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CPI-EXTENSIONS: OVERRINGS OF INTEGRAL DOMAINS WITH SPECIAL PRIME SPECTRUMS

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1. Introduction. Throughout this paper the term *ring* will denote a commutative ring with unity and the term *integral domain* will denote a ring having no nonzero divisors of zero. The set of all prime ideals of a ring R can be viewed as a topological space, called the *prime spectrum of* R, and abbreviated Spec (R), where the topology used is the Zariski topology [**1**, Definition 4, § 4.3, p. 99]. The set of all prime ideals of R can also be viewed simply as a poset – that is, a partially ordered set – with respect to set inclusion. We will use the phrase *the pospec of* R, or just Pospec(R), to refer to this partially ordered set.

Various questions involving the existence of a ring R whose pospec is order-isomorphic to a given partially ordered set S have been investigated recently. Kaplansky [**8**, Theorems 9 and 11, p. 6] gave two necessary conditions on S, and raised the question of their sufficiency. Lewis [**11**, p. 433] gives an example due to Hochster that shows that the conditions stated by Kaplansky are not sufficient. Some sufficient conditions have been established by Lewis. In particular, he shows that each finite partially ordered set S is order-isomorphic to the pospec of some ring R, and in case S has a unique minimal element, R may be taken to be an integral domain [**11**, Theorems 2.9 and 2.10, pp. 427-428]. Lewis also shows that when S is a tree satisfying Kaplansky's two conditions and has a finite number of minimal elements, then S is order-isomorphic to the pospec of some ring R and, again, if S has a unique minimal element, R can be taken to be an integral domain [**11**, Theorem 3.1 and Corollary 3.5, pp. 429 and 433]. Hochster [**7**, Proposition 8, p. 54] shows that given a poset which is the pospec of some ring, the poset with the reverse ordering is also the pospec of some ring.

In this paper, we obtain additional existence results along these lines but of a slightly different character. In particular, given the pospec of a domain D, we show the existence of an overring of D whose pospec is an easily describable portion of the pospec of D. To be more precise, a method is presented in Section 2 such that for each prime ideal P of D, an overring (called the CPI-extension of D with respect to P) can be constructed whose pospec is order-isomorphic under the contraction map to the set of all prime ideals of D comparable to P (Corollary 2.8). This order-isomorphism is, in fact, a homeomorphism from the prime spectrum of this overring to the set of all prime ideals of D comparable to P, considered as a subspace of Spec(D) (Theorem 2.6).

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In Section 3, we show that the CPI-extension construction can also be carried out with respect to a nonprime ideal, but the prime ideal structure of the resulting overring is not as closely related to the prime ideal structure of D as in the prime case. In particular, the contraction map is neither always a homeomorphism from the prime spectrum of the overring nor always an order-isomorphism from the pospec (Example 3.14). However, the pospec of the overring can be viewed in two pieces, each of which is order-isomorphic under contraction to a portion of the pospec of D. Furthermore, contraction defined on the entire pospec of the overring is one-to-one (Theorem 3.6).

Many natural questions arising from the study of the CPI-extension are considered. For example, we investigate the questions of when the CPI-extension is an integrally closed domain, a Prüfer domain, a Noetherian domain, etc. Also, conditions under which the CPI-extension is a proper extension are studied, and the relation of the CPI-extension to various standard domain-theoretic constructions is discussed. Supporting examples are presented at the end of each of Sections 2 and 3 showing that various improvements cannot be made on our results.

The authors thank the referee for pointing out how Section 2 could be simplified using the results appearing in [**6**].

2. The prime case. Throughout this section the following notation will be used:

(i) D denotes an integral domain,

(ii) P denotes a prime ideal of D,

(iii) $\bar{D}$ denotes the residue class ring D/P,

(iv) Q denotes the quotient field of $\bar{D}$,

(v) φ denotes the canonical homomorphism from D onto $\bar{D}$, and

(vi) Ψ denotes the epimorphism from D_P onto Q defined by $\Psi(a/b) = \varphi(a)/\varphi(b)$. (The argument that establishes that Ψ is an epimorphism is straightforward. It is also straightforward to show that PD_P is the kernel of Ψ).

2.1 *Definition.* The pre-image of $\bar{D}$ under Ψ is called the *CPI-extension of D with respect to P* and is denoted by $C(D, P)$.

The prefix CPI in this context refers to the fact that $C(D, P)$ is the "complete pre-image" of $\bar{D}$, complete in the sense of being $\Psi^{-1}(\bar{D})$ where the extension of φ to Ψ has been taken as far as it can be taken meaningfully. This construction arises naturally in a number of contexts where one is trying to relate some structure of a domain D to the structure of a homomorphic image of D. For example, Boisen and Larsen [**3**] used this construction to form a Prüfer domain with a specified Prüfer ring as a homomorphic image. Similarly, Boisen and Sheldon [**4**, Theorem 1] used this construction to obtain a valuation domain with a specified valuation ring as one of its homomorphic images.

The CPI-extension is closely related to a number of standard constructions in domain theory. The most obvious of these is that of localization since when P is a maximal ideal of D, $C(D, P) = D_P$. This can be seen by observing that when P is a maximal ideal, $\bar{D}$ is a field and hence $\Psi(D_P) = Q = \bar{D}$ and so $C(D, P) = \Psi^{-1}(\bar{D}) = D_P$. However, when P is a non-maximal prime ideal, $C(D, P)$ is a proper subring of D_P, but is still closely related to D_P as some of the results in this section show.

Another related construction is the composite of valuation domains [**12**, p. 35]. In the construction of the composite V of two valuation domains V_1 and V_2, we require V_1 to be a valuation domain with quotient field equal to the residue class field of V_2. Then V is defined to be the complete pre-image in V_2 of V_1 under the natural mapping of V_2 to its residue class field. While the formation of this construction is somewhat different from that of the CPI-extension, the spirit is very similar. In fact, in the case where $\bar{D}$ and D_P are both valuation domains, $C(D, P)$ is precisely the composite of $\bar{D}$ and D_P. To illustrate this situation in a nontrivial context, we cite the example of a Prüfer domain D that is not a valuation domain, but which has a nonmaximal prime ideal P contained in only one maximal ideal. In this case both $\bar{D}$ and D_P are valuation domains, and hence $C(D, P)$ is their composite. (See also Example 2.11).

Finally we note that the CPI-construction resembles the $D + M$ construction [**2**] in a number of ways. For one thing, it is straightforward to see that the set $C(D, P)$ equals the set $D + PD_P$, but, unlike $D + M$, this sum is never direct. The true analog to the case of $D + M$ would be when D contains a copy of $\bar{D}$ and $C(D, P)$ is the direct sum of this copy and PD_P; however, our general construction method fails to guarantee that such a structure exists in $C(D. P)$. On the other hand, PD_P has a property in $C(D, P)$ which is much like one of the significant properties of M in $D + M$, namely that of being comparable to every ideal of $C(D, P)$ (Proposition 2.3). Moreover, $C(D, P)$ inherits certain properties from $\bar{D}$ and D_P (see Propositions 3.11 and 3.12), much as $D + M$ inherits its properties from D and the valuation domain $(D + M)_M$.

2.2 *Remark.* Gilmer and Ohm in [**6**] considered domains $S = D_0 + A$ where D_0 is a subring of a domain E and A is an ideal of E. As observed in the previous paragraph, $C(D, P) = D + PD_P$ is of this form (where $D_P = E$). Two results that appear in [**6**] are of interest to us in our study of CPI-extensions. For convenience, we now translate these results into our notation.

(a) If Q is a primary ideal of $C(D, P)$ such that $Q \subseteq PD_P$, then Q is an ideal in D_P.

(b) If B is an ideal of $C(D, P)$ such that $B \not\subseteq PD_P$, then $B = (B \cap D) + PD_P$.

Statement 2.2(a) is a restatement of Proposition 5.1(a) [**6**] and statement 2.2(b) is a restatement of Proposition 5.1(f) [**6**].

As an immediate consequence of Remark 2.2(b) we obtain the following result.

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2.3 Proposition. *Every ideal of $C(D, P)$ is comparable to PD_P.*

We begin our study of CPI-extensions by considering two of the standard questions that can be asked of any extension construction: when does it yield a proper extension and what happens if the extension process is performed twice?

2.4 Theorem. *The extension $C(D, P)$ equals D if and only if every ideal of D is comparable to P.*

Proof. ($\Rightarrow$) Since PD_P lies over P and is contained in $C(D, P)$ and since $C(D, P) = D$, we have $PD_P = P$. Hence, by Proposition 2.3, every ideal in D is comparable to P.

($\Leftarrow$) Now assume P is comparable with every ideal of D. We want to show that $D = C(D, P)$. Since $C(D, P)$ equals $D + PD_P$, it will suffice to show $PD_P \subseteq D$. We note that each element of PD_P is of the form p/s for some p in P and s in $D \backslash P$. Since $sD \nsubseteq P$, then $P \subseteq sD$, and hence $p/s \in D$; we therefore conclude that $PD_P \subseteq D$, and the proof is complete.

Example 2.11 will show that the condition that P is comparable to each of the prime ideals of D is not equivalent to the condition that P is comparable to each of the ideals of D. Consequently, knowledge about the position that P occupies in the pospec is insufficient to determine whether $C(D, P) = D$.

To consider the effect of applying the CPI-extension process twice, we need a more precise formulation of the question. In particular, we must decide which prime ideal of $C(D, P)$ is to be used to form the CPI-extension of $C(D, P)$. The natural choice for such a prime ideal seems to be $P \cdot C(D, P)$, the extension of P to $C(D, P)$, but it turns out that $P \cdot C(D, P)$ is not, in general, a prime ideal of $C(D, P)$. (See Example 2.10.) The next result shows that PD_P is the natural choice to consider in the sense that it is the only prime ideal of $C(D, P)$ lying over P (that is, $PD_P \cap D = P$). With this choice, we see in the next result that the second application of the CPI-extension process fails to yield a proper extension.

2.5 Proposition. *The only prime ideal of $C(D, P)$ contracting to P is PD_P. Moreover, the CPI-extension of $C(D, P)$ with respect to PD_P is $C(D, P)$ itself.*

Proof. Let P' denote a prime ideal of $C(D, P)$ lying over P. If $PD_P \subseteq P'$, by Remark 2.2(b) we know that $P' = (P' \cap D) + PD_P = P + PD_P = PD_P$. If $P' \subsetneq PD_P$, then Remark 2.2(a) implies that P' is an ideal of D_P. But the only ideal of D_P that contracts to P is PD_P. The final statement of the proposition now follows from Propositions 2.3 and 2.4.

We now state the main result of this section which characterizes Spec $(C(D, P))$.

2.6 Theorem. *The prime spectrum of $C(D, P)$ is homeomorphic to the subspace of the prime spectrum of D consisting of all prime ideals of D that are comparable to P.*

This theorem will follow from the next proposition and its corollary which describe more specifically the relationship between the ideal structure of $C(D, P)$ and that of D.

2.7 PROPOSITION. *Contraction of ideals from $C(D, P)$ to D defines the following order-isomorphisms with respect to set inclusion:*

(i) *between the ideals of $C(D, P)$ containing PD_P and the ideals of D containing P, and*

(ii) *between the primary ideals of $C(D, P)$ contained in PD_P and the primary ideals of D contained in P.*

Furthermore, in both of these order-isomorphisms, prime ideals correspond to prime ideals.

Proof. (i) By Remark 2.2(b), each ideal B of $C(D, P)$ containing PD_P is of the form $(B \cap D) + PD_P$ and so this result follows.

(ii) Since $D_P = C(D, P)_{PD_P}$, the standard properties of localization imply that contraction determines an order-isomorphism between the set of primary ideals of D_P contained in PD_P and the set of primary ideals of $C(D, P)$ contained in PD_P. (In view of Remark 2.2(a), this implies that the primary ideals of $C(D, P)$ contained in PD_P are the primary ideals of D_P.) Similarly, contraction yields an order-isomorphism between the set of all primary ideals of D_P contained in PD_P and the primary ideals of D contained in P. Composing these order-isomorphisms we have the desired order-isomorphism. Moreover, since the original order-isomorphisms were determined by contraction from D_P to $C(D, P)$ and D respectively, and since D is contained in $C(D, P)$, it follows that the composed order-isomorphism is determined by contraction from $C(D, P)$ to D.

To prove the final sentence in the theorem we need merely note that in both of the above cases the final mappings were formed by composing correspondences that are known to preserve prime ideals.

2.8 COROLLARY. Pospec $(C(D, P))$ *is order-isomorphic to the partially ordered set of all primes in* Pospec(D) *that are comparable to P.*

For convenience in the following proof and elsewhere in this paper, we will refer to a prime ideal of D that is a contraction of a prime ideal of $C(D, P)$ simply as a *contracted prime ideal.*

Proof. By Proposition 2.3, every prime ideal of $C(D, P)$ is comparable to PD_P; consequently, each prime ideal of $C(D, P)$ is in one of the categories discussed in Proposition 2.7. From this we conclude that the set of contracted primes of D is precisely the set of prime ideals that are comparable to P. To prove the corollary, it will suffice to prove that contraction is an order-isomorphism from Pospec $(C(D, P))$ to the set of contracted primes of D. We know from Proposition 2.7 that it is an order-isomorphism both on the set of primes of $C(D, P)$ containing PD_P and on the set of primes of $C(D, P)$ contained in PD_P. Since all primes of $C(D, P)$ are comparable to PD_P, it follows that both the one-to-one and the order-preserving properties carry over when we consider contraction as a mapping on the union of these two sets, that is, on Pospec $(C(D, P))$.

Proof of Theorem 2.6. As we noted in the proof of Corollary 2.8, contraction defines a one-to-one correspondence between the set of prime ideals of $C(D, P)$ and the set of prime ideals of D comparable to P. To show that this mapping is in fact a homeomorphism of the corresponding topological spaces, we begin by noting that if θ is the inclusion mapping from D into $C(D, P)$, then θ induces the continuous mapping $\text{Spec}(\theta) : \text{Spec}(C(D, P)) \rightarrow \text{Spec}(D)$ (see [**9**, p. 259]) and we note that $\text{Spec}(\theta)$ is simply the contraction mapping. Thus we need only show that contraction is a closed mapping to conclude that it is a homeomorphism between $\text{Spec}(C(D, P))$ and the set of prime ideals of D comparable to P. Suppose that $\{P_\alpha\}$ is a closed set in $\text{Spec}(C(D, P))$ and let $I = \bigcap_\alpha P_\alpha$. We want to show that $\{P_\alpha{}^c\}$, the set of contracted primes, is closed. Let Q denote a prime ideal of $C(D, P)$ such that its contraction Q^c contains $\bigcap P_\alpha{}^c = (\bigcap P_\alpha) \cap D = I \cap D$. We need to show that Q must be in $\{P_\alpha\}$. In view of part (i) of Proposition 2.7, if either Q or I is not contained in PD_P, then $I \cap D \subseteq Q \cap D$ implies that $I \subseteq Q$. If both I and Q are contained in PD_P, then I is the intersection of the P_α's that are contained in PD_P and so, by Remark 2.2(a), I and Q are ideals of D_P. Therefore, since $I \cap D \subseteq Q \cap D$, we must have $I \subseteq Q$. Consequently, in either case $I \subseteq Q$. Since $\{P_\alpha\}$ is closed, Q is some P_α as required.

In view of Proposition 2.3, we may think of the ideal structure of $C(D, P)$ as having two parts – that above PD_P and that below PD_P—and Proposition 2.7 indicates that the structure above PD_P mimics that of $\bar{D}$ while that below resembles the ideal structure of D_P. Consequently, it is not surprising that in some cases when $\bar{D}$ and D_P both have a certain property, then $C(D, P)$ has the same property. For example, it is shown in a more general context in the next section (Propositions 3.11 and 3.12) that when both $\bar{D}$ and D_P are Prüfer domains, so is $C(D, P)$, and similarly when both are integrally closed, then so is $C(D, P)$. However, a number of other properties do not carry over from $\bar{D}$ and D_P to $C(D, P)$. For example, when D is a polynomial ring in more than one indeterminate over a field and when P is a minimal prime ideal generated by one of the indeterminates, then it follows from Theorem 2.6 that $C(D, P)$ has a unique minimal prime ideal and has Krull dimension greater than one. Such a domain can be neither Noetherian (by Krull's Principal Ideal Theorem [**8**, Theorem 142, p. 104]) nor a unique factorization domain nor even a Krull domain (since a Krull domain is the intersection of its localizations at its minimal primes, [**10**, Proposition 8.18, p. 182]) even though both D_P and $\bar{D}$ are Noetherian unique factorization domains. From the next result, we obtain detailed information about all such examples in which $C(D, P)$ fails to be

Noetherian while both D_P and $\bar{D}$ are Noetherian. Specifically, it is shown that the bad behavior must be restricted to those ideals of $C(D, P)$ contained in PD_P.

2.9 PROPOSITION. *Let A denote an ideal of $C(D, P)$ properly containing PD_P and let $\bar{A}$ denote the image of A in $\bar{D}$. Then A is a finitely generated (respectively, principal) ideal if and only if $\bar{A}$ is a finitely generated (respectively, principal) ideal.*

Proof. Clearly if A is finitely generated or principal then so is $\bar{A}$. Suppose that $\bar{A}$ is finitely generated, say $\bar{A} = (\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n)$. The ideal $(a_1, a_2, \ldots, a_n)$ of $C(D, P)$, where a_i is a preimage of $\bar{a}_i$, contains PD_P since it is not contained in PD_P. Therefore, $(a_1, \ldots, a_n)$ is an ideal of $C(D, P)$ containing PD_P whose image is $\bar{A}$. Consequently $(a_1, \ldots, a_n) = A$ by the correspondence theorem. A similar argument can be used to establish the assertion for principal ideals.

From Proposition 2.7, we see that the contraction mapping from the set of all ideals of $C(D, P)$ into the set of ideals of D is very well-behaved, being one-to-one both on the ideals of $C(D, P)$ containing PD_P and on the primary ideals of $C(D, P)$ contained in PD_P. Of course, in the latter case we really can expect no more of the contraction mapping than is true of the contraction mapping from a localization to the base ring, since when P is maximal ideal of D, then $C(D, P) = D_P$. However, it turns out that the contraction mapping from $C(D, P)$ to D fails to duplicate all of the good behavior that contraction mappings from localizations have. Specifically, contraction from D_P to D followed by extension to D_P is the identity mapping, but the following example shows that the same property does not hold for contraction from $C(D, P)$ to D followed by extension to $C(D, P)$ even for ideals inside PD_P. In fact, the example shows that at the ideal PD_P itself, contraction followed by extension does not yield PD_P.

2.10 *Example.* Let $D = F[X, Y]$ where F is a field and let $P = (Y)$. Then it is straightforward to show that

$$C(D, P) = \left\{\frac{f(X) + Yg(X, Y)}{h(X) + Yk(X, Y)} \mid f(X), h(X) \in F[X],\right.$$

$$\left. g(X, Y), k(X, Y) \in F[X, Y], h(X) \neq 0, h(X) \text{ divides } f(X) \text{ in } F[X]\right\},$$

and

$$PD_P = \left\{\frac{Yg(X, Y)}{h(X) + Yk(X, Y)} \mid h(X) \neq 0, g(X, Y), k(X, Y) \in F[X, Y]\right\}.$$

From the above observations, we see that $Y/X \in PD_P$. However, since $1/X \notin C(D, P)$, $Y/X \notin P \cdot C(D, P)$. Therefore, $(PD_P \cap D) \cdot C(D, P) = P \cdot C(D, P) \neq PD_P$. Furthermore, $P \cdot C(D, P)$ cannot be a prime ideal by Proposition 2.7.

The final example of this section serves a dual purpose. First, it shows that it is possible for $C(D, P)$ to be a proper integral extension of D. This underscores the fact that a CPI-extension is in general basically different from a localization (which is never a proper integral extension), even though it has some of the properties of a localization, and it equals D_P when P is a maximal ideal. The second purpose of this example is to show that Theorem 2.4 is no longer true when only prime ideals of D are considered in the second condition of the theorem. In other words, it is not necessarily true that $C(D, P)$ equals D when every prime ideal of D is comparable to P.

2.11 *Example.* The domain D and the prime ideal P defined below are the same as those labeled D and P in Example 1.6 of [**5**]. They are constructed as follows:

(i) F denotes a field and X and Y are polynomial indeterminates;

(ii) D_1 denotes $F[X, \{Y^kX^m | k, m \in Z, k > 0\}]$, M_1 denotes the maximal ideal of D_1 consisting of all expressions in D_1 with zero constant term, and V denotes $(D_1)_{M_1}$;

(iii) D_0 denotes $F[X, \{Y^kX^m | k, m \in Z, k > 0, m \geqq -k^2\}]$, M_0 denotes $M_1 \cap D_0$, and P_0 denotes the ideal of D_0 generated by $\{Y^nX^{-n} | n > 0\}$; and

(iv) D denotes $(D_0)_{M_0}$, M denotes M_0D, and P denotes P_0D.

Among the known properties of this example are the facts that V is integral over D and that the pospec of D is $(0) \subsetneqq P \subsetneqq M$ [**5**, Lemma 1.8 and Proposition 1.9]. Since it is evident that V is a proper overring of D, the example will be completed by the following proposition.

2.12 Proposition. *Following the notation just established, $D_P = F[X, Y]_{(Y)}$ and $C(D, P) = V$.*

Proof. First, we note that $F[X, Y] \subseteq D_0$ and $F[X, Y]\backslash(Y) \subseteq D_0\backslash P_0$, and we conclude that

$$F[X, Y]_{(Y)} \subseteq (D_0)_{P_0} = ((D_0)_{M_0})_{P_0(D_0)_{M_0}} = D_P.$$

To show the reverse containment, let t denote an element of D_P. Then we may write t as a/b, where $a \in D_0$ and $b \in D_0\backslash P_0$. Note that for a sufficiently large positive integer m, $aX^m \in F[X, Y]$ and $bX^m \in F[X, Y]\backslash(Y)$; consequently, $a/b = aX^m/bX^m \in F[X, Y]_{(Y)}$. So $D_P \subseteq F[X, Y]_{(Y)}$ and we have established the first equality in the proposition.

To prove $C(D, P) = V$, we begin by recalling the earlier observation that in general $C(D, P) = D + PD_P$. Let a denote an element of PD_P. Then a can be written as bc, where $b \in P$ and $c \in D_P$. By the first part of this proof, we know that $D_P = F[X, Y]_{(Y)}$, so c can be written as a fraction whose numerator is a polynomial in $F[X, Y]$ and whose denominator is a polynomial in $F[X, Y]$ with at least one nonzero monomial not involving Y. Since b is an element of $P_0(D_0)_{M_0}$, it can be written as a fraction that, after multiplying numerator and denominator by an appropriate power of X, has numerator in $YF[X, Y]$

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and denominator in $F[X, Y]$ with at least one nonzero term not involving Y. Multiplying b and c together, we conclude that a can be written as

$$a = Y \cdot f(X, Y)/(g(X) + Y \cdot h(X, Y))$$

with $g(X) \neq 0$. Let X^k, $k \geqq 0$, be the smallest power of X in $g(X)$. Multiplying numerator and denominator by X^{-k}, we can express a as a fraction whose numerator is in D_1 and denominator in $D_1 \backslash M_1$. So $a \in (D_1)_{M_1} = V$, and hence, $PD_P \subseteq V$. Since it is clear from the definition of D and V that $D \subseteq V$, we have shown that $D + PD_P \subseteq V$.

To show the reverse containment, we will show that $D_1 \subseteq D + PD_P$ and that $1/s_1 \in D + PD_P$ for each $s_1 \in D_1 \backslash M_1$. If $d_1 \in D_1$, then it is an F-linear combination of nonnegative powers of X and monomials of the form $Y^k X^m$ with $k > 0$ and $m \in Z$. Since powers of X are in D and monomials of the form $Y^k X^m$ are in PD_P, we conclude that $d_1 \in D + PD_P$ and hence $D_1 \subseteq D + PD_P$. Now let s_1 denote an element of $D_1 \backslash M_1$. So s_1 is of the same form as d_1 with the additional condition of having a nonzero constant term. Say $s_1 = g(X) + YX^{-m} f(X, Y)$ where $g(X)$ is a polynomial with nonzero constant term, $f(X, Y)$ is a polynomial, and m is a nonnegative integer. Then

$$\frac{1}{s_1} - \frac{1}{g(X)} = -\frac{Yf(X, Y)}{g(X)(g(X)X^m + Yf(X, Y))},$$

and therefore $1/s_1 - 1/g(X) \in PD_P$. But $1/g(X) \in D$, so $1/s_1 \in D + PD_P$ as required, and the proof that $V = D + PD_P = C(D, P)$ is complete. (Note that the motivation to pick $1/g(X)$ as the candidate for an element in D that differs from $1/s_1$ by an element of PD_P arises from the observation that under the homomorphism Ψ, Y goes to zero and hence, $\Psi(1/s_1) = \Psi(1/g(X))$.

3. The general case. The steps in the construction of the CPI-extension of an integral domain with respect to a prime ideal presented in the previous section can be performed with certain modifications with respect to ideals in general. The resulting domain has many properties similar to those found in the CPI-extension for the prime case and will be studied in this section. Throughout this section the following notation will be used:

(i) D denotes an integral domain,

(ii) A denotes an ideal of D,

(iii) $\bar{D}$ denotes D/A and φ denotes the canonical homomorphism from D onto $\bar{D}$.

(iv) S denotes $\{d \in D | \varphi(d)$ is a regular element in $\bar{D}\}$ (A *regular* element is defined to be an element that is not a zero divisor. It is straightforward to show that S is a multiplicative system in D and consequently we can form the quotient overring D_S.), and

(v) Ψ denotes the epimorphism from D_S onto T, the total quotient ring of $\bar{D}$, defined by $\Psi(d/s) = \varphi(d)/\varphi(s)$. (The fact that Ψ is an epimorphism is straightforward to establish.)

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3.1 *Definition.* The preimage of $\bar{D}$ under Ψ is called the *CPI-extension* of D with respect to A and is denoted by $C(D, A)$.

It is easy to show that the ideal AD_S, denoted by A_S, is the kernel of Ψ and hence, an ideal of $C(D, A)$. This ideal plays much the same role played by PD_P in the CPI construction described in the previous section. Specifically, it satisfies the condition of being a CPI-ideal as formally defined below.

3.2 *Definition.* The ideal A is a *CPI-ideal* of D in case $C(D, A) = D$.

3.3 Proposition. *The ideal A_S is a CPI-ideal of $C(D, A)$.*

Proof. Let t be an element of $C(D, A)$ such that $\Psi(t)$ is a regular element in $\bar{D}$. Using the fact that $\bar{D} = D/A = C(D, A)/A_S$, we see that $\Psi(t) = \Psi(s)$ for some $s \in S$ and hence, $t - s \in A_S$. Therefore, $t - s = a/s'$ where $a \in A$ and $s' \in S$ and so $t = (a + ss')/s'$. Since $\varphi(a + ss') = \varphi(ss')$ is a regular element of $\bar{D}$, $a + ss' \in S$. Thus, $(a + ss')^{-1} \in D_S$ which implies that $t^{-1} \in D_S$. Therefore, if $T = \{t \in C(D, A) | \Psi(t)$ is a regular element in $\bar{D}\}$, then $C(D, A)_T = D_S$ and since $C(D, A)$ is already the complete preimage of the extension of φ to D_S, $C(D, A)$ is the CPI-extension of $C(D, A)$ with respect to A_S.

Proposition 3.5 will show that the properties of contraction from $C(D, P)$ to D shown in Proposition 2.7 carries over to the general case with A_S playing the role of PD_P. However, the application of this result will be less complete than in the prime case since it is not true in general that all of the ideals of $C(D, A)$ are comparable to A_S (Example 3.13). We first state a result that is analogous to the comparability result stated for the prime case in Proposition 2.3.

3.4 Proposition. *If B is an ideal of $C(D, A)$, then either B contains A_S or B does not intersect S.*

Proof. Suppose $B \cap S \neq \emptyset$ and let $s \in B \cap S$. Let $a/s_1 \in A_S$ where $a \in A$ and $s_1 \in S$. Then $s(a/ss_1) = a/s_1 \in B$ since $a/ss_1 \in C(D, A)$ and so $A_S \subseteq B$ as required.

3.5 Proposition. *Contraction of ideals from $C(D, A)$ to D defines the following order-isomorphisms with respect to set inclusion:*

(i) *between the ideals of $C(D, A)$ containing A_S and the ideals of D containing A, and*

(ii) *between the primary ideals of $C(D, A)$ not intersecting S and the primary ideals of D not intersecting S. In addition, when restricted to those primary ideals of $C(D, A)$ contained in A_S the contraction map is onto the primary ideals of D contained in A.*

Furthermore, in both these correspondences, prime ideals correspond to prime ideals.

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Proof. (i) We first note that the restriction of ψ to $C(D, A)$ is a homomorphism from $C(D, A)$ onto $\bar{D}$ with kernel A_S and φ is a homomorphism from D onto $\bar{D}$ with kernel A. By applying the correspondence theorem to each of these homomorphisms, we obtain the pairing $\psi^{-1}(B)$ with $\varphi^{-1}(B)$ for each ideal B of $\bar{D}$, yielding a one-to-one correspondence between the ideals of $C(D, A)$ containing A_S and the ideals of D containing A. Furthermore, since φ is the restriction of ψ to D,

$$\begin{aligned}\psi^{-1}(B) \cap D &= \{x \in D | \psi(x) \in B\}\\ &= \{x \in D | \varphi(x) \in B\}\\ &= \varphi^{-1}(B)\end{aligned}$$

and so this correspondence is realized through contraction. Since $\varphi^{-1}(B) \subseteq \varphi^{-1}(C)$ if and only if $\psi^{-1}(B) \subseteq \psi^{-1}(C)$, the correspondence is an order-isomorphism.

To prove (ii), we observe that the argument given to establish part (ii) of Proposition 2.7 can be employed here with D_S playing the role of D_P and with the observation that $C(D, A)_S = D_S$. This adapted argument shows that the contraction mapping yields an order-isomorphism between the primary ideals of $C(D, A)$ not intersecting S and the primary ideals of D not intersecting S. We now wish to show that this correspondence restricted to those primary ideals of $C(D, A)$ contained in A_S is onto the set of primary ideals of D contained in A. Since $A = \ker \varphi$ and $A_S = \ker \Psi$, it is clear that $A_S \cap D = A$. Therefore, each primary ideal of $C(D, A)$ contained in A_S contracts to a primary ideal of D contained in A. If B is an ideal of $C(D, A)$ not contained in A_S. then there exists an element $b/s \in B$ where $b \in D \backslash A$. Since $b = s(b/s) \in B \cap D$, $B \cap D \not\subseteq A$. Therefore, the contraction of the primary ideals of $C(D, A)$ contained in A_S is onto the primary ideals of D contained in A.

Finally, prime ideals correspond to prime ideals for the same reasons given in Proposition 2.7.

In Section 2 we showed that when P is a prime ideal, the prime spectrum of $C(D, P)$ is homeomorphic to the subspace of Spec(D) consisting of the contracted primes. In Example 3.14, we will show that this is not the case for CPI-extensions with respect to ideals in general. Nor in this example is the pospec of $C(D, A)$ order-isomorphic to the poset of contracted primes of D. However, the next result affirms that there is still a strong relation between Pospec$(C(D, A))$ and the poset of contracted primes, since contraction is a one-to-one order-preserving correspondence from the former to the latter (the inverse of which sometimes fails to be order-preserving) and since on the subsets of Pospec$(C(D, A))$ suggested by Proposition 3.4, contraction is an order-isomorphism.

To simplify the statement of the next theorem, we introduce the following notation. Let $\mathscr{P}_1$ denote the set of all prime ideals of $C(D, A)$ containing A_S, let $\mathscr{P}_2$ denote the set of all prime ideals of $C(D, A)$ not intersecting S, and let $\mathscr{P}$ denote $\mathscr{P}_1 \cup \mathscr{P}_2$. (Note that $\mathscr{P}_1$ and $\mathscr{P}_2$ are not necessarily disjoint.) Also, let $\mathscr{C}_1$ denote the set of prime ideals of D containing A, let $\mathscr{C}_2$ denote the set of prime ideals of D not intersecting S, and let $\mathscr{C}$ denote $\mathscr{C}_1 \cup \mathscr{C}_2$.

3.6 THEOREM. *Using the notation introduced above, the following statements are true.*

(i) *$\mathscr{P}$ is the set of all prime ideals of $C(D, A)$.*
(ii) *$\mathscr{C}$ is the set of all contracted primes of D.*
(iii) *contraction is a one-to-one order-preserving mapping from $\mathscr{P}$ onto $\mathscr{C}$.*
(iv) *contraction is an order-isomorphism from $\mathscr{P}_1$ to $\mathscr{C}_1$.*
(v) *contraction is an order-isomorphism from $\mathscr{P}_2$ to $\mathscr{C}_2$.*

Proof. Part (i) follows from Proposition 3.4 and part (ii) follows from Proposition 3.5. To show that contraction is one-to-one on $\mathscr{P}$, suppose that P and Q are prime ideals of $C(D,A)$ such that $P \cap D = Q \cap D$. If $A \subseteq P \cap D$, then since $(A_S)^2 = (AD_S)^2 = A(AD_S) \subseteq A \cdot C(D, A)$, we have $A_S \subseteq P$ and $A_S \subseteq Q$. By Proposition 3.5, $P = Q$. If $A \not\subseteq P \cap D$, then $A_S \not\subseteq P$ and so, by Proposition 3.4, $P \cap S = \emptyset$ and similarly $Q \cap S = \emptyset$. Therefore, again by Proposition 3.5, we conclude that $P = Q$. So contraction is one-to-one. Moreover, it clearly maps onto $\mathscr{C}$, by the definition of contracted primes, and contraction is always an order-preserving map. Therefore, part (iii) is established. Parts (iv) and (v) follow from Proposition 3.5.

We now turn our attention to the description of the poset $\mathscr{C}$ of contracted primes of D as a subset of Pospec(D). The preceding result shows that a prime ideal P is in $\mathscr{C}$ if and only if either P is comparable with A or $(P + A)/A$ consists entirely of zero divisors in $\bar{D}$. Unlike in the prime case, $\mathscr{C}$, and consequently $\mathscr{P}$, cannot be characterized purely in terms of the pospec of D together with the relative position of A (Example 3.13). However, some partial information in this vein can be obtained from the following results.

3.7 PROPOSITION. *If P is a prime ideal of D comaximal with A, then $P \notin \mathscr{C}$.*

Proof. Since $P + A = D$ and $\varphi(D) = \bar{D}$, we conclude that $\varphi(P) = \bar{D}$. Hence $P \cap S \neq \emptyset$. Clearly $A \not\subseteq P$ and so $P \notin \mathscr{C}$.

3.8 COROLLARY. *Let M denote a maximal ideal of D. Then $M \in \mathscr{C}$ if and only if $A \subseteq M$.*

Proof. Immediate.

Next we consider the question of when an ideal is a CPI-ideal. We already have by Theorem 2.4 an ideal-theoretic characterization of those prime ideals that are CPI-ideals, namely that they be comparable to every ideal of D. Example 3.13 will show that this characterization does not carry over to the general case, and in fact, we have no characterization in the general case. We now present two partial results along these lines:

3.9 Proposition. *If A is a CPI-ideal of D, then A is contained in the Jacobson radial of D.*

Proof. Since $D = C(D, A)$, clearly all of the maximal ideals of D are trivially contractions of ideals from $C(D, A)$ (that is, each of the maximal ideals of D is in $\mathscr{C}$). Consequently, by Corollary 3.8, A is contained in all of the maximal ideals of D.

It is clear, even in the prime case, that the condition that A is contained in the Jacobson radical of D is not sufficient to conclude that A is a CPI-ideal. The next result gives a stronger condition which is sufficient.

3.10 Proposition. *If A is contained in the Jacobson radical of D and if $\bar{D}$ is a total quotient ring, then A is a CPI-ideal.*

Proof. We prove that $D = C(D, A)$ by proving the stronger fact that S is the set of units in D and hence $D = D_S$. Since each maximal ideal M of D contains A, its image in $\bar{D}$ is a proper ideal and hence consists entirely of zero divisors. Thus, the image of every nonunit of D is a zero divisor in $\bar{D}$. Consequently S is the set of units in D.

In view of Proposition 3.9, the condition that $\bar{D}$ be a total quotient ring is clearly not sufficient, by itself, for us to conclude that A is a CPI-ideal. We further note that $K[X]/(X^2)$ and $K[[X]]/(X^2)$, where K is a field, are isomorphic total quotient rings and yet (X^2) is not a CPI-ideal of $K[X]$ while (X^2) is a CPI-ideal of $K[[X]]$. Hence, the structure of $\bar{D}$ when $\bar{D}$ is a total quotient ring also appears to be inconclusive in determining whether A is a CPI-ideal.

As promised in Section 2, we now show that the properties of being a Prüfer domain or being integrally closed are imposed on $C(D, A)$ if D_S and $\bar{D}$ have the corresponding properties. Since $\bar{D}$ is not necessarily an integral domain, the property of $\bar{D}$ corresponding to being a Prüfer domain is that of being a *Prüfer ring*—that is, a ring in which every finitely generated ideal containing a regular element is an invertible ideal. We note that an integral domain is a Prüfer domain if and only if it is a Prüfer ring.

3.11 Proposition. *If D_S and $\bar{D}$ are Prüfer rings, then $C(D, A)$ is a Prüfer domain.*

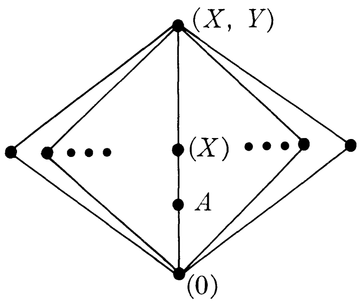
Proof. The proof of Theorem 2 in Boisen and Larsen [**3**, p. 88] allows us to conclude that $C(D, A)$ is a Prüfer domain once it is shown that each element x of D_S such that $\Psi(x)$ is a unit in T, is itself a unit in D_S. Let $x = d/s$ where $d \in D$ and $s \in S$. If $\Psi(x)$ is a unit in T, then $\Psi(d)\Psi(s^{-1}) = \Psi(ds^{-1}) = \Psi(x)$ is a unit in T. Therefore $\Psi(d)$ is a unit in T and consequently $\Psi(d)$ is a regular element in $\bar{D}$. Therefore, $d \in S$ and so $s/d \in D_S$ which implies that $x = d/s$ is a unit of D_S.

3.12 PROPOSITION. *If D_S and $\bar{D}$ are both integrally closed, then $C(D, A)$ is integrally closed.*

Proof. Let y be an element of K, the quotient field of D. Suppose that there exists $e_0, e_1, \ldots, e_{n-1} \in C(D, A)$ such that $y^n + e_{n-1}y^{n-1} + \ldots + e_0 = 0$. Since D_S is integrally closed, $y \in D_S$. Let $\bar{y}$ denote $\Psi(y)$ which is an element of the total quotient ring of $\bar{D}$. Also for each i, let $\bar{e}_i = \Psi(e_i)$, which is an element of $\bar{D}$ since $\Psi(C(D, A)) = \bar{D}$. Then $\bar{y}^n + \bar{e}_{n-1}\bar{y}^{n-1} + \ldots + \bar{e}_0 = \bar{0}$ in $\bar{D}$. Since $\bar{D}$ is integrally closed, we have $\bar{y} \in \bar{D}$. Consequently, $y \in C(D, A)$, and therefore, $C(D, A)$ is integrally closed.

Next we present an example that shows that the position an ideal A occupies relative to the pospec of D fails to determine the poset $\mathscr{C}$ of contracted primes from $C(D, A)$. In other words, the poset $(\text{Pospec}(D)) \cup \{A\}$ does not determine $\mathscr{C}$. This is in contrast to the prime case in which the set of contracted primes is precisely the set of primes comparable to P. This example also shows that a CPI-ideal need not be comparable with all of the ideals of D (or even all of the prime ideals of D).

3.13 *Example.* Let F denote a field and let $D = F[X, Y]_{(X,Y)}$. Then $A_1 = (X^2, XY)$ is a CPI-ideal while $A_2 = (X^2)$ is not a CPI-ideal. Both A_1 and A_2 are situated relative to the pospec of D in the position denoted by A in the figure below. While all prime ideals of D are contracted primes from $C(D, A_1)$, the prime ideal (Y) is not the contraction of a prime from $C(D, A_2)$.



Proof. Since $\sqrt{(X^2)} = \sqrt{(X^2, XY)} = (X)$, both A_1 and A_2 are situated relative to the pospec of D as described. Since the product of X with any nonunit of D is an element of A_1, D/A_1 is a total quotient ring. By Proposition 3.10, A_1 is a CPI-ideal. Since $C(D, A_1) = D$, it is clear that every prime ideal of D is the contraction of a prime ideal of $C(D, A_1)$. Since $A_2 = (X^2)$ is not comparable to (Y) and since $Y + A_2$ is clearly a regular element in D/A_2, by Theorem 3.6 (See the comment directly following Theorem 3.6), (Y) is not the contraction of a prime ideal of $C(D, A_2)$.

3.14 *Example.* Let F denote a field and let $D = F[W, X, Y, Z]$. Label the ideals $(X - YZ, WX, WZ)$, (W, X, Y), and (X) of D as A, P, and Q, respec-

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tively. Then P and Q are both contractions of prime ideals of $C(D, A)$. Let P' and Q' denote the prime ideals of $C(D, A)$ such that $P' \cap D = P$ and $Q' \cap D = Q$. Then $Q' \not\subseteq P'$ even though $Q \subseteq P$. Consequently, contraction is neither an order-isomorphism nor a homeomorphism.

Proof. Since P contains A, it follows from Theorem 3.6, part (ii), that P is a contracted prime ideal of D. Also, since $WX \in A$ and $W \notin A$, we have that $\varphi(X)$ is a zero divisor of $\bar{D}$, so $Q \cap S = \emptyset$. Again by Theorem 3.6, part (ii), Q is a contracted prime ideal of D. The example will be complete once we show $Q' \not\subseteq P'$. We will show that $X/Y \in Q' \backslash P'$. We begin by showing that $Y \in S$. Suppose that $Y \cdot e(W, X, Y, Z)$ is an element of A for some polynomial $e(W, X, Y, Z)$ in D. Then

$$(*) \quad Y \cdot e(W, X, Y, Z) = (X - YZ) \cdot f + WX \cdot g + WZ \cdot h$$

for some $f, g, h \in D$. Substituting zero for W in the above expression, we get

$$Y \cdot e(0, X, Y, Z) = (X - YZ) \cdot f(0, X, Y, Z)$$

By unique factorization in D, $X - YZ$ must divide $e(0, X, Y, Z)$, and consequently, $e(0, X, Y, Z) \in A$. Since $e(0, X, Y, Z)$ is the sum of those terms of e which do not involve W, we may, without loss of generality, assume that every monomial term of e involves W. Now substituting zero in for X and Z in $(*)$, we obtain the fact that $e(W, 0, Y, 0) = 0$. It follows at once that every monomial term of e involves X or Z. So every term of e is divisible by either WX or WZ, and hence, $e \in A$. Consequently, we have shown that whenever $Yd \in A$, then $d \in A$ which is precisely what it means for Y to map to a regular element in $\bar{D}$. Therefore $Y \in S$.

Now $X/Y \in D_S$, and $(X/Y) - Z$ which equals $(X - YZ)/Y$, is in A_S. Hence, $X/Y \in D + A_S = C(D, A)$, and $\Psi(X/Y) = \Psi(Z)$. Since $Z \notin P'$ and since $P' = \Psi^{-1}(\Psi(P'))$, it must be the case that $X/Y \notin P'$. Now since $Y(X/Y) \in Q'$ and $Y \notin Q'$, we have that $X/Y \in Q'$. So $X/Y \in Q' \backslash P'$, as we wanted to show.

To prove the final sentence in the statement of the example, we note that the contraction map fails to be an order-isomorphism since its inverse fails to preserve order. Furthermore, contraction fails to be a homeomorphism since it fails to preserve closed sets. Specifically, the closed set of all prime ideals of $C(D, A)$ containing Q' does not contain P', whereas its image in $\text{Spec}(D)$ is a set containing Q but not P, which cannot be a closed set in the relative topology.

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