\}$, where $\psi_j(s, r) \in L^2(\tilde{X}, \sigma \otimes \rho)$ and $\|\psi(s, r)\|^2 = \sum_{j=1}^{n} |\psi_j(s, r)|^2$ for every $\langle s, r \rangle$. Hence $\mathfrak{M}$ may be realized as a subspace of the direct sum of at most countable number of $L^2(\tilde{X}, \sigma \otimes \rho)$, and $f \sim \{\psi_j(s, r)\}$ implies

$$(3.16) \qquad \begin{cases} \text{i)} \ \|f\|_{\mathfrak{H}}^2 = \sum_{j=1}^{n} \int_{\tilde{X}} |\psi_j(s, r)|^2 d\sigma(s) d\rho(r) \ \ (n \leqq \infty) \\ \text{ii)} \ U_a f \sim \{\psi_j(a^{-1}s, r)\} \quad \text{for any} \quad u_a \in \mathbf{U}_0 \\ \text{iii)} \ V_b f \sim \{(b, \langle s, r \rangle) \psi_j(s, r)\} \quad \text{for any} \quad v_b \in \mathbf{V}. \end{cases}$$

For any $u_a \in \mathbf{U}$, there exists a sequence $\{u_{a_n}\} \subset \mathbf{U}_0$ such that $u_{a_n} \to u_a$, and $U_{a_n} f \sim \{\psi_j(a_n^{-1}s, r)\}$ for any $f \sim \{\psi_j(s, r)\}$. Since the representation $U(g)$ is strongly continuous, we may easily show that $U_a f \sim \{\psi_j(a^{-1}s, r)\}$ for any $f \sim \{\psi_j(s, r)\}$. Namely (3.16) ii) holds for any $u_a \in \mathbf{U}$. Hereafter we shall write $\|\cdot\|$ instead of $\|\cdot\|_{\mathfrak{H}}$.

Let now the cyclic unitary representation $\{\mathfrak{H}, U(g), f^\circ\}$ be irreducible. Then either $\mathfrak{M}$ or $\mathfrak{N}$ must be $\{0\}$. If $\mathfrak{M} = \{0\}$, then $\{\mathfrak{N}, U_a\}$ is an irreducible representation of the group $\mathbf{U}$ and $V_b = I$ in $\mathfrak{N}$ for all $v_b \in V$. Hence the normal elementary p. d. function $\Phi(g)$ corresponding to the irreducible representation $\{\mathfrak{H}, U(g)\}$ $(\mathfrak{H} = \mathfrak{N})$ is a character $\chi(a)$ stated in Theorem 1 iii). Conversely such a representation $\{\mathfrak{H}, U(g)\}$ of $\mathbf{G}$ is evidently irreducible. Next suppose that $\mathfrak{N} = \{0\}$; then the unitary space $\mathfrak{H}_0$ stated above is of one dimension and there exists a point $r_0 \in R$ such that $\rho(\{r_0\}) > 0$ and $\rho(R - \{r_0\}) = 0$. Hence the irreducible representation $\{\mathfrak{H}, U(g)\}$ and the corresponding normal elementary p. d. function are of the form stated in Theorem 1 i). The irreducibility of such representation is proved by means of Lemma 3. Thus, i), iii) and iv) of Theorem 1 is established.

Next we shall prove ii). If the representation $\{\mathfrak{H}_1, U_1(g)\}$ corresponding to r_1 is unitary equivalent to $\{\mathfrak{H}_2, U_2(g)\}$ corresponding to $r_2 (\neq r_1)$, then $(U_1(g)f_1, f_1) = (U_2(g)f_2, f_2)$ for certain $f_1 \in \mathfrak{H}_1$ and $f_2 \in \mathfrak{H}_2$. Hence, if we consider the direct sum $\{\mathfrak{H}, U(g)\} = \{\mathfrak{H}_1, U_1(g)\} \oplus \{\mathfrak{H}_2, U_2(g)\}$ and put $f = f_1 + f_2$, then $\{U(g)f \,/\, g \in \mathbf{G}\}$ does not span $\mathfrak{H}$ by Theorem 8 in [3]. But we may prove by Lemma 3 that $\{U(g)f \,/\, g \in \mathbf{G}\}$ spans $\mathfrak{H}$. Hence we get Theorem 1 ii).

§4. Proof of Theorem 2 and supplementary remarks. In this paragraph, we shall make use of the results obtained in §3. If $\{\mathfrak{H}, U(g), f^\circ\}$ is any cyclic unitary representation of $\mathbf{G}$, then the space $\mathfrak{H}$ is decomposable to the direct sum of two $U(g)$-invariant subspaces $\mathfrak{N}$ and $\mathfrak{M}$, as stated in §3; the space $\mathfrak{M}$ is realized as the space of $\mathfrak{H}_0$-valued functions $\psi(s, r) = \{\psi_j(s, r)\}$ on $S \times R$ and the norm $\|f\|$ of the element $f \in \mathfrak{M}$ and unitary operators U_a (for $u_a \in \mathbf{U}$) and V_b (for $v_b \in \mathbf{V}$) are given by (3.16).

In the case that the cyclic unitary representation $\{\mathfrak{H}, U(g), f^\circ\}$ is not necessarily irreducible, both $\mathfrak{M}$ and $\mathfrak{N}$ may be $\neq \{0\}$. If $\mathfrak{N} \neq \{0\}$, then $\{\mathfrak{N}, U(g)\}$ is a cyclic unitary representation of the group $\mathbf{U}$, and consequently is the direct sum $\bigoplus_{k=1}^{N} \{\mathfrak{N}_{l_k}, U_{l_k}(g)\}$ $(N \leqq \infty)$ as stated in Theorem 2 i). If $\mathfrak{M} \neq \{0\}$, then $\{\mathfrak{M}, U(g)\}$ is cyclic and is decomposable to the direct sum of $\{\mathfrak{M}_j, U(g)\}$, $j = 1, 2, \ldots, n$ $(\leqq \infty)$, where $\mathfrak{M}_j$ is a subspace of $L^2(\tilde{X}, \sigma \otimes \rho)$ and $U(g)$ is defined by (3.16) for every j. If

$$f^\circ = \sum_{j=0}^{n} \psi_j^\circ \qquad \psi_0^\circ \in \mathfrak{N}, \quad \psi_j^\circ \in \mathfrak{M}_j \ (j \geqq 1),$$

then $\{\mathfrak{M}_j, U(g), \psi_j^\circ\}$, $j = 1, 2, \ldots, n$, are cyclic unitary representation of $\mathbf{G}$. Put $J_j(r) = \int_S |\psi_j^\circ(s, r)|^2 d\sigma(s)$, $\rho_j(\varDelta) = \int_\varDelta J_j(r) d\rho(r)$ for $\varDelta \subset R$ and

$$(4.1) \qquad \tilde{\psi}_j(s, r) = \begin{cases} \psi_j(s, r)/J_j(r) & \text{if } J_j(r) \neq 0 \\ 0 & \text{if } J_j(r) = 0, \end{cases}$$

and define the unitary operator $U(g) = U_a V_b$ (for $g = u_a v_b$) by $U_a \tilde{\psi}_j(s, r) = \tilde{\psi}_j(a^{-1}s, r)$ and $V_b \tilde{\psi}_j(s, r) = (b, \langle s, r \rangle) \tilde{\psi}_j(s, r)$. Then the unitary representation $\{L^2(\tilde{X}, \sigma \otimes \rho), U(g)\}$ (defined by (3.16)) is unitary equivalent to $\{L^2(\tilde{X}, \sigma \otimes \rho_j), U(g)\}$ (defined above) by means of the mapping $\psi_j(s, r) \to \tilde{\psi}_j(s, r)$. If we put $f_j(s, r) = \tilde{\psi}_j^\circ(s, r)$, then $\{U(g) f_j \mid g \in G\}$ spans $L^2(\tilde{X}, \sigma \otimes \rho_j)$ by Lemma 3. Hence we may consider that $\mathfrak{M}_j = L^2(\tilde{X}, \sigma \otimes \rho_j)$. Clearly the functions $f_j(s, r)$, $j = 1, 2, \ldots$, satisfy the conditions 1°) and 2°) in Theorem 2 i). By Theorem 8 in [3], the direct sum $\{\mathfrak{M}, U(g)\} = \bigoplus_{j=1}^{n} \{\mathfrak{M}_j, U(g)\}$ is cyclic if and only if $f_j(s, r)$, $j = 1, 2, \ldots$, satisfy the condition 3°) also. Thus $\{\mathfrak{H}, U(g), f^\circ\}$ must be of the form as stated in Theorem 2, and the corresponding p. d. function $\Psi(g)$ is given by (1.5), and consequently (1.6) is evident.

Conversely let us consider the unitary representation $\{\mathfrak{H}, U(g), f^\circ\}$ stated in Theorem 2 i). $\{\mathfrak{M}_j, U(g), f_j\}$, $j = 1, 2, \ldots$, are cyclic as stated above. Consequently p. d. functions $\Psi_j(g) = (U(g) f_j, f_j)$, $j = 1, 2, \ldots$, are mutually disjoint[15] from the assumptions 1°), 2°) and 3°). Hence the direct sum $\bigoplus_{j=1}^{n} \{\mathfrak{M}_j, U(g), f_j\}$ is cyclic as is earily proved by making use of Theorem 8 in [3]. Similar argument shows that the direct sum $\bigoplus_{k=1}^{N} \{\mathfrak{N}_{l_k}, U_{l_k}(g)\}$ also is cyclic. Since $U_{l_k}(v_b) = I$ in $\bigoplus_{k=1}^{N} \mathfrak{N}_{l_k}$ for all $v_b \in \mathbf{V}$ and $U(v_b) \equiv V_b \neq I$ in $\mathfrak{M}_j$ for all $v_b \neq e$, we may prove

[15] See [3] § 12.

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by making use of Theorem 8 in [3] again that $\{\mathfrak{H}, U(g), f^\circ\}$ is a cyclic unitary representation of **G**. And hence (1.5) follows at once. Thus Theorem 2 is established.

Supplementary remarks. In the proofs of Theorems 1 and 2, we make use of the following fact. The group **G** has the property (1.1), where the group **U** may be replaced by any group the types of whose unitary representations are well known (for example, a maximally almost periodic Lie group), and either the character group X of the commutative group **V** or $\tilde{X} = X - \{\chi_0\}$ is a topological product space $S \times R$, where S is invariant under the transformation $\chi \rightarrow T_a\chi$ defined by $(u_a v_b u_a^{-1}, \chi) = (v_b, T_a\chi)$ and may be considered as a group isomorphic to the group **U**. The group **G′** (stated in § 0) also satisfies the above conditions.

As for the group of all congruent transformations in the n-dimensional euclidean space E^n for $n \geqq 3$, the space S is not a group but a factor space $SO(n)/SO(n-1)$ while $\mathbf{U} = SO(n)$. Hence we must consider the space of functions $\psi(u, r)$ on $\mathbf{U} \times R$ instead of the space of functions $\psi(s, r)$ on $S \times R$ (in § 3). It seems to be difficult to find irreducible invariant subspaces in the space of functions on $\mathbf{U} \times R$, since the similar argument to Lemma 3 is impossible.

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