Proceedings of the Edinburgh Mathematical Society (1979), 22, 87–89 ©

T-IDEALS AND c-IDEALS

by AARON KLEIN
(Received 27th May 1977)

1.

Given a ring R we consider the category $\tilde{R}$ of R-rings (rings A with given ring-homomorphisms $R \to A$), and R-homomorphisms (ring-homomorphisms that form commutative triangles with the given maps from R). All rings are associative and have 1, all homomorphisms send 1 to 1.

We define a c-R-ring as an object A in $\tilde{R}$ with a family of maps $\{\rho_x \in \hom_{\tilde{R}}(A, A) \mid x \in A\}$. Equivalently, a c-R-ring is an R-ring A with a binary operation $a \cdot b (= a\rho_b)$ on A satisfying

$$(a + a') \cdot b = (a \cdot b) + (a' \cdot b); \quad (aa') \cdot b = (a \cdot b)(a' \cdot b); \hat{r} \cdot b = \hat{r} \tag{1}$$

for all $a, a', b \in A$, $r \in R$. Here $\hat{r}$ denotes the image of r under $R \to A$ and the ring multiplication on A is denoted by juxtaposition.

We call the third operation R-composition and we denote by $c\tilde{R}$ the category whose objects are the c-R-rings and whose maps are those maps of $\tilde{R}$ which preserve R-composition.

Let A be a c-R-ring and $K \subset A$. We call K a *c-ideal* in A if K is the kernel of a map in $c\tilde{R}$. The following is implied.

Theorem 1. *$K \subset A$ is a c-ideal in A if and only if:*
(i_1) *K is an ideal in the ring A,*
(i_2) *$k \cdot a \in K$ for all $k \in K$, $a \in A$, and*
(i_3) *$a \cdot (k + a') - a \cdot a' \in K$ for all $a, a' \in A$, $k \in K$.*

2.

We emphasise the importance of c-ideals by relating them to the well-known T-ideals (**1**, 2.2) in free algebras.

Let Λ be a commutative ring with 1 and $V = \Lambda\{x_s\}_{s\in S}$ the free associative Λ-algebra with 1 over a set S (the notation follows (**1**)). Let $V^{|S|}$ be the direct product of $|S|$ copies of V. We may write elements of $V^{|S|}$ as vectors of polynomials in the (non-commuting) indeterminates $\{x_s \mid s \in S\}$, namely

$$\mathbf{f} = (f_s(\mathbf{x}))_{s\in S} \text{ with } \mathbf{x} = (x_s)_{s\in S},$$

with component-wise addition and multiplication. We define a composition on $V^{|S|}$

$$\mathbf{f} \circ \mathbf{g} = (f_s(\mathbf{g}))_{s\in S} \text{ for } \mathbf{f} = (f_s(\mathbf{x})), \mathbf{g} = (g_s(\mathbf{x})). \tag{2}$$

It follows that $V^{|S|}$ with the (associative) composition (2) may be viewed as an object of the category $c\tilde{\Lambda}$.

Theorem 2. (*i*) *If K is a c-ideal in $V^{|S|}$ with the composition* (2), *then K is a subdirect sum of $|S|$ copies of a T-ideal J of the free algebra V and it contains the direct sum $\Sigma_{|S|} J$ of $|S|$ copies of J; thus K is a dense sub-direct sum.*

(*ii*) *For any T-ideal J of the free algebra V, the direct product $J^{|S|}$ of $|S|$ copies of J is a c-ideal in $V^{|S|}$. If the index-set S is infinite and $J \neq 0$ then the direct sum $\Sigma_{|S|} J$ is not a c-ideal in $V^{|S|}$.*

Proof. (i) Let K be a c-ideal in $V^{|S|}$ and for any $r \in S$ denote by J_r the r-th projection of K into V. We show that J_r is a T-ideal in the free algebra V. Take $h \in J_r$ and any $\mathbf{h} \in K$ with $h_r = h$. For arbitrary $\mathbf{f} \in V^{|S|}$ we have $\mathbf{h} \circ \mathbf{f} \in K$ by (i_2) of Theorem 1, hence $(\mathbf{h} \circ \mathbf{f})_r \in J_r$. But $(\mathbf{h} \circ \mathbf{f})_r = h_r(\mathbf{f}) = h(\mathbf{f})$ and this proves that J_r is indeed a T-ideal. We use the property (i_3), Theorem 1, of K to show that all J_r are equal. Let r, t be any two indices in S and h any element of J_r. Take again $\mathbf{h} \in K$ with $h_r = h$ and $\mathbf{f} = (\delta_{st} x_r)_{s \in S}$. Then the vector $\mathbf{f} \circ (\mathbf{h} + O) - \mathbf{f} \circ O$ is in K so its t-component, namely $f_t(\mathbf{h}) = h$, is in J_t. This proves $J_r \subset J_t$ and, since r and t were arbitrary, it follows that all the projections of K are equal, say to J. Now, the set of vectors $\{(\delta_{st} h)_{s \in S} \mid t \in S, h \in J\}$ generates the direct sum $\Sigma_{|S|} J$. We show that all these vectors are in K, hence $\Sigma_{|S|} J \subset K$. Fix $t \in S$ and put $\mathbf{f} = (\delta_{st} x_s)_{s \in S}$. With any $h \in J$ take $\mathbf{h} \in K$ with $h_t = h$. Then $\mathbf{f} \circ (\mathbf{h} + O) - \mathbf{f} \circ O$ is in K and this vector is precisely $(\delta_{st} h)_{s \in S}$.

(ii) Assume J is a T-ideal in the free algebra V and let K be a direct product of $|S|$ copies of J. We show that K is a c-ideal in $V^{|S|}$. The condition (i_1) of Theorem 1 is evident and (i_2) follows since J is a T-ideal. To establish that K meets (i_3) we have to show that $f(\mathbf{h} + \mathbf{g}) - f(\mathbf{g}) \in J$ for all $\mathbf{h} \in K$ and for any $f \in V$, $\mathbf{g} \in V^{|S|}$. It suffices to show it for monomials $f = \lambda x_{s_1} \ldots x_{s_m}$ in V. Thus we prove that $\lambda (h_{s_1} + g_{s_1}) \ldots (h_{s_m} + g_{s_m}) - \lambda g_{s_1} \ldots g_{s_m}$ is in J. Upon expanding, the term $g_{s_1} \ldots g_{s_m}$ cancels out and we arrive to a sum of monomials, each of them involving at least one component of $\mathbf{h}$ as a factor. Since all the components of $\mathbf{h}$ are in J and J is an ideal, the result follows.

To establish the last assertion in (ii), let $L = \Sigma_{|S|} J$, J a non-zero T-ideal in V. Take any $\mathbf{h} \in L$ with a certain non-zero component h_{s_0}. Now, consider $\mathbf{f} \in V^{|S|}$ with all components equal to x_{s_0}. If L satisfies (i_3) of Theorem 1, then $\mathbf{f} \circ (\mathbf{h} + O) - \mathbf{f} \circ \mathbf{h}$ has to be in L. Yet all the components of $\mathbf{f} \circ (\mathbf{h} + O) - \mathbf{f} \circ \mathbf{h}$ are equal to the non-zero polynomial h_{s_0}, hence this vector cannot be in L if S is infinite. So, in this case L cannot be a c-ideal.

Remark. The assumption that V has 1 is not essential. It puts V in $\tilde{\Lambda}$, but the theorem is true even without 1 and the proof remains unaltered.

3.

For infinite S, the question whether there are c-ideals in $V^{|S|}$ which are not direct products of T-ideals, namely $J^{|S|}$, remains open. In this form, the problem is related to a well-known open problem (2) concerning varieties of algebras, as follows. We assume that $|S| \geqq |\Lambda|$, so $|\Lambda\{x_s\}_{s \in S}| = |S|$.

Theorem 3. *If there is in $V = \Lambda\{x_s\}_{s\in S}$ a T-ideal J which is not finitely generated as a T-ideal, then there is a c-ideal in $V^{|S|}$ which is not a direct product of T-ideals.*

Proof. We may assume that J is generated by a set of polynomials $\{p_s \mid s \in S\}$. For finite subsets $H \subset S$ denote by J_H the T-ideal in V generated by $\{p_s \mid s \in H\}$. Let K_H be a direct product of $|S|$ copies of J_H. Then K_H is a c-ideal by (ii) of Theorem 2, and so we obtain a directed set of c-ideals $\{K_H \mid H \text{ finite}, H \subset S\}$ in $V^{|S|}$. It follows that $K = \bigcup K_H$ is a c-ideal in $V^{|S|}$. However, K is a subdirect sum of $|S|$ copies of J, but not the whole $J^{|S|}$ since $\mathbf{p} = (p_s)_{s\in S} \notin K$.

REFERENCES

(1) C. PROCESI, *Rings with polynomial identities* (M. Dekker, N.Y., 1973).

(2) W. SPECHT, Gesetze in Ringen 1, *Mat. Zeit.* **52** (1950), 557–589.

DEPARTMENT OF MATHEMATICS
BAR-ILAN UNIVERSITY
RAMAT GAN, ISRAEL