Canad. Math. Bull. Vol. 29 (3), 1986

ON INDECOMPOSABLE PROJECTIVE MODULES

BY

JOHN D. O'NEILL

ABSTRACT. If P is an indecomposable projective R-module generated by a countable set X, then, for some countable subring S of R, P contains an indecomposable projective S-module generated by X. The subring S may be chosen to inherit many standard ring-theoretic properties from R.

Let R be a ring and let P be an indecomposable projective R-module. It is well-known (by Theorem 1 in [3]) that P has a countable generating set X. We show here that, for some countable subring S of R, P contains an indecomposable projective S-module generated by X. Moreover S may be chosen so that many standard ring-theoretic properties, if possessed by R, are inherited by S.

In this note all rings are associative with unity and all modules are left unital modules. A ring is local (semilocal) if the sum of any two non-units in it is a non-unit (if it contains only a finite number of maximal ideals). By $J(R)$, $U(R)$ and $N(R)$ we mean the Jacobson, nil and prime radical of R respectively. Other terminology may be found in [1 and 4].

THEOREM. *Let R be a ring and let P be an indecomposable projective R-module generated by the countable set X. Then R contains a countable subring S such that P contains an indecomposable projective S-module M generated by X and S has the following properties:*

(a) *An element in S is left or right invertible or is a left or right zero-divisor in S if it is in R.*

(b) *If $I \neq J$ are left ideals in S, then $RI \neq RJ$ in R. The corresponding statements for right and two-sided ideals are also true.*

(c) *If R has the ascending or descending chain condition on left, right or two-sided ideals, then S has the corresponding chain condition.*

(d) *If R is a prime, local or semilocal ring, then so is S.*

(e) *If $J(R)$, $U(R)$ or $N(R)$ is zero, then $J(S)$, $U(S)$ or $N(S)$ is zero respectively.*

(f) *If R is a commutative domain and R is a principal ideal domain or a Dedekind, Prüfer or Bezout ring, then S is the corresponding type of ring.*

Received by the editors February 11, 1985, and, in revised form, May 17, 1985.
AMS Subject Classification (1980): 16A50

CONSTRUCTION OF S AND M. Write $P \oplus Q = \oplus_1^\infty Re_n$, a free R-module, and let Y be a countable generating set in Q. Each element in X and Y is a linear combination over R of the e_n's. Express each e_n as a linear combination (not necessarily unique) of elements in X and Y. Let S_0 be the subring of R generated by the coefficients in these expressions of the e_n's and of the elements in X and Y. Then S_0 is countable and we let M_0 and N_0 be the S_0-modules generated over S_0 by X and Y respectively.

We next let S_1 be the countable subring of R generated by S_0 and elements selected from R, if possible, as follows:

(1) For each finite subset in S_0, say $x_1, \ldots, x_n$, and each y in S_0 select finite sets $\{a_i\}$, $\{b_i\}$ and $\{c_i, d_i\}$ so that $\Sigma a_i x_i$, $\Sigma x_i b_i$, $\Sigma c_i x_i d_i$ each equal y,

(2) For each x and y in S_0 select a so that $xay \neq 0$,

(3) For each x in S_0 and each positive integer n select a_n so that $(a_n x)^n \neq 0$.

(4) For each x in S_0 select a and b so that axb is not nilpotent.

(5) If R is a commutative domain, for each finite subset F in S_0 choose a finite set E from R so that, in the ring generated by E and S_0, the ideal generated by F is invertible and also principal if possible.

(6) For each two finite subsets, say $\{u_1, \ldots, u_n\}$ and $\{v_1, \ldots, v_m\}$ in $M_0 \cup N_0$, choose, if possible, elements a_i and b_i from R so that $\Sigma a_i u_i = \Sigma b_i v_i \neq 0$.

In the above selection process at each step we chose just one finite (countable in step 3) set of elements from R for each finite subset. As a result the ring S_1 generated by S_0 and the a's, b's, c's, d's and E's chosen in the steps above is a countable ring. Let M_1 and N_1 be the S_1-modules generated over S_1 by X and Y respectively. We now inductively construct ring S_n and S_n-modules M_n and N_n for each positive integer n by repeating the above process relative to S_{n-1}, M_{n-1}, and N_{n-1}. Let $S = \cup_0^\infty S_n$, $M = \cup_0^\infty M_n$ and $N = \cup_0^\infty N_n$.

PROOF OF THE THEOREM. We have $M \oplus N = \oplus_1^\infty Se_n$. Also S is countable and M and N are generated over S by X and Y respectively. We claim M is S-indecomposable. Suppose $M = A \oplus B$ as an S-module. Since X is in M, $P = RA + RB$. Since P is R-indecomposable, $\Sigma a_i u_i = \Sigma b_i u_i \neq 0$ for some a_i, b_i in R and u_i in A and v_i in B. But, by step (6) of our construction, there are elements a_i' and b_i' in S such that $0 \neq \Sigma a_i' u_i = \Sigma b_i' v_i$ and $A \cap B \neq 0$, a contradiction. So M is S-indecomposable (if Q is R-indecomposable, N is also S-indecomposable).

We now verify the second part of the Theorem. We shall refer repeatedly to the steps in our Construction above.

(a) This follows from step 1 with $n = 1$ and with $y = 1$ or 0 in R.

(b) We just treat the left ideal case. Suppose $y \in J \backslash I$. By step 1 then $y \notin RI$. So $RI \neq RJ$.

(c) This follows from (b).

(d) Suppose R is a prime ring and x and y are nonzero elements in S. Then $xRy \neq 0$ (see p. 164 ex. 10 in [1]). By step (2) then $xSy \neq 0$ and S is prime. Suppose R is local and x, y are non-units in S. By (a) above x and y are non-units in R and, since R is local, $x + y$ is a non-unit in R and also in S. Therefore S is local. Suppose R is semilocal. If I is a maximal ideal in S, then RIR is proper in R by step (b) and $I \subseteq RIR \subseteq M \subsetneqq R$ for some maximal ideal M in R. As a result $I = M \cap S$. Thus, if R has only a finite number of maximal ideals, so does S.

(e) Let $\{I_k\}$, $k \in K$, be the set of maximal left ideals in R. By arguing as in (d) for left (instead of two-sided) ideals we see that the maximal left ideals in S are precisely the ideals $I_k \cap S$ for $k \in K$. If $J(R) = 0$, then

$$J(S) = \bigcap_K (I_k \cap S) = \left(\bigcap_K I_k\right) \cap S = J(R) \cap S = 0.$$

Suppose $U(R) = 0$ and x is a non-zero element in S. The ideal $R_x R$ is not nil in R. By (4) $S_x S$ is not nil in S so $U(S) = 0$. If $N(R) = 0$, then R contains no non-zero left nilpotent ideal (see ex. 14 on p. 176 in [1]). Neither does S by step (3). So $N(S) = 0$.

(f) Suppose I is an ideal in S generated by a finite set F. If RI is invertible or principal in R, then I is invertible or principal in S by step (5). Also by (c) above S is Noetherian if R is. Now (f) follows from the definitions of the particular rings (as may be found in [4]).

COROLLARY. *Any ring R contains a countable subring S with properties* (a)−(f) *listed in the Theorem above.*

PROOF. Let S_0 be the subring of R generated by 1. Let S_1 be the subring of R generated by S_0 and elements selected from R as in steps (1)−(5) in the Construction above. Similarly construct S_n for $n = 2, 3, \ldots$ by building on S_{n-1}. Let $S = \bigcup_0^\infty S_n$. Now apply the proof of the Theorem beginning with the second paragraph.

EXAMPLE. Let R be the ring of real-valued continuous functions on the unit interval. R contains an ideal P which is an indecomposable projective R-module which is not finitely generated (see p. 31 of [2]). By our theorem R contains a countable subring S and P contains a countable subset M such that M is an indecomposable projective S-module which is not finitely generated.

REFERENCES

1. F. Anderson and K. Fuller, *Rings and Categories of Modules*, Springer-Verlag, N.Y. 1973.
2. H. Bass, *Big projective modules are free*, Illinois J. of Math., **7** (1963), pp. 24−31.
3. I. Kaplansky, *Projective modules*, Ann. of Math., **68** (1958), pp. 372−377.
4. I. Kaplansky, *Commutative Rings*, Allyn and Bacon, Boston, 1970.

UNIVERSITY OF DETROIT
DETROIT, MICHIGAN 48221