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POTENTIALS OF A FROBENIUS-LIKE STRUCTURE

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Abstract. This paper proves the existence of potentials of the first and second kind of a Frobenius like structure in a frame, which encompasses families of arrangements. The frame uses the notion of matroids. For the proof of the existence of the potentials, a power series ansatz is made. The proof that it works requires that certain decompositions of tuples of coordinate vector fields are related by certain elementary transformations. This is shown with a nontrivial result on matroid partition.

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1. Introduction and main results. A Frobenius manifold comes equipped locally with a potential. If one gives a definition that does not mention this potential explicitly, one nevertheless obtains it immediately by the following elementary fact: Let z_i be the coordinates on $\mathbb{C}^n$ and $\partial_i = \frac{\partial}{\partial z_i}$ be the coordinate vector fields. Let M be a convex open subset of $\mathbb{C}^n$ and $\mathcal{T}_M$ be the holomorphic tangent bundle of M. Let $A : \mathcal{T}_M^3 \to \mathcal{O}_M$ be a symmetric map such that also $\partial_i A(\partial_j, \partial_k, \partial_l)$ is symmetric in i, j, k, l. Then, a potential $F \in \mathcal{O}_M$ with $\partial_i\partial_j\partial_k F = A(\partial_i, \partial_j, \partial_k)$ exists. On Frobenius manifolds, see [**1, 4**].

This paper is devoted to a nontrivial generalization of this fact. The generalization turns up in the theory of families of arrangements as in [**7**, chap. 3]. The geometry there looks at first view similar to the geometry of Frobenius manifolds, but at second view, it is quite different.

At first view, one finds in both cases data $(M, K, \nabla^K, C, S, \zeta)$ with the following properties: M is an open subset of $\mathbb{C}^n$ (with coordinates z_i and coordinate vector fields $\partial_i = \frac{\partial}{\partial z_i}$). $K \to M$ is a holomorphic vector bundle with a flat holomorphic connection ∇^K. C is a Higgs field, i.e., an $\mathcal{O}_M$-linear map

$$C : \mathcal{O}(K) \to \Omega^1_M \otimes \mathcal{O}(K) \tag{1.1}$$

such that all the endomorphisms $C_X : K \to K$, $X \in \mathcal{T}_M$, commute: $C_X C_Y = C_Y C_X$, and C and ∇^K satisfy the integrability condition

$$\nabla^K_{\partial_i} C_{\partial_j} = \nabla^K_{\partial_j} C_{\partial_j} \qquad \text{for all } i, j \in \{1, \dots, n\} \tag{1.2}$$

(which is equivalent to $\nabla^K(C) = 0$, see Remark 4.1). S is a ∇^K-flat symmetric nondegenerate and Higgs field invariant pairing. ζ is a global nowhere vanishing section of K.

At second view, one sees the differences. In the case of a Frobenius manifold, M is the Frobenius manifold, $\mathrm{rk}\, K = n$, and (much stronger) $C_\bullet \zeta : \mathcal{T}_M \to \mathcal{O}(K)$ is an isomorphism and all the sections $C_{\partial_i}\zeta$ are ∇^K-flat. One obtains an identification of TM with K and of the coordinate vector fields ∂_i with the flat sections $C_{\partial_i}\zeta$.

In the case of a family of arrangements, $\mathrm{rk}\, K \geq n$, and the ∇^K-flat sections in K have the following much more surprising form. Define $J := \{1, \dots, n\}$. A family of arrangements in $\mathbb{C}^k$ with $k < n$ as in [**7**, ch. 3] comes equipped with vectors $(v_i)_{i \in J}$ in $M(1 \times k, \mathbb{C}) = \{$row vectors of length k with values in $\mathbb{C}\}$ such that $\langle v_1, \dots, v_n \rangle = M(1 \times k, \mathbb{C})$. A subset $\{i_1, \dots, i_k\} \subset J$ is called *maximal independent* if $v_{i_1}, \dots, v_{i_k}$ is a basis of $M(1 \times k, \mathbb{C})$. The sections $C_{\partial_{i_1}} \dots C_{\partial_{i_k}} \zeta$ in K for such subsets $\{i_1, \dots, i_k\}$ are ∇^K-flat.

The purpose of this paper is to show that also in this situation a potential exists, which resembles the potential of a Frobenius manifold. This is nontrivial. The proof combines the integrability condition (1.2) with intricate combinatorial considerations, which are due to the complicated form of the ∇^K-flat sections.

Theorem 1.2 is the main result. Definition 1.1 gives the frame and the used notions. The frame is in two mild aspects more general than the data above in the case of arrangements. First, S is more general, and second, the maximal independent subsets $\{i_1, \dots, i_k\} \subset J$ are maximal independent with respect to an arbitrary matroid (J, F) of rank k. See Definition 2.1 for the notion of a matroid.

DEFINITION 1.1.

(a) A *Frobenius like structure of order* $(n, k, m) \in \mathbb{Z}^3_{>0}$ with $n \geq k$ is a tuple $(M, K, \nabla^K, C, S, \zeta, (J, F))$ with the following properties. M, K, ∇^K, C, ζ and J are as above. S is a ∇^K-flat m-linear form $S : \mathcal{O}(K)^m \to \mathcal{O}_M$, which is Higgs field invariant, i.e.,

$$S(C_X s_1, s_2, \dots, s_m) = S(s_1, C_X s_2, \dots, s_m) = \dots = S(s_1, s_2, \dots, C_X s_m) \quad (1.3)$$

for $s_1, s_2, \dots, s_m \in \mathcal{O}(K)$ and $X \in \mathcal{T}_M$. (J, F) is a matroid with rank $r(J) = k$. For any maximal independent subset $\{i_1, \dots, i_k\} \subset J$ the section $C_{\partial_{i_1}} \dots C_{\partial_{i_k}} \zeta$ is ∇^K-flat.

(b) Some notations: For any subset $I = \{i_1, \dots, i_k\} \subset J$, the differential operator $\partial_I := \partial_{i_1} \dots \partial_{i_k}$ and the endomorphism $C_I := C_{\partial_{i_1}} \dots C_{\partial_{i_k}} : \mathcal{O}(K) \to \mathcal{O}(K)$ are well defined (they do not depend on the chosen order of the elements $i_1, \dots, i_k$).

(c) In the situation of (a), a *potential of the first kind* is a function $Q \in \mathcal{O}_M$ with

$$\partial_{I_1} \dots \partial_{I_m} Q = S(C_{I_1}\zeta, \dots, C_{I_m}\zeta) \quad (1.4)$$

for any m maximal independent subsets $I_1, \dots, I_m \subset J$. A *potential of the second kind* is a function $L \in \mathcal{O}_M$ with

$$\partial_i \partial_{I_1} \dots \partial_{I_m} L = S(C_{\partial_i} C_{I_1}\zeta, \dots, C_{I_m}\zeta) \quad (1.5)$$

for any m maximal independent subsets $I_1, \dots, I_m \subset J$ and any $i \in J$.

THEOREM 1.2. *Let* $(M, K, \nabla^K, C, S, \zeta, (J, F))$ *be a Frobenius-like structure of some order* $(n, k, m) \in \mathbb{Z}^3_{>0}$. *Then, locally (i.e., near any* $z \in M \subset \mathbb{C}^n$*) potentials of the first and second kind exist.*

Notice that by formulas (1.4) and (1.5), the potential of the first kind determines the matrix elements of the m-linear form S on the flat sections $C_{\partial_{i_1}} \dots C_{\partial_{i_k}}\zeta$, and the potential of the second kind determines the matrix elements of the Higgs operators C_{∂_i} acting on the flat sections $C_{\partial_{i_1}} \dots C_{\partial_{i_k}}\zeta$. Thus, all information on the m-linear form and the Higgs operators is packed into the two potential functions.

At the end of the paper, several remarks discuss the case of arrangements and the relation to Frobenius manifolds. In the case of arrangements, one has an $(n, k, 2)$-Frobenius type structure, but also other ingredients, which lead to richer geometry. In the case of a Frobenius manifold, one has an $(n, 1, 2)$-Frobenius type structure. The potential L above generalizes the potential of a Frobenius manifold. For generic arrangements, a global explicit construction of the potentials Q and L had been given in [**8**]. Recently, this was generalized in [**5**] to all families of arrangements as in [**7**, ch. 3].

Section 2 cites a nontrivial result of Edmonds [**2**, Theorem 4] on matroid partition and adds some considerations. Section 3 applies an implication of it to a combinatorial situation, which in turn is needed in the proof of the main Theorem 1.2 in Section 4. Section 4 concludes with some remarks.

We thank a referee of an earlier version [**3**] of this paper for pointing us to the result on matroid partition. This led to the present version of the paper that uses matroids. The second author thanks the Max-Planck-Institut für Mathematik (MPI) in Bonn for hospitality during his visit in 2015–2016.

2. Matroid partition. DEFINITION 2.1 For example, [**2**]. A *matroid* (E, F) is a finite set E together with a nonempty family $F \subset \mathcal{P}(E)$ of subsets of E, called *independent sets*, such that the following holds:

(i) Every subset of an independent set is independent.
(ii) For every subset $A \subset E$, all maximal independent subsets of A have the same cardinality, called the rank $r(A)$ of A.

For example, if V is a vector space and $(v_e)_{e\in E}$ is a tuple of elements,which generates V, one obtains a matroid where a subset $B \subset E$ is independent if and only if the tuple $(v_b)_{b\in B}$ is a linearly independent tuple of vectors. In the case of a family of arrangements, such a matroid will be used.

The following result on matroid partition was proved by Edmonds [**2**].

THEOREM 2.2 [**2**, Theorem 1]. *Let (E, F_i), $i = 1, \dots, m$, be matroids that are defined on the same set E. Let $r_i(A)$ be the rank of $A \subset E$ relative to (E, F_i). The following two conditions are equivalent:*
(α) The set E can be partitioned into a family $\{I_i\}_{i=1,\dots,m}$ of sets $I_i \in F_i$.
(β) Any set $A \subset E$ satisfies

$$|A| \le \sum_{i=1}^{m} r_i(A). \tag{2.1}$$

The implication $(\alpha) \Rightarrow (\beta)$ is immediate: Suppose that $\{I_i\}_{i=1,\dots,m}$ is a partition of E with $I_i \in F_i$. Then, for any $A \subset E$

$$A = \dot{\bigcup}_{i=1}^{m} A \cap I_i, \quad |A| = \sum_{i=1}^{m} |A \cap I_i| \le \sum_{i=1}^{m} r_i(A).$$

But the implication $(\beta) \Rightarrow (\alpha)$ is nontrivial. The proof in [**2**] is an involved inductive algorithm.

We are interested in the more special situation in Theorem 2.6. Before, two lemmata are needed.

DEFINITION 2.3 [**2**].

(a) A minimal dependent set of elements of a matroid is called a *circuit*.
(b) For any number $l \in \mathbb{Z}_{\geq 0}$ and any finite set E with $|E| \geq l$, the set $F^{(l,E)} := \{I \subset E \mid |I| \leq l\}$ defines obviously a matroid $(E, F^{(l,E)})$, the *uniform matroid* of rank l.

LEMMA 2.4 [**2**, Lemma 2]. The union of any independent set I and any element e of a matroid contains at most one circuit of the matroid.

LEMMA 2.5. Let (E, F) be a matroid. Let $A_1, A_2 \subset E$ be subsets. For $i = 1, 2$, let $I_i \subset A_i$ be a maximal independent subset of A_i. Suppose that $I_1 \cup I_2$ is an independent set. Then, $I_1 \cup I_2$ is a maximal independent subset of $A_1 \cup A_2$, and $I_1 \cap I_2$ is a maximal independent subset of $A_1 \cap A_2$.

Proof. Suppose that for some element $b \in (A_1 \cup A_2) - (I_1 \cup I_2)$ the union $I_1 \cup I_2 \cup \{b\}$ is independent. Then, for some $i \in \{1, 2\}$, $b \in A_i$. But $I_i \cup \{b\}$ is a larger independent subset of A_i than I_i, a contradiction. This proves that $I_1 \cup I_2$ is a maximal independent subset of $A_1 \cup A_2$.

Suppose that for some element $b \in (A_1 \cap A_2) - (I_1 \cap I_2)$ the union $(I_1 \cap I_2) \cup \{b\}$ is independent. If $b \in I_i$, then $b \notin I_j$ where $\{i, j\} = \{1, 2\}$. Then, $I_j \cup \{b\}$ is an independent subset of A_j, a contradiction to the maximality of I_j. Therefore, $b \notin I_1 \cup I_2$. Thus, for $i = 1, 2$, the set $I_i \cup \{b\} \subset A_i$ is dependent as it is larger than I_i. Therefore, it contains a circuit $C_i \subset I_i \cup \{b\}$. Obviously $C_i \cap (I_i - I_j) \neq \emptyset$, where $\{i, j\} = \{1, 2\}$. Thus, $C_1 \neq C_2$. Both are circuits in $(I_1 \cup I_2) \cup \{b\}$, a contradiction to Lemma 2.4. This proves that $I_1 \cap I_2$ is a maximal independent subset of $A_1 \cap A_2$. $\square$

THEOREM 2.6. *Let (E, F_i), $i = 1, \ldots, m$, be matroids that are defined on the same set E and that satisfy together (α) and (β) in Theorem 2.2. Suppose that $F_m = F^{(l,E)}$ for some $l \in \mathbb{Z}_{\geq 0}$ with $l \leq |E|$. Suppose that the set*

$$G := \{A \subset E \mid |A| = l + \sum_{i=1}^{m-1} r_i(A)\} \tag{2.2}$$

contains the set E.

(*a*) *Then, this set G is closed under the operations union and intersection of sets. Especially, it contains a set called $A_{min} \subset E$, which is the unique minimal element of G with respect to the partial order given by inclusion. Of course $A_{min} \neq \emptyset$ if and only if $l \geq 1$.*
(*b*) *Now suppose $l \geq 1$. Then, $A_{min} = A_{par}$, where A_{par} is the set*

$$A_{par} := \{b \in E \mid \exists \text{ a partition } \{I_i\}_{i=1,\ldots,m} \text{ of } E \\ \text{such that } I_i \in F_i \text{ and } b \in I_m\}. \tag{2.3}$$

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Proof.

(a) Choose a partition $\{I_i\}_{i=1,\dots,m}$ of E with $I_i \in F_i$. For any subset $A \subset E$, it induces a partition $A = \dot{\bigcup}_{i=1}^m A \cap I_i$ of A into subsets $(A \cap I_i) \in F_i$. If $A \in G$, then by (2.2) each set $A \cap I_i$ is a maximal independent subset of A with respect to the matroid (E, F_i). As $|A| \geq l$, especially $|A \cap I_m| = l$. As E itself is in G, $|I_m| = l$, and thus $A \cap I_m = I_m$ for any set $A \in G$.
Let $A_1, A_2 \in G$. For any $i = 1, \dots, m$, Lemma 2.5 applies to the maximal independent sets $A_1 \cap I_i$ and $A_2 \cap I_i$ of A_1 respectively A_2 relative to the matroid (E, F_i), because also $(A_1 \cup A_2) \cap I_i \in F_i$. Therefore, $(A_1 \cup A_2) \cap I_i$ is a maximal independent subset of $A_1 \cup A_2$ relative to (E, F_i), and $(A_1 \cap A_2) \cap I_i$ is a maximal independent subset of $A_1 \cap A_2$ relative to (E, F_i). Also, $I_m = A_1 \cap I_m = A_2 \cap I_m$ shows

$$I_m = (A_1 \cup A_2) \cap I_m = (A_1 \cap A_2) \cap I_m.$$

Now, $A_1 \cup A_2 \in G$ and $A_1 \cap A_2 \in G$ are obvious. Therefore, G is closed under the operations union and intersection of sets.

(b) $A_{par} \subset A_{min}$: Fix an arbitrary element $b \in A_{par}$. Choose a partition $\{I_i\}_{i=1,\dots,m}$ of E with $I_i \in F_i$ and $b \in I_m$. Recall $A_{min} \cap I_m = I_m$. Thus, $b \in A_{min}$.
$A_{min} \subset A_{par}$: Fix an arbitrary element $b \in A_{min}$. Define $\widetilde{E} := E - \{b\}$. Any set $A \subset \widetilde{E}$ does not contain A_{min}, because $b \in A_{min}$. Therefore, any set $A \subset \widetilde{E}$ satisfies $A \notin G$ and

$$|A| \leq -1 + l + \sum_{i=1}^{m-1} r_i(A). \tag{2.4}$$

Consider the matroids $(\widetilde{E}, \widetilde{F}_i)$, where $\widetilde{F}_i := \{I \in F_i \mid b \notin I\}$ for $i \in \{1, \dots, m-1\}$ and $\widetilde{F}_m := F^{(l-1,\widetilde{E})}$. For $i \in \{1, \dots, m-1\}$ the rank of $A \subset \widetilde{E}$ relative to $(\widetilde{E}, \widetilde{F}_i)$ is equal to the rank $r_i(A)$ of A relative to (E, F_i).
By (2.4) and Theorem 2.2, a partition $\{\widetilde{I}_i\}_{i=1,\dots,m}$ of $\widetilde{E}$ with $\widetilde{I}_i \in \widetilde{F}_i$ exists. Now, the sets $I_i := \widetilde{I}_i$ for $i = 1, \dots, m-1$, and $I_m := \widetilde{I}_m \cup \{b\}$ form a partition of E with $I_i \in F_i$. This shows $b \in A_{par}$.

$\square$

3. An equivalence between index systems. In this section, we fix three positive integers $n, k, m \in \mathbb{Z}_{>0}$ with $n \geq k$ and a matroid (J, F) with underlying set $J = \{1, \dots, n\}$, rank function $r : \mathcal{P}(J) \to \mathbb{Z}_{\geq 0}$ and rank $r(J) = k$.

NOTATIONS 3.1. As usual $\mathbb{Z}^J := \{\text{maps} : J \to \mathbb{Z}\}$ and $\mathbb{Z}^J_{\geq 0} := \{\text{maps} : J \to \mathbb{Z}_{\geq 0}\}$. The set $\mathbb{Z}^J$ is an additive group, and the set $\mathbb{Z}^J_{\geq 0}$ is an additive monoid.

For $j \in J$ denote by $[j] \in \mathbb{Z}^J_{\geq 0}$ the map with $[j](j) = 1$ and $[j](i) = 0$ for any $i \neq j$. Then, any map $T \in \mathbb{Z}^J$ can be written as $T = \sum_{j=1}^n T(j) \cdot [j]$. For $T \in \mathbb{Z}^J$ denote $|T| := \sum_{j=1}^n T(j) \in \mathbb{Z}$. The support of $T \in \mathbb{Z}^J$ is $\operatorname{supp} T := \{j \in J \mid T(j) \neq 0\}$. The map

$$d_H : \mathbb{Z}^J \times \mathbb{Z}^J \to \mathbb{Z}_{\geq 0}, \quad (T_1, T_2) \mapsto \sum_{j \in J} |T_1(j) - T_2(j)| \tag{3.1}$$

is a metric on $\mathbb{Z}^J$. On $\mathbb{Z}^J$ one has the partial ordering $\leq$ with

$$S \leq T \iff S(j) \leq T(j) \quad \forall j \in J. \tag{3.2}$$

Any map $T \in \mathbb{Z}^J_{\geq 0}$ with $|T| = t \in \mathbb{Z}_{\geq 0}$ is called a *system of elements of* J or simply a *system* or a t-*system*. If S and T are systems with $S \leq T$, then S is a *subsystem* of T.

DEFINITION 3.2. Here, $l \in \mathbb{Z}_{\geq 0}$. Here, all systems are systems of elements of J.

(a) A system $T \in \mathbb{Z}^J_{\geq 0}$ is a *base* if $\operatorname{supp} T \in F$ and $|T| = k$ (so the support $\operatorname{supp} T$ is a maximal independent subset of J and all $T(a) \in \{0; 1\}$).
(b) A *strong decomposition* of an $(mk + l)$-system T is a decomposition $T = T^{(1)} + \cdots + T^{(m+1)}$ into m k-systems $T^{(1)}, \ldots, T^{(m)}$ and one l-system $T^{(m+1)}$ such that $T^{(1)}, \ldots, T^{(m)}$ are bases (and $T^{(m+1)}$ is an arbitrary l-system; e.g., if $l = 0$, then $T^{(m+1)} = 0$ automatically).
(c) An $(mk + l)$-system is *strong* if it admits a strong decomposition.
(d) A *good decomposition* of an N-system T with $N \geq mk + 1$ is a decomposition $T = T_1 + T_2$ into two systems such that T_2 is a strong $(mk + 1)$-system of elements of J.
(e) Two good decompositions $T_1 + T_2 = T$ and $S_1 + S_2 = T$ of an N-system T with $N \geq mk + 1$ are *locally related*, notation: $(S_1, S_2) \sim_{loc} (T_1, T_2)$, if there are strong decompositions $S_2^{(1)} + \cdots + S_2^{(m+1)} = S_2$ of S_2 and $T_2^{(1)} + \cdots + T_2^{(m+1)} = T_2$ of T_2 with $S_2^{(j)} = T_2^{(j)}$ for $1 \leq j \leq m$. Of course, $\sim_{loc}$ is a reflexive and symmetric relation.
(f) Two good decompositions $T_1 + T_2 = T$ and $S_1 + S_2 = T$ of an N-system T with $N \geq mk + 1$ are *equivalent*, notation: $(S_1, S_2) \sim (T_1, T_2)$, if there is a sequence $\sigma_1, \sigma_2, \ldots, \sigma_r$ for some $r \in \mathbb{Z}_{\geq 1}$ of good decompositions of T such that $\sigma_1 = (S_1, S_2)$, $\sigma_r = (T_1, T_2)$, and $\sigma_j \sim_{loc} \sigma_{j+1}$ for $j = 1, \ldots, r - 1$. Of course, $\sim$ is an equivalence relation.

The main result of this section is the following theorem.

THEOREM 3.3. *Let $T \in \mathbb{Z}^J_{\geq 0}$ be an N-system for some $N \geq mk + 1$, which has good decompositions. Then, all its good decompositions are equivalent.*

The theorem will be proved after the proofs of Corollary 3.4 and Lemma 3.5. Corollary 3.4 is a corollary of Theorem 2.6.

COROLLARY 3.4. *Fix a strong $(mk + l)$-system $T \in \mathbb{Z}^J_{\geq 0}$ with $l \in \mathbb{Z}_{\geq 0}$. Then, for any $B \subset J$*

$$\sum_{j \in B} T(j) \leq l + m \cdot r(B). \tag{3.3}$$

The set

$$G(T) := \{B \subset \operatorname{supp} T \mid \sum_{j \in B} T(j) = l + m \cdot r(B)\} \tag{3.4}$$

contains $\operatorname{supp} T$ *and is closed under the operations union and intersection of sets. Especially, it contains a set called $A_{min}(T) \subset$* $\operatorname{supp} T$, *which is the unique minimal element* *with respect to inclusion. In the case $l \geq 1$, define the set*

$$A_{dec}(T) := \{b \in J \mid \exists \text{ a strong decomposition } T = T^{(1)} + \cdots + T^{(m+1)} \text{ with } b \in \operatorname{supp} T^{(m+1)}\}. \tag{3.5}$$

Then, $A_{min}(T) = A_{dec}(T)$.

Proof. We will construct from T certain lifts of the matroids (J, F) and $(J, F^{(l,J)})$ to matroids on the set $E := \{1, 2, \ldots, mk + l\}$ and go with them into Theorem 2.6. Choose a map $f : E \to J$ with $|f^{-1}(j)| = T(j)$. Define the sets

$$F_1 = \ldots = F_m := \{A \subset E \mid f|_A : A \to J \text{ injective}, f(A) \in F\} \subset \mathcal{P}(E),$$
$$F_{m+1} := F^{(l,E)} \subset \mathcal{P}(E).$$

Then, (E, F_i) for $i \in \{1, \ldots, m+1\}$ is a matroid. Together they satisfy (α) in Theorem 2.2 (with $m+1$ instead of m) because T is a strong $(mk+l)$-system. We go into Theorem 2.6 with $m+1$ instead of m.

That T is a strong $(mk+l)$-system, gives also $E \in G$ and (3.3).

Therefore, the set A_{min} in Theorem 2.6 is well defined. The set A_{par} is well defined, anyway. One sees easily

$$r_1(A) = \ldots = r_m(A) = r(f(A)) \quad \text{for } A \subset E,$$
$$G = \{f^{-1}(B) \mid B \in G(T)\}.$$

Therefore, $G(T)$ contains $\operatorname{supp} T$ and is closed under the operations union and intersection of sets. Now, one sees also easily

$$A_{min} = f^{-1}(A_{min}(T)), \quad A_{par} = f^{-1}(A_{dec}(T)),$$

and thus $A_{min}(T) = A_{dec}(T)$. □

LEMMA 3.5. *Let S and $T \in \mathbb{Z}^J_{\geq 0}$ be two strong $(mk+1)$-systems. At least one of the following two alternatives holds:*

(α) T has a strong decomposition $T = T^{(1)} + \cdots + T^{(m+1)}$ with $T^{(m+1)} = [i]$ for some $i \in \operatorname{supp} T$ with $T(i) > S(i)$.

(β) For any strong decomposition $S = S^{(1)} + \cdots + S^{(m+1)}$ a strong decomposition $T = T^{(1)} + \cdots + T^{(m+1)}$ with $T^{(m+1)} = S^{(m+1)}$ exists.

Proof. Suppose that (α) does not hold. Then, for any $i \in A_{dec}(T)$ $S(i) \geq T(i)$. Especially,

$$\sum_{i \in A_{dec}(T)} S(i) \geq \sum_{i \in A_{dec}(T)} T(i) = 1 + m \cdot r(A_{dec}(T)).$$

The equality uses $A_{dec}(T) = A_{min}(T) \in G(T)$. Now (3.3) for S instead of T shows that $\geq$ can be replaced by $=$. Therefore, $A_{dec}(T) \in G(S)$. Any element of $G(S)$ contains $A_{min}(S)$. This and the equality $A_{dec}(S) = A_{min}(S)$ give

$$A_{dec}(S) = A_{min}(S) \subset A_{dec}(T).$$

Thus, (β) holds. □

Proof of Theorem 3.3 Let (S_1, S_2) and (T_1, T_2) be two different good decompositions of an N-system T of elements of J (with $N \geq mk + 1$). Then, S_2 and T_2 are strong $(mk + 1)$-systems of elements of J. At least one of the two alternatives (α) and (β) in Lemma 3.5 holds for S_2 and T_2.

First case, (α) *holds:* Let $T_2 = T_2^{(1)} + \cdots + T_2^{(m+1)}$ be a strong decomposition with $T_2^{(m+1)} = [i]$ for some $i \in \operatorname{supp} T_2$ with $T_2(i) > S_2(i)$. Then, a $j \in \operatorname{supp} T$ with $T_1(j) > S_1(j)$ and $T_2(j) < S_2(j)$ exists. The decomposition

$$T = R_1 + R_2 \quad \text{with } R_1 = T_1 - [j] + [i], \quad R_2 = T_2 + [j] - [i] \tag{3.6}$$

is a good decomposition of T because $T_2^{(1)} + \cdots + T_2^{(m)} + [j]$ is a strong decomposition of R_2. The good decompositions (R_1, R_2) and (T_1, T_2) are locally related, $(R_1, R_2) \sim_{loc} (T_1, T_2)$, and thus equivalent,

$$(R_1, R_2) \sim (T_1, T_2). \tag{3.7}$$

Furthermore,

$$d_H(R_2, S_2) = d_H(T_2, S_2) - 2. \tag{3.8}$$

Second case, (β) *holds:* Let $T_2 = T_2^{(1)} + \cdots + T_2^{(m+1)}$ and $S_2 = S_2^{(1)} + \cdots + S_2^{(m+1)}$ be strong decompositions of T_2 and S_2 with $T_2^{(m+1)} = S_2^{(m+1)} = [a]$ for some $a \in \operatorname{supp} T$. Two elements $b, c \in \operatorname{supp} T$ with $T_1(b) > S_1(b)$, $T_2(b) < S_2(b)$, and $T_1(c) < S_1(c)$, $T_2(c) > S_2(c)$ exist. Consider the decompositions of T and S,

$$T = R_1 + R_2 \quad \text{with } R_1 = T_1 - [b] + [a], R_2 = T_2 + [b] - [a], \tag{3.9}$$

$$S = Q_1 + Q_2 \quad \text{with } Q_1 = S_1 - [c] + [a], Q_2 = S_2 + [c] - [a]. \tag{3.10}$$

They are good decompositions because R_2 has the strong decomposition $R_2 = T^{(1)} + \cdots + T^{(m)} + [b]$ and Q_2 has the strong decomposition $Q_2 = S^{(1)} + \cdots + S^{(m)} + [c]$. The local relations

$$(R_1, R_2) \sim_{loc} (T_1, T_2) \quad \text{and} \quad (Q_1, Q_2) \sim_{loc} (S_1, S_2)$$

and the equivalences

$$(R_1, R_2) \sim (T_1, T_2) \quad \text{and} \quad (Q_1, Q_2) \sim (S_1, S_2) \tag{3.11}$$

hold. Furthermore,

$$d_H(R_2, Q_2) = d_H(T_2, S_2) - 2. \tag{3.12}$$

The properties (3.7), (3.8), (3.11) and (3.12) show that in both cases the equivalence classes of (S_1, S_2) and (T_1, T_2) contain good decompositions whose second members are closer to one another with respect to the metric d_H than T_2 and S_2. This shows that (S_1, S_2) and (T_1, T_2) are in one equivalence class. $\square$

4. Potentials of the first and second kind. The main part of this section is devoted to the proof of Theorem 1.2. At the end, some remarks on the relation to families of arrangements and Frobenius manifolds are made.

REMARK 4.1. Here, a coordinate free formulation of the integrability condition (1.2) will be given. For M, ∇^K and C as in the introduction, $\nabla^K(C) \in \Omega^2_M \otimes \mathcal{O}(\mathrm{End}(K))$ is the 2-form on M with values in $\mathrm{End}(K)$ such that for $X, Y \in \mathcal{T}_M$

$$\nabla^K(C)(X, Y) = \nabla^K_X(C_Y) - \nabla^K_Y(C_X) - C_{[X,Y]}. \tag{4.1}$$

Now, (1.2) is equivalent to $\nabla^K(C) = 0$

Proof of Theorem 1.2 Let $(M, K, \nabla^K, C, S, \zeta, (J, F))$ be a Frobenius-like structure of some order $(n, k, m) \in \mathbb{Z}^3_{>0}$.

We need some notations. If $T \in \mathbb{Z}^J_{\geq 0}$ is a system of elements of J, then

$$(z-x)^T := \prod_{i\in J}(z_i - x_i)^{T(i)} \quad \text{for any } x \in \mathbb{C}^n,$$
$$T! := \prod_{i\in J} T(i)!, \quad \partial_T := \prod_{i\in J} \partial_{z_i}^{T(i)}, \quad C_T := \prod_{i\in J} C_{\partial_{z_i}}^{T(i)}.$$

Thus, if S and T are systems of elements of J, then

$$\partial_T(z-x)^S = \begin{cases} 0 & \text{if } T \not\leq S, \\ \frac{S!}{(S-T)!}\cdot (z-x)^{S-T} & \text{if } T \leq S, \end{cases} \tag{4.2}$$

for any $x \in \mathbb{C}^n$.

The existence of a (not just local, but even global) potential Q of the first kind is trivial. The function

$$Q := \sum_{T \text{ with } (*)} \frac{1}{T!}\cdot S(C_T\zeta, \zeta, \ldots, \zeta)\cdot z^T \quad (m \text{ times } \zeta) \tag{4.3}$$
$$(*) \ : \ T \in \mathbb{Z}^J_{\geq 0} \text{ is a strong } mk\text{-system (Definition 3.1(c))}$$

works. It is a homogeneous polynomial of degree mk and contains only monomials that are relevant for (1.2). In fact, one can add to this Q an arbitrary linear combination of the monomials z^T for the mk-systems T that are not strong, so that are not relevant for (1.2).

The existence of a potential L of the second kind is not trivial. Let some $x \in M$ be given. We make the power series ansatz

$$L := \sum_{T\in\mathbb{Z}^J_{\geq 0}} a_T \cdot (z-x)^T, \tag{4.4}$$

where the coefficients a_T have to be determined. If T satisfies $|T| \leq mk$ or if it satisfies $|T| \geq mk+1$, but does not admit a good decomposition (Definition 3.1 (d)), then the conditions (1.3) are empty for $a_T(z-x)^T$ because of (4.2), so then a_T can be chosen arbitrarily, e.g., $a_T := 0$ works.

Now, consider T with $|T| \geq mk+1$, which admits good decompositions. Then, each good decomposition $T = T_1 + T_2$ gives via (1.3) a candidate

$$a_T(T_1, T_2) := \frac{1}{T!}\cdot (\partial_{T_1} S(C_{T_2}\zeta, \zeta, \ldots, \zeta))(x), \tag{4.5}$$

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for the coefficient a_T of $(z-x)^T$ in L. We have to show that the candidates $a_T(T_1, T_2)$ for all good decompositions (T_1, T_2) of T coincide.

Suppose that two good decompositions (T_1, T_2) and (S_1, S_2) are locally related, $(T_1, T_2) \sim_{loc} (S_1, S_2)$ (Definition 3.1 (e)), but not equal. Then, there are strong decompositions $T_2 = T_2^{(1)} + \cdots + T_2^{(m)} + [a]$ and $S_2 = T_2^{(1)} + \cdots + T_2^{(m)} + [b]$ with $a \neq b$, and thus also $T_1 - [b] = S_1 - [a] \in \mathbb{Z}_{\geq 0}^J$ holds. Because any $T_2^{(j)}$, $j \in \{1, \ldots, m\}$, is independent, $C_{T_2^{(j)}}\zeta$ is ∇^K-flat. This and (4.3) give

$$\begin{aligned}
&\partial_{z_b} S(C_{T_2}\zeta, \zeta, \ldots, \zeta)\\
&= \partial_{z_b} S(C_{\partial_{z_a}} C_{T_2^{(1)}}\zeta, C_{T_2^{(2)}}\zeta, \ldots, C_{T_2^{(m)}}\zeta)\\
&= S(\nabla^K_{\partial_{z_b}}(C_{\partial_{z_a}}) C_{T_2^{(1)}}\zeta, C_{T_2^{(2)}}\zeta, \ldots, C_{T_2^{(m)}}\zeta)\\
&= S(\nabla^K_{\partial_{z_a}}(C_{\partial_{z_b}}) C_{T_2^{(1)}}\zeta, C_{T_2^{(2)}}\zeta, \ldots, C_{T_2^{(m)}}\zeta)\\
&= \partial_{z_a} S(C_{\partial_{z_b}} C_{T_2^{(1)}}\zeta, C_{T_2^{(2)}}\zeta, \ldots, C_{T_2^{(m)}}\zeta)\\
&= \partial_{z_a} S(C_{S_2}\zeta, \zeta, \ldots, \zeta).
\end{aligned} \tag{4.6}$$

This implies

$$a_T(T_1, T_2) = a_T(S_1, S_2), \tag{4.7}$$

so the locally related good decompositions (T_1, T_2) and (S_1, S_2) give the same candidate for a_T. Thus, all equivalent (Definition 3.1 (f)) good decompositions give the same candidate for a_T. By Theorem 3.3, all good decompositions of T are equivalent. Therefore, they all give the same candidate for a_T. Thus, a potential L of the second kind exists as a formal power series as in (4.4).

It is in fact a convergent power series because of the following. There are finitely many strong $(mk+1)$-systems T_2. Each determines the coefficients a_T for all $T \geq T_2$. We put $a_T := 0$ for T, which do not admit good decompositions. The part of L in (4.4) that is determined by some strong $(mk+1)$-system T_2 is a convergent power series. Thus, L is the *union* of finitely many overlapping convergent power series. It is easy to see that it is itself convergent. This finishes the proof of Theorem 1.2. □

REMARK 4.2. In [**7**, chap. 3], families of arrangements are considered, which give rise to Frobenius-like structures $(M, K, \nabla^K, C, S, \zeta, (J, F))$ of order $(n, k, 2)$, see the special case of generic arrangements in [**6, 8**].

Start with two positive integers k and n with $k < n$ and with a matrix $B := (b_i^j)_{i=1,\ldots,n; j=1,\ldots,k} \in M(n \times k, \mathbb{C})$ with $\operatorname{rank} B = k$. Define $J := \{1, \ldots, n\}$. Here, the matroid (J, F) is the *vector matroid* (also called *linear matroid*) of the tuple $(v_i)_{i\in J}$ of row vectors $v_i := (b_i^j)_{j=1,\ldots,k}$ of the matrix B. More precisely, a subset $A \subset J$ is independent, if the tuple $(v_i)_{i\in A}$ is a linearly independent system of vectors.

Consider $\mathbb{C}^n \times \mathbb{C}^k$ with the coordinates $(z, t) = (z_1, \ldots, z_n, t_1, \ldots, t_k)$ and with the projection $\pi : \mathbb{C}^n \times \mathbb{C}^k \to \mathbb{C}^n$. Define the functions

$$g_i := \sum_{j=1}^{k} b_i^j \cdot t_j, \quad f_i := g_i + z_i \quad \text{for } i \in J \tag{4.8}$$

on $\mathbb{C}^n \times \mathbb{C}^k$.

We obtain on $\mathbb{C}^n \times \mathbb{C}^k$ the arrangement $\mathcal{C} = \{H_i\}_{i\in J}$, where H_i is the zero set of f_i. Let $U(\mathcal{C}) := \mathbb{C}^n \times \mathbb{C}^k - \bigcup_{i\in J} H_i$ be the complement. For every $x \in \mathbb{C}^n$, the arrangement $\mathcal{C}$ restricts to an arrangement $\mathcal{C}(x)$ on $\pi^{-1}(x) \cong \mathbb{C}^k$. For almost all $x \in \mathbb{C}^n$ the arrangement $\mathcal{C}(x)$ is *essential* (definition in **[7]**) with normal crossings. The subset $\Delta \subset \mathbb{C}^n$, where this does not hold, is a hypersurface and is called the *discriminant*, see **[7, Subsection 3.2]**. Define $M := \mathbb{C}^n - \Delta$.

A set $I = \{i_1, \dots, i_k\} \subset J$ is maximal independent, i.e., $(v_{i_1}, \dots, v_{i_k})$ is a basis of $M(1 \times k, \mathbb{C})$, if and only if for some (or equivalently for any) $x \in \mathbb{C}^n$ the hyperplanes $H_{i_1}(x), \dots, H_{i_k}(x)$ are transversal.

Let $a = (a_1, \dots, a_n) \in (\mathbb{C}^*)^n$ be a system of *weights* such that for any $x \in M$ the weighted arrangement $(\mathcal{C}(x), a)$ is *unbalanced*: See **[7]** for the definition of *unbalanced*, e.g., $a \in \mathbb{R}^n_{>0}$ is unbalanced, also a generic system of weights is unbalanced. The *master function* of the weighted arrangement $(\mathcal{C}, a)$ is

$$\Phi_a(z, t) := \sum_{i\in J} a_i \log f_i. \tag{4.9}$$

Several deep facts are related to this master function. We use some of them in the following. See **[7]** for references.

For $z \in M$ all critical points of Φ_a are isolated, and the sum μ of their Milnor numbers is independent of the unbalanced weight a and the parameter $z \in M$. The bundle

$$K := \bigcup_{z\in M} K_z \quad \text{with } K_z := \mathcal{O}(U(\mathcal{C}) \cap \pi^{-1}(z))/\left(\frac{\partial \Phi_a}{\partial t_j} \mid j = 1, \dots, k\right) \tag{4.10}$$

over M is a vector bundle of μ-dimensional algebras.

It comes equipped with the section ζ of unit elements $\zeta(z) \in K_z$, a Higgs field C, a *combinatorial connection* ∇^K and a pairing S. The Higgs field $C : \mathcal{O}(K) \to \Omega^1_M \otimes \mathcal{O}(K)$ is defined with the help of the period map

$$\Psi : TM \to K, \quad \partial_{z_i} \mapsto \left[\frac{\partial \Phi_a}{\partial z_i}\right] = \left[\frac{a_i}{f_i}\right] =: p_i \tag{4.11}$$

by

$$C_{\partial_{z_i}}(h) := p_i \cdot h \qquad \text{for } h \in K_z. \tag{4.12}$$

Because of

$$0 = \left[\frac{\partial \Phi_a}{\partial t_j}\right] = \sum_{i=1}^n b_i^j p_i, \tag{4.13}$$

the Higgs field vanishes on the vector fields $X_j := \sum_{i=1}^n b_i^j \partial_i, j \in \{1, \dots, k\}$,

$$C_{X_j} = 0 \qquad \text{for } j \in \{1, \dots, k\}. \tag{4.14}$$

In fact the whole geometry of the family of arrangements is invariant with respect to the flows of these vector fields.

The sections $\det(b_i^j)_{i\in I, j=1,\dots,k}\cdot C_I\zeta$ for all maximal independent sets $I=\{i_1,\dots,i_k\}\subset J$ generate the bundle K, and they satisfy only relations with constant coefficients in $\mathbb{Z}$. The combinatorial connection ∇^K is the unique flat connection such that the sections $C_I\zeta$ for $I\subset J$ maximal independent are ∇^K-flat. The sections $\det(b_i^j)_{i\in I, j=1,\dots,k}\cdot C_I\zeta$ for $I\subset J$ maximal independent generate a ∇^K-flat $\mathbb{Z}$-lattice structure on K.

The pairing S comes from the Grothendieck residue with respect to the volume form

$$\frac{dt_1\wedge\ldots\wedge dt_k}{\prod_{j=1}^k \frac{\partial\Phi_a}{\partial t_j}}. \tag{4.15}$$

It is symmetric, nondegenerate, ∇^K-flat, multiplication invariant and Higgs field invariant.

The existence of potentials of the first and second kind for families of arrangements was conjectured in **[6]**. If all the $k\times k$ minors of the matrix $B=(b_i^j)$ are nonzero, the potentials were constructed in **[6]**, cf. **[8]**. In **[5]**, this was generalized to all cases in Remark 4.2. The potentials are given by explicit formulas in terms of the linear functions defining the hyperplanes in $\mathbb{C}^n$ composing the discriminant.

REMARK 4.3.

(i) The situation in Remark 4.2 is in several aspects richer than a Frobenius-like structure of type (n,k,m). The bundle K is a bundle of algebras. The sections $C_I\zeta$ for maximal independent sets $I\subset J$ generate the bundle. The sections $\det(b_i^j)_{i\in I, j=1,\dots,k}\cdot C_I\zeta$ generate a flat $\mathbb{Z}$-lattice structure in K. The Higgs field vanishes on the vector fields $X_1,\dots,X_k$. The m-linear form S is a pairing $(m=2)$ and is nondegenerate. We will not discuss the $\mathbb{Z}$-lattice structure, but we will discuss some logical relations between the other enrichments and some implications of them.

(ii) Let $(M,K,\nabla^K,C,S,\zeta,V,(v_1,\dots,v_n))$ be a Frobenius-like structure of order (n,k,m). Suppose that it satisfies the *generation condition*

(GC) The sections $C_I\zeta$ for maximal independent sets $I\subset J$ generate the bundle K. (4.16)

Let μ be the rank of K. Then, for any $x\in M$, the endomorphisms C_X, $X\in T_xM$, generate a μ-dimensional commutative subalgebra $A_z\subset\mathrm{End}(K_x)$, and any endomorphism that commutes with them is contained in this subalgebra. This gives a rank μ bundle A of commutative algebras. And, the map

$$A\to K,\quad B\mapsto B\zeta, \tag{4.17}$$

is an isomorphism of vector bundles and induces a commutative and associative multiplication on K_x for any $x\in M$, with unit field $\zeta(x)$. Therefore, the special section ζ and the generation condition (GC), which exist and hold in Remark 4.2, give the multiplication on the bundle K there.

(iii) In the situation in (ii) with the condition (GC), the m-linear form is multiplication invariant because it is Higgs field invariant. The condition (GC) implies also that

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it is symmetric:

$$S(C_{I_1}\zeta, C_{I_2}\zeta, \dots, C_{I_m}\zeta) = S(C_{I_{\sigma(1)}}\zeta, C_{I_{\sigma(2)}}\zeta, \dots, C_{I_{\sigma(m)}}\zeta)$$

for any maximal independent sets $I_1, \dots, I_m$ and any permutation $\sigma \in S_m$.

(iv) The following special case gives rise to Frobenius manifolds without Euler fields. Consider a Frobenius-like structure $(M, K, \nabla^K, C, S, \zeta, (J, F))$ of order $(n, 1, 2)$ with nondegenerate pairing S, ∇^K-flat section ζ, the uniform matroid $(J, F) = (J, F^{(1,J)})$ and the condition that the map $C_\bullet\zeta : TM \to K$ is an isomorphism. Then, the sections $C_{\partial_i}\zeta$ generate the bundle K and are ∇^K-flat. Here, M becomes a Frobenius manifold (without Euler field) whose flat structure is the naive flat structure of $\mathbb{C}^n \supset M$. The potential L is the potential of the Frobenius manifold.

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