INTRINSIC FUNCTIONS ON SEMI-SIMPLE ALGEBRAS

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1. Introduction. Rinehart **(5)** has introduced and motivated the study of the class of intrinsic functions on a linear associative algebra $\mathfrak{A}$, with identity, over the real field R or the complex field C. In this paper we shall consider a semi-simple algebra $\mathfrak{A} = \mathfrak{A}_1 \oplus \ldots \oplus \mathfrak{A}_t$ over R or C with simple components $\mathfrak{A}_i$. Let $\mathbf{G}$ be the group of all automorphisms or anti-automorphisms of $\mathfrak{A}$ which leave the ground field elementwise invariant, and let $\mathbf{H}$ be the subgroup of $\mathbf{G}$ such that $\Omega\mathfrak{A}_i = \mathfrak{A}_i$ $(i = 1, 2, \ldots, t)$ for each Ω in $\mathbf{H}$.

DEFINITION 1. *The single-valued function F, with domain $\mathbf{D}$ and range in $\mathfrak{A}$, is called an $\mathbf{H}$-intrinsic function on $\mathbf{D}$ if:*

(1) $\Omega\mathbf{D} = \mathbf{D}$ *for each* Ω *in* $\mathbf{H}$,

(2) Z *in* $\mathbf{D}$ *implies* $F(\Omega Z) = \Omega F(Z)$ *for all* Ω *in* $\mathbf{H}$.

If $\mathbf{H} = \mathbf{G}$, then F is said to be intrinsic on $\mathbf{D}$. Note that every intrinsic function on a semi-simple algebra $\mathfrak{A}$ is also $\mathbf{H}$-intrinsic, but not the converse. If $\mathfrak{A}$ is simple, however, then $\mathbf{H} = \mathbf{G}$ and every $\mathbf{H}$-intrinsic function is trivially intrinsic.

Intrinsic functions have been characterized by Rinehart **(5)** for the algebra of complex numbers over the real field and for the algebra Q of real quaternions. An essentially complete characterization of continuous intrinsic functions on the total matrix algebras C_n (n by n matrices with complex elements) over C and R_n (n by n matrices with real elements) over R has also been achieved by Rinehart **(6)** and on Q_n (n by n matrices with real quaternion elements) over R by Cullen **(2)**.

It is known that if $\mathfrak{A} = \mathfrak{A}_1 \oplus \ldots \oplus \mathfrak{A}_t$ is semi-simple over R with the $\mathfrak{A}_i$ as simple direct summands, then each $\mathfrak{A}_i$ is isomorphic to C_{ni}, R_{ni}, or Q_{ni}; whereas $\mathfrak{A}$ semi-simple over C implies that the $\mathfrak{A}_i$ are isomorphic to C_{ni} **(1)**.

In this paper we shall use **(4)** and the characterizations in **(2, 5, 6)** to characterize those intrinsic functions on a general semi-simple algebra over the real or complex field which induce (single-valued) functions on the direct summands. This study was motivated by the attempt to extend the notion of an n-ary function to a general semi-simple algebra.

2. Literature to date. There are several (essentially equivalent) methods of extending a function f of a complex variable to a function F on a linear associative algebra $\mathfrak{A}$; cf. **(7)**. Such functions F are called primary functions on $\mathfrak{A}$ with stem function f.

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Consider the following definition from **(6)** and **(2)**.

DEFINITION 2. *Given a function $f(z, \sigma_1, \ldots, \sigma_{n-1})$ with domain Σ contained in E_C^n (Euclidean complex n-space), and range in C. Define the n-ary function F, induced on R_n (C_n or Q_n) by f, to have functional values*

$$F(A) = f_A(A) = f_A(A, \sigma_1[A], \ldots, \sigma_{n-1}[A])$$

where $\sigma_i[A]$ is the ith *symmetric function of the characteristic roots of A. F is defined at A if the distinct characteristic roots $\alpha_1, \ldots, \alpha_k$ of A satisfy:*

(1) $(\alpha_i, \sigma_1[A], \ldots, \sigma_{n-1}[A]) \in \Sigma \quad (i = 1, 2, \ldots, k)$,

(2) $f(z, \sigma_1[A], \ldots, \sigma_{n-1}[A])$ *as a function of z is analytic at each α_i of index greater than one.*

($f_A(A)$ is the primary functional value of the extension of the scalar function $f_A(z) = f_A(z, \sigma_1[A], \ldots, \sigma_{n-1}[A])$, considered as a function of z only).

Note that the primary functions are n-ary functions that do not depend on the parameters σ_i.

In **(6)**, Rinehart proved the following theorems that motivated the study of n-ary functions.

THEOREM 2.1. *An intrinsic function F on C_n induces a single-valued function $f(\lambda, \sigma_1, \ldots, \sigma_{n-1})$ mapping a subset of E_C^n into the complex plane. The function f is defined at any point P: $(\lambda_0, \sigma_1^0, \ldots, \sigma_{n-1}^0)$ for which there exists a non-derogatory matrix A in the domain of F with λ_0 as a characteristic root and with characteristic polynomial*

$$c(x, A) = x^n - \sigma_1^0 x^{n-1} + \ldots + (-1)^{n-1} \sigma_{n-1}^0 x + (-1)^n \sigma_n^0.$$

The value of f at P is independent of the choice of the non-derogatory matrix A, and is given by $\lambda_0[F(A)] = L_A(\lambda_0)$, where $L_A(x)$ is a polynomial such that $L_A(A) = F(A)$ and $\lambda[B]$ denotes a characteristic root of B.

THEOREM 2.2. *Let F be an intrinsic function on R_n (or C_n) with domain* **D** *and let A belong to* **D**. *Let $f(z, \sigma_1, , \ldots, \sigma_{n-1})$ be the function on E_C^n to the complex plane induced by F. Let $f_A(z)$ denote the function of z only,*

$$f_A(z) = f(z, \sigma_1, \ldots, \sigma_{n-1}),$$

where $\sigma_i = \sigma_i[A]$, the ith *symmetric function of the characteristic roots of A. Then F(A) must be given by the primary function value $f_A(A)$ if either:*

Case I: A has distinct characteristic roots, or

Case II: A has repeated characteristic roots, A is interior to **D**, *F is continuous at A, and $f_A(z)$ is analytic in a z-neighbourhood of the repeated characteristic roots of A.*

Thus Rinehart has shown that intrinsic functions on R_n (or C_n) subject to the conditions of the preceding theorem are n-ary functions. Cullen **(2)** has shown that similar conditions also imply that an intrinsic function on Q_n is an n-ary function; thus intrinsic functions have been characterized on these three matrix algebras as being essentially n-ary functions.

In **(4)** the authors prove the following:

THEOREM 2.3. *Let $\mathfrak{A}$ be an algebra over R (or C) and let F be an intrinsic function from $\mathfrak{A}$ to $\mathfrak{A}$. Let $\mathfrak{M}$ be an algebra over the same field isomorphic to $\mathfrak{A}$. F induces on $\mathfrak{M}$ a function G with functional values*

$$G(A) = \phi F(\phi^{-1} A) = \phi F(\alpha),$$

where ϕ is an isomorphism of $\mathfrak{A}$ onto $\mathfrak{M}$ and $\phi(\alpha) = A$. Schematically:

$$\begin{array}{ccc} \mathfrak{A} & \xrightarrow{\ F\ } & \mathfrak{A} \\ \phi \downarrow & G & \downarrow \phi \\ \mathfrak{M} & \longrightarrow & \mathfrak{M} \end{array}$$

if F is intrinsic on $\mathfrak{A}$, then G is intrinsic on $\mathfrak{M}$.

Since the isomorphism ϕ maps the zero of $\mathfrak{A}$ into the zero of $\mathfrak{M}$ and also

$$\phi\left(\sum_{k=1}^{t} c_k \alpha_k\right) = \sum_{k=1}^{t} c_k \phi(\alpha_k),$$

we have

THEOREM 2.4. *If $\phi(\alpha) = A$, then the minimum polynomial of α equals the minimum polynomial of A over the same field.*

3. Intrinsic functions on simple algebras. Let $\mathfrak{A}$ be an n-dimensional simple algebra over R (or C) and F an intrinsic function on $\mathfrak{A}$. As discussed in § 1, $\mathfrak{A}$ is isomorphic to a matrix algebra $\mathfrak{M}$ ($\mathfrak{M} = C_n$, R_n, or Q_n) and, by Theorem 2.3, F induces an intrinsic function G on $\mathfrak{M}$. Let α belong to $\mathfrak{A}$, $\phi(\alpha) = A$ (where $\phi(\mathfrak{A}) = \mathfrak{M}$), and define a norm on $\mathfrak{A}$ by the isomorphism ϕ, i.e. $||\alpha|| = ||A||$, where $||A|| = 1/n \sup A_{ij}$ for $A = (A_{ij})$. $\mathfrak{A}$ is then a normed ring and the concepts of continuity, interior point, etc. are well defined. We shall extend the definition of n-ary function to simple algebras in general by the following:

DEFINITION 3. *F is said to be an n-ary function on a simple algebra $\mathfrak{A}$ if its domain and range are contained in $\mathfrak{A}$, and for each α in the domain of F, $F(\alpha) = g_A(\alpha)$ where*

$$g_A(z) = g_A(z, \sigma_1[A], \ldots, \sigma_{n-1}[A]) \tag{1}$$

is the stem function induced by $A = \phi(\alpha)$ on C, in accordance with Theorem 2.1, and $g_A(\alpha)$ is the primary functional value of the extension of $g_A(z)$ to $\mathfrak{A}$.

Using this extended definition of n-ary functions on simple algebras we now characterize intrinsic functions on simple algebras as follows:

THEOREM 3.4. *If $\mathfrak{A}$ is an n-dimensional simple algebra over C, then an intrinsic function F on $\mathfrak{A}$ is an n-ary function on $\mathfrak{A}$ if for each α in the domain of F, either*

Case I: *the minimum polynomial of α over C has n distinct roots, or*

Case II: *α is interior to the domain of F, F is continuous at α, and the scalar function $g_A(z)$ is analytic in a z-neighbourhood of the roots of the minimum polynomial of α.*

Proof. By Theorem 2.3, the induced function G on $\mathfrak{M} = C_n$ is intrinsic. G induces by Theorem 2.1 a scalar function (1) on C.

Case I above implies, by Theorem 2.4, that A is non-derogatory, and further that A has distinct characteristic roots.

Case II above implies, by isomorphism of algebras, that A is interior to the domain of the induced function G on $\mathfrak{M} = C_n$, G is continuous at A, and the stem function $g_A(z)$ is analytic in a z-neighbourhood of *all* the roots of A.

In either case, Theorem 2.2 implies that $G(A)$ is the primary value $g_A(A)$ with stem function $g_A(z)$.

Now g_A, being primary on $\mathfrak{M} = C_n$, is also a poly-function **(4)**, and thus there exists a polynomial $L_A(x)$ such that $L_A(A) = g_A(A)$ and

$$F(\alpha) = \phi^{-1}g_A(A) = \phi^{-1}L_A(A) = L_A(\phi^{-1}A) = L_A(\alpha) = g_A(\alpha).$$

That every intrinsic function on a simple algebra over C is not an n-ary function can be shown by the following:

Example 1. Define

$$F(A) = \begin{cases} 0 & \text{if } \det A = a_1 + a_2 i,\ a_1 \text{ rational}, \\ I & \text{if } \det A = a_1 + a_2 i,\ a_1 \text{ irrational}, \end{cases}$$

for A in $\mathfrak{A} = C_2$ as an algebra over C. This example is similar to Example 4 in **(3)** and the method there shows that $F(A)$ is intrinsic on C_2 over C but not n-ary on C_2 over C.

Note that if α satisfies Case I of Theorem 3.4, then the stem function (1) can be considered as a function

$$g_\alpha(z) = g(z, \sigma_1[\alpha], \ldots, \sigma_{n-1}[\alpha]),$$

where $\sigma_i[\alpha]$ is the ith symmetric function of the roots of the minimum polynomial of α, since the minimum polynomial of α equals the characteristic polynomial of A. If α satisfies Case II, however, in general there is no such relationship between the $\sigma_i[\alpha]$ and the $\sigma_i[A]$.

If $\mathfrak{A}$ is simple over R, then an intrinsic function F on $\mathfrak{A}$ is n-ary if in addition to the conditions of Theorem 3.4, we also have that the stem function $g_A(z)$ is an intrinsic function of z at the eigenvalues of A (or equivalently at the roots of the minimum polynomial of α). This added condition ensures that the functional value $g_A(A)$ is a real polynomial in A, and thus is an element of $\mathfrak{M}$; cf. **(3)**.

We now prove the following theorem, which is a consistent extension of similar results in **(2, 6)** for matrix algebras over R or C.

THEOREM 3.5. *Every n-ary function F on a simple algebra* $\mathfrak{A}$ *is an intrinsic function on* $\mathfrak{A}$.

Proof. The function $g_A(z)$ of Definition 4, when extended to the total matrix algebra isomorphic to $\mathfrak{A}$, yields an n-ary function, which by **(2, 6)** is also intrinsic. Theorem 2.3 then yields the desired result.

4. Intrinsic functions on semi-simple algebras. Let $\mathfrak{A}$ be an n-dimensional semi-simple algebra over R (or C); then $\mathfrak{A} = \mathfrak{A}_1 \oplus \ldots \oplus \mathfrak{A}_t$, where the $\mathfrak{A}_i$ are n_i-dimensional simple algebras over R (or C). Let F be a function with domain $\mathbf{D} = \mathbf{D}_1 \oplus \ldots \oplus \mathbf{D}_t$ and range in $\mathfrak{A}$. For each $\alpha = \alpha_1 + \ldots + \alpha_t$, $\alpha_i \in \mathbf{D}_i$, there are unique β_i in $\mathfrak{A}_i$ such that $F(\alpha) = \beta_1 + \ldots + \beta_t$.

If F is Hausdorff-differentiable, then F_i, the restriction of F to $\mathfrak{A}_i$, satisfies $F_i(\alpha_i) = \beta_i$, $i = 1, 2, \ldots, t$, cf. **(9)**, and more recently **(8)**. In general F_i need not map $\mathfrak{A}_i$ into $\mathfrak{A}_i$ and in fact the correspondence $\alpha_i \rightarrow \beta_i$ may not even define a (single-valued) function. If the correspondence $\alpha_i \rightarrow \beta_i$ is a well-defined function F_i $(i = 1, 2, \ldots, t)$, then we shall write $F = F_1 \oplus \ldots \oplus F_t$. Note that F_i maps the ith direct summand into itself.

We shall extend the concept of n-ary functions to semi-simple algebras by the following:

DEFINITION 4. *The function* $F = F_1 \oplus \ldots \oplus F_t$ *on* $\mathfrak{A} = \mathfrak{A}_1 \oplus \ldots \oplus \mathfrak{A}_t$ is *a direct sum of* n_i*-ary functions if each of the functions* F_i *is* n_i*-ary on the direct summand* $\mathfrak{A}_i$ $(i = 1, 2, \ldots, t)$.

In **(4)** it is shown that if $F = F_1 \oplus \ldots \oplus F_t$ is $\mathbf{H}$-intrinsic on $\mathfrak{A} = \mathfrak{A}_1 \oplus \ldots \oplus \mathfrak{A}_t$, then the F_i are intrinsic on the direct summands. Since an intrinsic function is $\mathbf{H}$-intrinsic, we also have that F intrinsic on $\mathfrak{A}$ implies that the F_i are intrinsic on the $\mathfrak{A}_i$. The converse of the former statement is true, but the converse of the latter is false. Using these results from **(4)** and Theorem 3.4, we have the following characterization of those intrinsic functions on semi-simple algebras which induce (single-valued) functions on the direct summands.

THEOREM 4.1. *If* $\mathfrak{A} = \mathfrak{A}_1 \oplus \ldots \oplus \mathfrak{A}_t$ *is an n-dimensional semi-simple algebra over* C *(or* R*) with simple direct summands* $\mathfrak{A}_i$ *of dimension* n_i*, and if* $F = F_1 \oplus \ldots \oplus F_t$ *is an intrinsic function on* $\mathfrak{A}$*, then* F *is a direct sum of* n_i*-ary functions* F_i *if for each* $\alpha = \alpha_1 + \ldots + \alpha_t$ *in the domain of* F *either*:

Case I: *the minimum polynomial of* α_i *over* C *(or* R*) has* n_i *distinct roots* $(i = 1, 2, \ldots, t)$, *or*

Case II: α_i *is interior to the domain of* F_i, F_i *is continuous at* α_i, *and the scalar function* $g_{A_i}(z)$ *induced by* α_i *is analytic in a z-neighbourhood of the roots of the minimum polynomial of* α_i $(i = 1, 2, \ldots, t)$. *(If* $\mathfrak{A}$ *is semi-simple* *over* R, *then we also require* $g_A(z)$ *to be intrinsic at the roots of the minimum polynomial of* α_i.)

If we try to replace the conditions of α_i in Theorem 4.1 by equivalent conditions on α, we notice the following problem: In general, α can be interior to the domain of F without α_i being interior to the domain of F_i. For this reason Theorem 4.1 seems to be as far as one might expect to be able to go in the direction of characterizing intrinsic functions on semi-simple algebras in terms of n-ary functions. We have the following additional theorems, however:

THEOREM 4.2. *If* $F = F_1 \oplus \ldots \oplus F_t$ *on* $\mathfrak{A} = \mathfrak{A}_1 \oplus \ldots \oplus \mathfrak{A}_t$ *is a direct sum of* n_i*-ary functions on* $\mathfrak{A}_i$, *then* F *is* **H**-*intrinsic on* $\mathfrak{A}$.

Proof. We have already shown that the n_i-ary functions F_i are intrinsic on $\mathfrak{A}_i$ (Theorem 3.5). By **(4)** this implies that $F = F_1 \oplus \ldots \oplus F_t$ is **H**-intrinsic on $\mathfrak{A}$.

In **(4)** an example is given of a function $F = F_1 \oplus \ldots \oplus F_t$ which is not intrinsic even though the F_i are intrinsic (in fact n_i-ary). However, using other results of **(3)**, we have

THEOREM 4.3. *If* $F = F_1 \oplus \ldots \oplus F_t$ *on* $\mathfrak{A} = \mathfrak{A}_1 \oplus \ldots \oplus \mathfrak{A}_t$ *is a direct sum of* n_i*-ary functions on* $\mathfrak{A}_i$ *and*

(1) *if* $\mathfrak{A}_i$ *is not isomorphic to* $\mathfrak{A}_j$ $(i \neq j)$, *or*

(2) *if for any automorphism or anti-automorphism* Ω' *of* $\mathfrak{A}$ *such that* $\Omega'\mathfrak{A}_i = \mathfrak{A}_j$ $(i \neq j)$ *it follows that* $F_j(\Omega\alpha_i) = \Omega' F_i(\alpha_i)$, *for all* α_i *in the domain of* F_i,

then F *is intrinsic on* $\mathfrak{A}$.

Proof. Condition (1) or (2) implies that F is intrinsic on $\mathfrak{A}$ if and only if the F_i are intrinsic on the $\mathfrak{A}_i$; **(4)**. The result follows from Theorem 3.5.

We now determine whether our Definition 4 (of a direct sum of n_i-ary functions) is consistent with the definition of n-ary functions on semi-simple algebras where both may be defined. Consider $\mathfrak{M}_n = \mathfrak{M}_{n1} \oplus \ldots \oplus \mathfrak{M}_{nt}$ $(t > 1)$ where $\mathfrak{M}_n$ is a subalgebra of R_n (C_n or Q_n) containing matrices which are direct sums $A = A_1 \oplus \ldots \oplus A_t$, with A_i belonging to $\mathfrak{M}_{ni} = R_{ni}$ (C_{ni} or Q_{ni}).

Let F be an n-ary function on R_n (C_n or Q_n) with domain $\mathbf{D} = \mathbf{D}_1 \oplus \ldots \oplus \mathbf{D}_t$ contained in $\mathfrak{M}_n$. That every n-ary function on $\mathfrak{M}_n$ does not induce single-valued functions F_i on the $\mathfrak{M}_{n_i}$ can be seen by the following:

Example 2. Consider the n-ary function F on C_4 with domain

$$\mathbf{D} = \{A \mid A = A_1 \oplus A_2, A_i \in C_2\}$$

and functional values $F(A) = \operatorname{tr}(A). I_4$. The stem function is $f_A(z) = \operatorname{tr} A$.

Let $B = I_2 \oplus I_2$ and $P = I_2 \oplus O_2$. Then if we assume that $F = F_1 \oplus F_2$, we have $F(B) = \text{tr}\, B \cdot I_4 \Rightarrow F_1(I_2) = 4I_2$ while

$$F(P) = \text{tr}\, P \cdot I_4 \Rightarrow F_1(I_2) = 2I_2.$$

Thus F_1 is not single-valued.

The following theorem sheds some light on the above example.

THEOREM 4.4. *The only n-ary functions on a direct sum of matrix algebras which induce single-valued functions on the direct summands are primary functions.*

Proof. Let F be an n-ary function on R_n (C_n or Q_n) with domain $\mathbf{D}$ contained in $\mathfrak{M}_n$ and with stem function $f_A(z, \sigma_1[A], \ldots, \sigma_{n-1}[A])$. Assume that F induces (single-valued) functions F_i on the direct summands,

$$F = F_1 \oplus \ldots \oplus F_t.$$

If f_A is dependent on the σ_i, then there exist matrices A and B in $\mathbf{D}$ with a common characteristic root z_1 such that :

$$f_A(z_1, \sigma_1[A], \ldots, \sigma_{n-1}[A]) \neq f_B(z_1, \sigma_1[B], \ldots, \sigma_{n-1}[B]).$$

Since F is intrinsic **(2, 6)**, we can assume that A and B are in Jordan canonical form. Without loss of generality we can also assume that z_1 is a root of A_1 (in fact that $[A_1]_{11} = z_1$) and of B_2 (in fact that $[B_2]_{11} = z_1$) where $A = A_1 \oplus \ldots \oplus A_t$ and $B = B_1 \oplus \ldots \oplus B_t$.

Noting that $F(A)$ is the primary functional value of the extension of $f_A(z)$ to $\mathfrak{A}$, it follows from **(7)** that:

$$[F_1(A_1)]_{11} = [f_A(A_1)]_{11} = f_A(z_1) \quad \text{and} \quad [F_2(B_2)]_{11} = [f_B(B_2)]_{11} = f_B(z_1).$$

Construct the matrix $C = C_1 \oplus \ldots \oplus C_t$ where $C_1 = A_1$, $C_j = B_j$ $(j \neq 1)$. $C \in \mathbf{D}$ since F_i has been defined at C_i $(i = 1, 2, \ldots, t)$.

Now $F(C) = \sum c_k C^k$ for complex c_k (note, however, that we cannot imply, nor is it necessary, that $c_k \in R$ if R is the ground field **(3)**). This implies that

$$[F_1(C_1)]_{11} = [F_1(A_1)]_{11} = [\sum c_k A_1^k]_{11} = \sum c_k z_1^k$$

and

$$[F_2(C_2)]_{11} = [F_2(B_2)]_{11} = [\sum c_k B_2^k]_{11} = \sum c_k z_1^k.$$

Thus

$$f_A(z_1) = [F_1(A_1)]_{11} = \sum c_k z_1^k = [F_2(B_2)]_{11} = f_B(z_1).$$

But $f_A(z_1) \neq f_B(z_1)$ and thus the F_i are not single-valued.

If F is primary, then it is known **(8)** that F induces single-valued functions on the direct summands which are also primary with the same stem function as F. This completes our proof.

COROLLARY 4.4. *The only n-ary functions on a direct sum of matrix algebras that are Hausdorff differentiable are primary functions.*

Proof. If F is Hausdorff differentiable, then F induces (single-valued) functions on the simple summands **(9)**.

Using the preceding theorem, we have:

THEOREM 4.5. *If* $F = F_1 \oplus \ldots \oplus F_t$ *(F_i well defined) is an n-ary function on* $\mathfrak{A} = \mathfrak{M}_n$, *then it is a direct sum of n_i-ary functions.*

Proof. Theorem 4.4 implies that F is primary on $\mathfrak{M}_n$, and this implies by **(4)** that the F_i are primary on the $\mathfrak{M}_{n_i}$. Every primary function being n-ary, we obtain our conclusion.

That functions $F = F_1 \oplus \ldots \oplus F_t$ on $\mathfrak{A} = \mathfrak{M}_n$ exist which are direct sums of n_i-ary functions (and thus intrinsic) but are not n-ary can be shown by the following:

Example 3. Consider the function F on C_4 with domain

$$\mathbf{D} = \{A \mid A = A_1 \oplus A_2, A_i \in C_2\}$$

and functional values $F(A) = F_1(A_1) \oplus F_2(A_2) = \operatorname{tr} A_1 \cdot I_2 \oplus A_2^2$. F_1 is 2-ary with stem function $f_{A_1}(z) = \operatorname{tr} A_1$ and F_2 is primary and thus 2-ary with stem function $f_{A_2}(z) = z^2$. Thus F is a direct sum of n_i-ary functions on $\mathfrak{M}_{C_4}$. F is not n-ary on C_4 however, since if f_A is the associated stem function, then for $A = A_1 \oplus A_1$ we have

$$f_A(A) = f_A(A_1) \oplus f_A(A_1) = \operatorname{tr} A_1 \cdot I_2 \oplus A_1^2,$$

which contradicts the uniqueness of the primary extension to C_2 of the function f_A.

We have thus derived a characterization of intrinsic functions on semi-simple algebras which induce (single-valued) functions on the direct summands, as direct sums of n_i-ary functions under similar conditions as those required in **(2, 6)**. Our extensions of the concept of n-ary functions to semi-simple algebras have been shown to be consistent with the original concepts where both are applicable.

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