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Compatibility of theta lifts and tempered condition

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Abstract. In this note, assuming the nonvanishing result of explicit theta correspondence for the symplectic–orthogonal dual pair over quaternion algebra $\mathbb{H}$, we show that, for metapletic–orthogonal dual pair over $\mathbb{R}$ and the symplectic–orthogonal dual pair over quaternion algebra $\mathbb{H}$, the theta correspondence is compatible with tempered condition by directly estimating the matrix coefficients, without using the classification theorem.

1 Introduction

Throughout this article, let $\psi(x) = e^{2i\pi x}$ be the nontrivial additive character of $\mathbb{R}$. Let dx be Lebesgue measure on $\mathbb{R}$ which is self-dual for Fourier transformation with respect to ψ. Unless we explicitly mention the contrary, by a representation of Lie group, we always mean a unitary Casselman–Wallach representation of finite length (admissible smooth Fréchet representation of moderate growth and $\mathscr{Z}$-finite; cf. [18, Chapter XI]), where $\mathscr{Z}$ is the center of the universal enveloping algebra of its complexified Lie algebra. An inner product on a representation is denoted by $(-,-)$. Let π be a representation. We denote by $\pi^\vee$ the space of continuous linear functionals on π, and it is given the strong topology (uniform convergence on bounded subsets). The smooth dual of π, i.e., the subspace of smooth vectors in $\pi^\vee$, is identified with $\overline{\pi}$.

Let $\mathbf{H}$ be a real reductive group G or its double cover $\widehat{G}$ (cf. [17, Section 2]). Among the irreducible (genuine) representations of $\mathbf{H}$, there is an important class of representations, whose matrix coefficients are controlled by the Harish-Chandra Ξ function. Such representations are called irreducible tempered (genuine) representation.

Let (G, G') be a reductive dual pair in $\mathrm{Sp}_{2m}(\mathbb{R})$. Let $\widehat{G}$ and $\widehat{G}'$ be the inverse images of G and G' in the metaplectic double covering group $\widehat{\mathrm{Sp}}_{2m}(\mathbb{R})$ by the covering map. For irreducible admissible representations π and π' of $\widehat{G}$ and $\widehat{G}'$, respectively, we say π and π' correspond if $\pi\widehat{\otimes}\pi'$ is a quotient of the Weil representation ω of $\widehat{\mathrm{Sp}}_{2m}(\mathbb{R})$, restricted to $\widehat{G} \times \widehat{G}'$. Note that the Weil representation is not a representation by our convention as it is not of finite length.

Let (W, V) be the underlying quadratic space over the field $(K, \sharp)$ of equal rank n. In this paper, we consider the following two cases:

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(A) If $K = \mathbb{R}$, then W is a $2n$-dimensional real symplectic vector space and V is a $2n+1$ real orthogonal spaces. Adams and Barbasch [1] show that the dual pair $(\mathrm{Sp}(W), \mathrm{SO}(p,q))$ with $p+q = 2n+1$ gives rise to a bijection between the genuine representations of metaplectic group and the representations of odd special orthogonal group of the same rank with certain discriminant. Note that there are two extensions of an irreducible representation π' of $\mathrm{SO}(V')$ to $\mathrm{O}(V')$ and there are precisely only one of such extensions in the domain of metaplectic theta correspondence.

The explicit theta correspondence is first introduced by Li [13], and in this case, the nonvanishing result of explicit theta correspondence is proved by Gan, Qiu, and Takeda in [5, Proposition 11.5] and by Ichino in [9, Proposition 7.1].

(B) If K is the quaternion algebra $\mathbb{H}$, then W is a skew-Hermitian space over $\mathbb{H}$ of rank n and V is a Hermitian over $\mathbb{H}$ of rank n or $n+1$. As in [12, Theorem 5.1], for any irreducible admissible representation of $\mathrm{Sp}(p,q)$ with $p+q=n$, there are nonzero theta lifts to $\mathrm{O}(V)$ both for n and $n+1$.

Note that the nonvanishing result for the explicit theta correspondence in case (B) is not yet proved. Hence, we assume the nonvanishing result in this case.

Moreover, in [1, 12], the authors explicitly determined the K-types of all the representations on both sides of the theta correspondence. Together with the classification of the irreducible tempered representations, one can deduce the following result.

Theorem 1.1 *With the same notation as above. The theta correspondence sends the tempered representations of* $\mathrm{Sp}(W)$ *to the tempered representations of* $\mathrm{O}(V)$.

The main purpose of this article is to prove Theorem 1.1 by directly estimating the matrix coefficients, without using the classification theorem. The approach of estimating matrix coefficients is widely used. In fact, Gan and Ichino use the estimations of matrix coefficients to prove the convergence of the inner product of two matrix coefficients of representation obtained by the theta correspondence for the dual pair $(\mathrm{O}_4, \mathrm{Sp}_4)$ over p-adic field (cf. [3, Section 9, Lemma 9.1]). In [4, Lemma D], they extend this to other dual pairs over p-adic field: $(\mathrm{U}(n), \mathrm{U}(m))$, $(\mathrm{Sp}_{2n}, \mathrm{O}_{2m+1})$, $(\mathrm{Sp}_{2n}, \mathrm{O}_{2m})$, and prove that in their setting, the discrete series condition is compatible with the theta correspondence. In the unitary case over $\mathbb{R}$, a corresponding estimation is given by Xue (cf. [19, Lemma 3.2]). In this note, we adapt their estimations to our case to prove that the temperedness of representations is compatible with theta correspondence. Note that the dual pairs in this note are not considered by them.

Remark 1.2 (1) In [7], He proved that the theta correspondence is compatible with unitary condition in the semistable range. Our setting, the equal rank case, is contained in the semistable range. But since the category of tempered representations is a subcategory of the category of unitary representations, we need to refine the estimations of matrix coefficients given by He [7, Theorems 6.2.1, 6.3.1, and 6.4.1-3] to achieve our goal. More precisely, to show that the theta correspondence is compatible with the tempered condition, we will need the $L^{2+\varepsilon}$-convergence of the matrix coefficients to prove our results.

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(2) In [4, Lemma D], Gan and Ichino only provide an estimation of the matrix coefficient from a bigger group to a slightly smaller group over p-adic field. In this note, we only provide an estimation from small group to big group over field of real numbers. If we do not assume the symmetry of the theta correspondence, our estimation can be viewed as a complement for the estimation of Gan and Ichino.

2 Tempered (genuine) representations

Let G be a real reductive group, and let $\widehat{G}$ be the double covering group of G. Let A_G be the maximal $\mathbb{R}$-split torus of G of rank r (i.e., $A_G(\mathbb{R}) \cong (\mathbb{R}^\times)^r$), and let M be the centralizer of A_G in G, which is exactly the Levi factor of a minimal parabolic subgroup P of G. We will write an element $a \in A_G(\mathbb{R})$ by $(a_1, \dots, a_r)$. We denote by $\Delta = R(A_G, P)$ the set of roots of A_G in the unipotent radical U of P. Set

$$A_G^+ = \{a \in A_G(\mathbb{R}) : |\alpha(a)| \leq 1, \forall \alpha \in \Delta\} = \{(a_1, \dots, a_r) : 0 < |a_1| \leq |a_2| \leq \dots \leq |a_r| \leq 1\}. \tag{2.1}$$

We denote by $\delta_{P,G}$ the modulus character of P. We fix a special maximal compact subgroup K of $G(\mathbb{R})$, and we have a Cartan decomposition of $G(\mathbb{R})$:

$$G(\mathbb{R}) = KA_G^+K.$$

For any integrable function f on $G(\mathbb{R})$, the following formula holds (cf. [10, Section 4]):

$$\int_{G(\mathbb{R})} f(g)dg = \int_{A_G^+} \nu(a) \int_{K\times K} f(k_1ak_2)dk_1dk_2da, \tag{2.2}$$

where ν is a positive function on A_G^+ such that $\nu(a) \leq C \cdot \delta_{P,G}^{-1}(a)$ for some constant C.

Harish-Chandra defined a special spherical function Ξ^G on $G(\mathbb{R})$, which can be used to control the growth of C^∞-functions on $G(\mathbb{R})$ with values in $\mathbb{C}$. We recall briefly its definition and some useful results.

We denote by $C^\infty(G(\mathbb{R}))$ the space of all complex-valued C^∞-functions on $G(\mathbb{R})$. Consider the normalized smooth-induced representation

$$i_P^G(1)^\infty := \{f \in C^\infty(G(\mathbb{R})) : f(pg) = \delta_{P,G}(p)^{1/2} f(g), \forall p \in P(\mathbb{R}), g \in G(\mathbb{R})\}$$

equipped with the scalar product

$$(f, f') = \int_K f(k)\overline{f'(k)}dk, \forall f, f' \in i_P^G(1)^\infty.$$

Let $e_K \in i_P^G(1)^\infty$ be the unique function such that $e_K(k) = 1$ for all $k \in K$. Then the Harish-Chandra spherical function Ξ^G is defined by

$$\Xi^G(g) = (i_P^G(1)(g)e_K, e_K), \forall g \in G(\mathbb{R}).$$

Note that if f and g are positive functions on a set X, we will say f is essential bounded by g, if there exists a $c > 0$ such that $f(x) \leq cg(x)$ for all $x \in X$. We will denote it by $f \ll g$. We say f and g are equivalent if f is essentially bounded by g and g is essentially bounded by *f*. The function Ξ^G is a bi-K-invariant function, and it is independent of the choice of the maximal compact subgroup K up to equivalence.

Fixing an embedding $\iota : G(\mathbb{R}) \to \mathrm{GL}_m(\mathbb{R})$, we define the height function

$$\sigma(g) = 1 + \sup\{\log|a_{i,j}|, \log|b_{i,j}|\},$$

where $(a_{i,j})$ is the matrix $\iota(g)$ and $(b_{i,j})$ is the corresponding matrix of $\iota(g^{-1})$. In particular, if $a = (a_1, \dots, a_r) \in A_G^+$, we have

$$\sigma(a) = 1 - \log|a_1| \geq 1. \tag{2.3}$$

We have the following well-known estimation of Ξ^G due to Harish-Chandra.

Lemma 2.1 [16, Theorem 30] *There exist constants* $A, B > 0$ *such that for any* $a \in A_G^+$, *we have*

$$A^{-1}\delta_{P,G}^{\frac{1}{2}}(a) \leq \Xi^G(a) \leq A\delta_{P,G}^{\frac{1}{2}}(a)\sigma(a)^B.$$

The double covering group $\widehat{G}$ of G is not an algebraic group, but behaves in many way like an algebraic group. In particular, we have the Cartan decomposition for $\widehat{G}$, i.e., $\widehat{G} = KA_{\widehat{G}}^+K$, where K is the inverse image of a special maximal compact subgroup of G and $A_{\widehat{G}}^+$ is the inverse image of A_G^+ in $\widehat{G}$. We define the corresponding Harish-Chandra spherical function by $\Xi^{\widehat{G}} = \Xi^G \circ p$, where p is the covering map.

Using Harish-Chandra's Ξ-function, we have the following definition of tempered representation for real reductive groups and metaplectic groups. Let $\mathbf{H}$ be the real reductive group G or the double covering group $\widehat{G}$ of G.

Definition 2.1 We say that a unitary representation $(\pi, \mathscr{H}_\pi)$ of $\mathbf{H}$ is tempered if, for any $e, e' \in \pi$, we have an inequality

$$|(\pi(g)e, e')| \leq A \cdot \Xi^{\mathbf{H}}(g)\sigma(g)^B, \forall g \in \mathbf{H}(\mathbb{R})$$

for some constants A, B.

Thanks to the work of Cowling, Haagerup, and Howe [2, 14], a representation of $\mathbf{H}$ is tempered if and only if its matrix coefficients are almost square-integrable functions (i.e., it belongs to $L^{2+\varepsilon}(\mathbf{H}(\mathbb{R}))$ for all $\varepsilon \in \mathbb{R}_{>0}$).

Let π be a tempered representation of $\mathbf{H}$. For any $v, v' \in \pi$ and $g \in \mathbf{H}(\mathbb{R})$, by definition of tempered representation, there exist constants A_1, B_1, such that

$$|(\pi(g)v, v')| \leq A_1 \cdot \Xi^{\mathbf{H}}(g)\sigma(g)^{B_1}.$$

Moreover, a more precise estimation is given by Sun [14]: there is a continuous seminorm ν_π on π such that

$$|(\pi(g)v, v')| \leq \Xi^{\mathbf{H}}(g)\nu_\pi(v)\nu_\pi(v'), \forall v, v' \in \pi. \tag{2.4}$$

We deduce, from Lemma 2.1 and the fact that the Harish-Chandra function $\Xi^{\mathbf{H}}$ is bi-K-invariant, that for any $g = k_1ak_2 \in KA_{\mathbf{H}}^+K$, we have

$$\Xi^{\mathbf{H}}(g) = \Xi^{\mathbf{H}}(k_1ak_2) = \Xi^{\mathbf{H}}(a).$$

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If $\mathbf{H} = G$, there exist two positive constants A_2 and B_2 such that

$$\Xi^{\mathbf{H}}(a) \leq A_2 \delta_{P,G}^{1/2}(a)\sigma(a)^{B_2}.$$

If $\mathbf{H} = \widehat{G}$, there exist two positive constants A_3 and B_3 such that

$$\Xi^{\mathbf{H}}(a) = \Xi^{G}(p(a)) \leq A_3 \delta_{P,G}^{1/2}(p(a))\sigma(p(a))^{B_3}. \tag{2.5}$$

Thus, for any $g = k_1 a k_2 \in K A_{\mathbf{H}}^{+} K$, there exist two positive constants A and B such that

$$|(\pi(g)v, v')| \leq A \delta_{P,G}^{1/2}(p(a))\sigma(p(a))^{B}. \tag{2.6}$$

3 Theta correspondence

In [13, Theorem 6.1], Li shows that if the dual pair (G_1, G_2) is in the stable range, then there is an explicit realization of the theta correspondence using the mixed model of Weil representation [13, Section 4]. The explicit realization of theta correspondence for unitary case is studied in [11], and for more general classical groups, it is studied in [5] and used by Xue in [19]. The explicit theta correspondence for our dual pairs has been described in [1, 12]. In this paragraph, we recall the explicit theta correspondence using the mixed model of the Weil representation and study the matrix coefficients of the explicit theta lift.

The Weil representations are depended on the choice of the additive character ψ. Since we have fixed it, we may omit it from the subscript of the Weil representation.

3.1 Mixed model of Weil representations

The mixed models of Weil representations for our dual pairs in the introduction are defined as follows:

(A) Let $(W, \langle,\rangle_W)$ be a $2n$-dimensional real symplectic vector space, and let $(V, \langle,\rangle_V)$ be a real quadratic space of dimension $2n+1$ with discriminant

$$\mathrm{disc}(V) = (-1)^n \det(V) > 0.$$

The space $(W \otimes V, \langle -,-\rangle_W \otimes \langle -,-\rangle_V)$ is a real symplectic space. We have a natural homomorphism

$$\widehat{\mathrm{Sp}}(W) \times \mathrm{O}(V) \to \widehat{\mathrm{Sp}}(W \otimes V). \tag{3.1}$$

Let ω_ψ be the Weil representation of $\widehat{\mathrm{Sp}}(W \otimes V)$ associated with $W \otimes V$. We denote by $\omega_{W,V}$ the representation of $\widehat{\mathrm{Sp}}(W) \times \mathrm{O}(V)$ by pulling back the Weil representation ω_ψ by the homomorphism (3.1).

Let r_V be the Witt index of V. Let V_0 be the anisotropic kernel of V, which is of dimension $2n+1-2r_V$. Let $P_V = M_V N_V$ be a minimal parabolic subgroup of $\mathrm{O}(V)$ stabilizing a full flag of $V^{\perp}$. Let $A_V \cong (\mathbb{R}^{\times})^{r_V}$ be the maximal split torus in M_V and define

$$A_V^{+} = \{(b_1, \dots, b_{r_V}) | 0 < b_1 \leq \cdots \leq b_{r_V} \leq 1\}.$$

We have two dual pairs $(\mathrm{Sp}(W), \mathrm{O}(V))$ and $(\mathrm{Sp}(W), \mathrm{O}(V_0))$. Let $\mathscr{S}_0$ be the Schrödinger model of the Weil representation ω_{W,V_0} of the dual pair $(\mathrm{Sp}(W), \mathrm{O}(V_0))$. Let $\mathscr{S} = \mathscr{S}(W^{r_V})\widehat{\otimes}\mathscr{S}_0$. Then the Weil representation $\omega_{W,V}$ for the dual pair $(\mathrm{Sp}(W), \mathrm{O}(V))$ can be realized on $\mathscr{S}$, called the mixed model of $\omega_{W,V}$. We view elements in $\mathscr{S}$ as Schwartz functions on W^{r_V} valued in $\mathscr{S}_0$.

Since $\mathrm{Sp}(W)$ is split, the maximal split torus $A_W \cong \mathbb{R}^n$ and we define

$$A_W^+ = \{(a_1, \ldots, a_n) | 0 < a_1 \leq \cdots \leq a_n \leq 1\}.$$

For any $a \in A_W^+$, $b \in A_V^+$, and $\phi \in \mathscr{S}$, we have

$$\omega_{W,V}(a,b)\phi(z,w) = \det(a)^{\frac{2n+1}{2}}\phi(b^{-1}za, wb). \tag{3.2}$$

(B) Let $(V, (,)^\sharp)$ be an n-dimension Hermitian space over $\mathbb{H}$, and let $(W, \langle,\rangle^\sharp)$ be an m-dimensional skew-Hermitian space W over $(\mathbb{H}, \sharp)$ with $m = n$ or $n-1$. The space $(W \otimes_{\mathbb{H}} V, \mathrm{Tr}_{\mathbb{H}/\mathbb{R}}(\langle,\rangle^\sharp \otimes (,)^\sharp))$ is a real symplectic space of dimension $4mn$. This defines an embedding of dual pair $(\mathrm{Sp}(W), \mathrm{O}(V))$:

$$\mathrm{Sp}(W) \times \mathrm{O}(V) \to \mathrm{Sp}_{4nm}(\mathbb{R}).$$

Let $\omega_{W,V}$ be the oscillator representation for the dual pair $(\mathrm{Sp}(W), \mathrm{O}(V))$, which is a representation of $\widehat{\mathrm{Sp}}_{4nm}(\mathbb{R})$. Let r_W and r_V be the Witt index of W and V, respectively. Let W_0 and V_0 be the corresponding anisotropic kernels of W and V. Then we have

$$\dim_{\mathbb{H}}(W_0) = m - 2r_W, \text{ and } \dim_{\mathbb{H}}(V_0) = n - r_V.$$

Let $P_W = M_W N_W$ be a minimal parabolic subgroup of $\mathrm{Sp}(W)$ stabilizing a full flag of $W_0^\perp$. Then $M_W \cong \mathrm{GL}_1(\mathbb{R})^{r_W} \times \mathrm{Sp}(W_0)$. Let $A_W \cong (\mathbb{R}^\times)^{r_W}$ be the maximal split torus in M_W, and let

$$A_W^+ = \{(a_1, \ldots, a_{r_W}) | 0 < a_1 \leq \cdots \leq a_{r_W} \leq 1\}.$$

We have three dual pairs

$$(\mathrm{Sp}(W), \mathrm{O}(V)), (\mathrm{Sp}(W_0), \mathrm{O}(V)), \text{ and } (\mathrm{Sp}(W_0), \mathrm{O}(V_0)).$$

Let $\mathscr{S}_{00}$ be the Schrödinger model of the Weil representation ω_{W_0,V_0} of the dual pair $(\mathrm{Sp}(W_0), \mathrm{O}(V_0))$. Let $\mathscr{S}_0 = \mathscr{S}(W_0^{r_V})\widehat{\otimes}\mathscr{S}_{00}$. Then the Weil representation $\omega_{W_0,V}$ for the dual pair $(\mathrm{Sp}(W_0), \mathrm{O}(V))$ can be realized on $\mathscr{S}_0$. Finally, the Weil representation $\omega_{W,V}$ of the dual pair $(\mathrm{Sp}(W), O(V))$ can be realized on $\mathscr{S} = \mathscr{S}(V^{r_W})\widehat{\otimes}\mathscr{S}_0$, called the mixed model of the Weil representation $\omega_{W,V}$. We view elements in $\mathscr{S}$ as Schwartz functions on $V^{r_W} \times W_0^{r_V}$ valued in $\mathscr{S}_{00}$. Define

$$A_V^+ = \{(b_1, \ldots, b_{r_V}) | 0 < b_1 \leq \cdots \leq b_{r_V} \leq 1\}.$$

For any $a \in A_W^+$, $b \in A_V^+$, and $\phi \in \mathscr{S}$, we have

$$\omega_{W,V}(a,b)\phi(z,w) = \det(a)^n \det(b)^{m-2r_W}\phi(b^{-1}za, wb). \tag{3.3}$$

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3.2 Matrix coefficients of Weil representations

In this paragraph, we give the estimation of matrix coefficients of Weil representations $\omega_{W,V}$ using the mixed model described in the previous paragraph.

Lemma 3.1 *For $\phi, \phi' \in \mathscr{S}(\mathbb{R})$ and $t \in \mathbb{R}^\times$, there exists some constant C such that*

$$\left|\int_{\mathbb{R}} \phi(tx)\phi'(x)dx\right| \leq C \cdot \Upsilon(t), \tag{3.4}$$

where C is a constant and $\Upsilon(t) = \begin{cases} 1, & \text{if } |t| \leq 1, \\ |t|^{-1}, & \text{if } |t| > 1. \end{cases}$

Proof By changing of variable, one can reduce to show that for $\phi, \phi' \in \mathscr{S}(\mathbb{R})$, there exists a constant C such that

$$\left|\int_{\mathbb{R}} \phi(tx)\phi'(x)dx\right| \leq C,$$

for all $0 < |t| \leq 1$. It follows from a direct estimation of the integration for three regions: $|x| \leq 1$, $1 < |x| \leq 1/t$, and $|x| \geq 1/t$. ∎

Using this lemma, we get the following important estimation.

Proposition 3.2 *For our dual pairs (G_1, G_2) over $(K, \natural)$ of equal rank n with underlying spaces (W, V) as in the introduction, there exists a constant C, such that:*

(A) *If $K = \mathbb{R}$, then for $(\hat{g}, h) \in \widehat{\mathrm{Sp}}(W) \times \mathrm{O}(V)$ and $\phi, \phi' \in \omega_{W,V}$, we have*

$$|(\omega_{W,V}(\hat{g}, h)\phi, \phi')| \leq C \cdot \prod_{i=1}^{n} |a_i|^{\frac{2n+1}{2}} \prod_{k=1}^{n}\prod_{j=1}^{r_V} \Upsilon(a_k b_j^{-1}). \tag{3.5}$$

(B) *If $K = \mathbb{H}$, then for $(g, h) \in \mathrm{Sp}(W) \times O(V)$ and $\phi, \phi' \in \omega_{W,V}$, we have*

$$|(\omega_{W,V}(g, h)\phi, \phi')| \leq C \cdot \prod_{i=1}^{r_W} |a_i|^n \prod_{j=1}^{r_V} |b_j|^{m-2r_W} \prod_{i=1}^{r_W}\prod_{j=1}^{r_V} \Upsilon(a_i b_j^{-1}). \tag{3.6}$$

Proof The two cases can be proved by the same argument, and we only show the case (A). For any $(\widehat{g}, h) \in \widehat{\mathrm{Sp}}(W) \times \mathrm{O}(V)$, by Cartan decomposition, we can write $(\widehat{g}, h) = (k_1 a k_2, k_1' b k_2')$, with k_i in the inverse image K_W of a special maximal compact subgroup of $\mathrm{Sp}(W)$, k_i' in a special maximal compact subgroup K_V of $\mathrm{O}(V)$, $a \in A_W^+$, and $b \in A_V^+$. Thus, for $\phi, \phi' \in \omega_{W,V}$, there exists some constant C_1 such that

$$|(\omega_{W,V}(\widehat{g}, h)\phi, \phi')| \leq C_1 \cdot \det(a)^{\frac{2n+1}{2}} (\phi(b^{-1} \cdot a), \phi').$$

Together with the previous lemma, we get the desired estimation. ∎

In the unitary case, the above estimation is refined by Xue [19]. Similarly, we can provide a more precise estimation for our dual pairs.

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Proposition 3.3 *Let (G_1, G_2) be one of our dual pairs over $(K, \natural)$ as in the introduction with underlying spaces (W, V). Then there exists a continuous seminorm $\nu_{\mathscr{S}}$ on $\omega_{W,V}$ such that:*

(A) *If $K = \mathbb{R}$, then we have*

$$|(\omega_{W,V}(\widehat{g}, h)\phi, \phi')| \leq \prod_{i=1}^{n} |a_i|^{\frac{2n+1}{2}} \prod_{k=1}^{n} \prod_{j=1}^{r_V} \Upsilon(a_k b_j^{-1}) \nu_{\mathscr{S}}(\phi) \nu_{\mathscr{S}}(\phi'). \tag{3.7}$$

(B) *If $K = \mathbb{H}$, then we have*

$$|(\omega_{W,V}(g, h)\phi, \phi')| \leq \prod_{i=1}^{r_W} |a_i|^n \prod_{j=1}^{r_V} |b_j|^{m-2r_W} \prod_{i=1}^{r_W} \prod_{j=1}^{r_V} \Upsilon(a_i b_j^{-1}) \nu_{\mathscr{S}}(\phi) \nu_{\mathscr{S}}(\phi'). \tag{3.8}$$

Proof Note that, in [19, Lemma 3.1], Xue proved a general result: let m be an integer and take $\phi, \phi' \in \mathscr{S}(\mathbb{R}^m)\widehat{\otimes}\mathscr{S}_{00}$, viewed as Schwartz functions valued in $\mathscr{S}_{00}$, and $\lambda = (\lambda_1, \cdots, \lambda_m) \in \mathbb{R}^m$. Then there is a seminorm ν on $\mathscr{S}(\mathbb{R}^m)\widehat{\otimes}\mathscr{S}_{00}$ such that

$$\left| \int_{\mathbb{R}^m} \langle \phi(\lambda_1 x_1, \ldots, \lambda_m x_m), \phi'(x_1, \ldots, x_m) \rangle dx_1 \cdots dx_m \right| \leq \prod_{i=1}^{m} \Upsilon(\lambda_i^{-1}) \nu(\phi) \nu(\phi'). \tag{3.9}$$

Together with the formulae (3.2) and (3.3), we can deduce our result. ■

3.3 Weil representation and theta lifts

Let ω be the Weil representation of one of our dual pairs (G_1, G_2) over $(K, \natural)$. If $K = \mathbb{R}$ (resp. $\mathbb{H}$), let π be an irreducible genuine representation of the double cover $\widehat{G_1}$ of G_1 (resp. an irreducible representation of G_1). Then the tensor product $\omega\widehat{\otimes}\overline{\pi}$ is a $\widehat{G_1} \times \widehat{G_2}$-module, where $\widehat{G_2}$ is the double covering group corresponding to G_2 and acting by ω and $\widehat{G_1}$ acts by $\omega\widehat{\otimes}\overline{\pi}$. The maximal isotropic quotient of ω with respect to π has the form $\pi \boxtimes \Theta(\pi)$ for some smooth representation $\Theta(\pi)$ of G_2, which is either 0 or of finite length. Let $\theta(\pi)$ be the maximal semisimple quotient of $\Theta(\pi)$. The topology on $\theta(\pi)$ is induced from the projective topology of the projective tensor product $\omega\widehat{\otimes}\overline{\pi}$. It is known by Howe [8] that $\theta(\pi)$ is either zero or irreducible.

If $K = \mathbb{H}$, we regard $\mathrm{Sp}(W)$ as a subgroup of $\mathrm{Sp}_{2m}(\mathbb{C})$. We may denote

$$\dim W = \begin{cases} 2n, & \text{if } K = \mathbb{R} \\ 2m, & \text{if } K = \mathbb{H} \end{cases} \text{and } \dim V = \begin{cases} 2n+1, & \text{if } K = \mathbb{R}, \\ 2n, & \text{if } K = \mathbb{H}. \end{cases} \tag{3.10}$$

Lemma 3.4 *Let (G_1, G_2) be one of our dual pairs over $(K, \natural)$ of equal rank n with underlying spaces (W, V). Let π be an irreducible genuine tempered representation of $\widehat{\mathrm{Sp}}(W)$ (if $K = \mathbb{R}$) or an irreducible tempered representation of $\mathrm{Sp}(W)$ (if $K = \mathbb{H}$). For any $v, v' \in \pi$ and $\phi, \phi' \in \omega_{W,V}$, there exist continuous semi-norms ν_π on π and $\nu_{\mathscr{S}}$ on $\omega_{W,V}$ such that*

$$\left| \int_{\mathrm{Sp}(W)} (\omega_{W,V}(g,1)\phi, \phi') \overline{(\pi(g)v, v')} dg \right| \leq \nu_\pi(v) \nu_\pi(v') \nu_{\mathscr{S}}(\phi) \nu_{\mathscr{S}}(\phi').$$

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Proof By estimations (3.6) and (2.5), there exist a continuous seminorm $\tilde{\nu}_\pi$ and positive constants A, B such that for any $a \in A_W^+$ and $v, v' \in \pi$,

$$|(\pi(a)v, v')| \le A_1 \delta_{P,\mathrm{Sp}(W)}^{\frac{1}{2}}(a)\sigma(a)^{B_1}\tilde{\nu}_\pi(v)\tilde{\nu}_\pi(v').$$

By (3.7) and (3.8), there exists a continuous seminorm $\tilde{\nu}_{\mathscr{S}}$ such that for $a \in A_W^+$ and $\phi, \phi' \in \omega_{W,V}$,

$$|(\omega_{W,V}(a,1)\phi, \phi')| \le \prod_{i=1}^{r_W} |a_i|^{\frac{\dim V}{2}} \tilde{\nu}_{\mathscr{S}}(\phi)\tilde{\nu}_{\mathscr{S}}(\phi').$$

Finally, by the formula (2.2), the integral is bounded by

(3.11)
$$\int_{A_W^+} \delta_{P,\mathrm{Sp}(W)}(a)^{\frac{1}{2}}(1+\log|a_i|)^B \prod_{j=1}^{r_W} |a_j|^{\frac{\dim V}{2}} da,$$
$$\int_{K_1\times K_1} \tilde{\nu}_\pi(\pi(k_1)v)\tilde{\nu}_\pi(\pi(k_1'^{-1})v')\tilde{\nu}_{\mathscr{S}}(\omega_{W,V}(k_1,1)\phi)\tilde{\nu}_{\mathscr{S}}(\omega_{W,V}(k_1'^{-1},1)\phi')dk_1dk_1',$$

where B is a positive constant and $\tilde{\nu}_\pi$ (resp. $\tilde{\nu}_{\mathscr{S}}$) is a continuous seminorm on π (resp. $\omega_{W,V}$).

For any $a \in A_W^+$, $\delta_{P,\mathrm{Sp}(W)}(a) = \prod_{i=1}^{r_W} |a_i|^{2n+2-2i}$. The integral

$$\int_{A_W^+} \prod_{i=1}^{r_W} |a_i|^{-\frac{1}{2}(2n+2-2i)} \left(1 - \sum_{i=1}^{r_W} \log|a_i|\right)^B \prod_{j=1}^{r_W} |a_j|^{\frac{\dim V}{2}} da$$

converges. Since K_1 is compact, the integral

$$\int_{K_1\times K_1} \tilde{\nu}_\pi(\pi(k_1)v)\tilde{\nu}_\pi(\pi(k_1'^{-1},1)v')\tilde{\nu}_{\mathscr{S}}(\omega_{W,V}(k_1,1)\phi)\tilde{\nu}_{\mathscr{S}}(\omega_{W,V}(k_1'^{-1},1)\phi')dk_1dk_1'$$

is bounded by

$$\mathrm{Vol}(K_1)^2\nu_\pi(v)\nu_\pi(v')\nu_{\mathscr{S}}(\phi)\nu_{\mathscr{S}}(\phi'),$$

where $\nu_\pi(v) = \sup_{k_1\in K_1} \tilde{\nu}_\pi(\pi(k_1)v)$ and $\nu_{\mathscr{S}}(\phi) = \sup_{k_1\in K_1} \tilde{\nu}_{\mathscr{S}}(\omega_{W,V}(k_1,1)\phi)$. Each sup term defines a continuous seminorm on the corresponding space by the uniform boundedness principle [15, Theorem 33.1]. ■

Proposition 3.5 *Let π be an irreducible genuine tempered representation of $\widehat{\mathrm{Sp}}(W)$ (if $K = \mathbb{R}$) or an irreducible tempered representation of $\mathrm{Sp}(W)$ (if $K = \mathbb{H}$). Take $v, v' \in \pi$ and $\phi, \phi' \in \omega_{W,V}$. The multilinear form on $\overline{\pi} \otimes \pi \otimes \omega_{W,V} \otimes \overline{\omega}_{W,V}$* [1]

(3.12) $$(v, v', \phi, \phi') \mapsto \int_{\mathrm{Sp}(W)} \overline{(\pi'(g)v, v')} \cdot (\omega_{W,V}(g,1)\phi, \phi')dg$$

is absolutely convergent and continuous.

Proof The absolute convergence and continuity follow from Lemma 3.4. ■

[1] We ignore the identification of multilinear form and the linear form via the tensor product.

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If $\theta_{W,V,\psi}(\pi) \neq 0$, then the integral (3.12) is not identically zero.

Note that the integral (3.12) defines a Hermitian form on $\overline{\pi} \otimes \omega_{W,V}$. In fact, for any $\phi, \phi' \in \omega_{W,V}$ and $v, v' \in \pi$,

$$\begin{aligned}
\overline{\langle v \otimes \phi, v' \otimes \phi' \rangle} &= \overline{\int_{\mathrm{Sp(W)}} \overline{(\pi(g)v, v')} \cdot (\omega_{W,V}(g,1)\phi, \phi')dg} \\
&= \int_{\mathrm{Sp(W)}} (\pi(g)v, v') \cdot \overline{(\omega_{W,V}(g,1)\phi, \phi')}dg \\
&= \int_{\mathrm{Sp(W)}} \overline{(v', \pi(g)v)} \cdot (\phi', \omega_{W,V}(g,1)\phi)dg \\
&= \int_{\mathrm{Sp(W)}} \overline{(\pi(g^{-1})v', v)} \cdot (\omega_{W,V}(g^{-1},1)\phi', \phi)dg \\
&= \int_{\mathrm{Sp(W)}} \overline{(\pi(g)v', v)} \cdot (\omega_{W,V}(g,1)\phi', \phi)dg \\
&= \langle v' \otimes \phi', v \otimes \phi \rangle,
\end{aligned} \tag{3.13}$$

which means (3.12) defines a Hermitian form on $\Theta_{W,V}(\pi)$. By [6, Theorem 1.1], this form is semipositivity. Moreover, we have the fact that if q is a nonzero semipositive definite Hermitian form on a vector space X, and L is the radical of q, then q descends to an inner product on X/L, still denote by q. To prove this, if there exists an $x \notin L$ such that $q(x,x) = 0$, then take some $y \in X$, which satisfies $q(x,y) \neq 0$. For $t \in \mathbb{C}$, then we have

$$q(tx + y, tx + y) = q(y, y) + 2\mathrm{Re}(t) \cdot q(x, y).$$

As t is an arbitrary complex number and $q(x,y) \neq 0$, we conclude that for a well-chosen complex number t, $q(tx + y, tx + y)$ can be a negative real number, which is a contradiction to the semipositivity of q.

Let R be the radical of semipositive Hermitian form defined by (3.12) as above. Then the nonzero semipositive definite Hermitian form q defines an inner product on $\Theta_{W,V}(\pi)/R$. Therefore, $\Theta_{W,V}(\pi)/R$ must be semisimple, and thus coincides with $\theta_{W,V}(\pi)$.

The explicit theta correspondence allows us to give the explicit matrix coefficients of $\theta_{W,V}(\pi)$ as follows.

Proposition 3.6 *Let (G_1, G_2) be one of our dual pairs over $(K, \natural)$ of equal rank n with underlying spaces (W, V) as in the introduction. Let π be an irreducible genuine tempered representation of $\widehat{\mathrm{Sp}}(W)$ (if $K = \mathbb{R}$) or an irreducible tempered representation of $\mathrm{Sp}(W)$ (if $K = \mathbb{H}$). Then, for $v, v' \in \pi$ and $\phi, \phi' \in \omega_{W,V}$, the function*

$$\Phi_{\phi,\phi',v,v'} : h \in O(V) \mapsto \int_{\mathrm{Sp(W)}} \overline{(\pi(g)v, v')} \cdot (\omega_{W,V}(g,h)\phi, \phi')dg$$

defines a matrix coefficient of $\theta_{W,V}(\pi)$.

4 Theta lifts for tempered representations

In this paragraph, we use the estimations of the matrix coefficients of various representations established in the previous sections to prove our main Theorem 1.1.

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To prove the theorem, it is suffices to show that the matrix coefficients of $\theta_{W,V}(\pi)$ are almost square-integrable functions. Moreover, since the representation $\theta_{W,V}(\pi)$ is irreducible, hence we only need to check that the matrix coefficients $\Phi_{\phi,\phi',v,v'}$ with $\phi,\phi' \in \omega_{W,V}$ and $v, v' \in \pi$ are almost square-integrable.

Hence, we need to prove for any $\varepsilon_0 \in \mathbb{R}_{>0}$, for any $\phi, \phi' \in \omega_{W,V}$, and for any $v, v' \in \pi$, the integral

$$\int_{O(V)} \left|\Phi_{\phi,\phi',v,v'}(h)\right|^{2+\varepsilon_0} dh = \int_{O(V)} \left| \left(\int_{\mathrm{Sp}(W)} (\omega_{W,V}(g,h)\phi, \phi')\overline{(\pi(g)v, v')} dg \right) \right|^{2+\varepsilon_0} dh$$

converges. In the following, we will prove a stronger condition: the integral

$$\int_{O(V)} \left(\int_{\mathrm{Sp}(W)} |(\omega_{W,V}(g,h)\phi,\phi')(\pi(g)v,v')|\, dg \right)^{2+\varepsilon_0} dh \tag{4.1}$$

converges.

Let r_W and r_V be the Witt index of W and V, respectively. Note that, for our dual pair of type I of equal rank n, if $K = \mathbb{H}$, the dimension of W over $\mathbb{H}$ can be $n-1$ or n. In the following, if $K = \mathbb{H}$, we will assume $\dim W = \dim V = n$ and the other case works in the same way.

4.1 Reduction using the estimation of matrix coefficients

Let $\mathrm{Sp}(W) = K_1 A_W^+ K_1$ and $\mathrm{O}(V) = K_2 A_V^+ K_2$ be the Cartan decomposition of $\mathrm{Sp}(W)$ and $\mathrm{O}(V)$, respectively. Let $g \in \mathrm{Sp}(W)$ and $h \in \mathrm{O}(V)$, then there exist $a = (a_1, \dots, a_{r_W}) \in A_W^+$, $b = (b_1, \dots, b_{r_V}) \in A_V^+$, $k_1, k_1' \in K_1$, and $k_2, k_2' \in K_2$ such that $g = k_1 a k_1'$ and $h = k_2 b k_2'$.

For any $\phi, \phi' \in \omega_{W,V}$ and $v, v' \in \pi$, by the estimations (2.6) and the estimation of the matrix coefficient of Weil representation (see (3.5) and (3.6)), we deduce that there exist positive constants A, B such that

$$\begin{aligned} &|(\omega_{W,V}(g,h)\phi,\phi')(\pi(g)v,v')| \\ \le & A\delta_{P,\mathrm{Sp}(W)}^{\frac{1}{2}}(a)\sigma(a)^B \prod_{i=1}^{r_W} |a_i|^{\frac{\dim V}{2}} \prod_{j=1}^{r_V} |b_j|^{\frac{\dim W}{2} - r_W} \prod_{i=1}^{r_W}\prod_{j=1}^{r_V} \Upsilon(a_i b_j^{-1}). \end{aligned} \tag{4.2}$$

To simplify the notation, for $a \in A_W^+$ and $b \in A_V^+$, we set

$$C_{W,V}(a,b) = \prod_{i=1}^{r_W} |a_i|^{\frac{\dim V}{2}} \prod_{j=1}^{r_V} |b_j|^{\frac{\dim W}{2} - r_W} \prod_{i=1}^{r_W}\prod_{j=1}^{r_V} \Upsilon(a_i b_j^{-1}).$$

Together with equation (2.2), we have

$$\begin{aligned} &\int_{\mathrm{Sp}(W)} |(\omega_{W,V}(g,h)\phi,\phi')(\pi(g)v,v')|\, dg \\ \le & A \int_{\mathrm{Sp}(W)} \delta_{P,\mathrm{Sp}(W)}^{\frac{1}{2}}(a)\sigma(a)^B C_{W,V}(a,b) dg \\ \le & A \int_{A_W^+} \delta_{P,\mathrm{Sp}(W)}^{-1}(a) \int_{K_1 \times K_1} \delta_{P,\mathrm{Sp}(W)}^{\frac{1}{2}}(a)\sigma(a)^B C_{W,V}(a,b) dk_1 da dk_1' \\ = & A \cdot \mathrm{Vol}(K_1)^2 \cdot \int_{A_W^+} \delta_{P,\mathrm{Sp}(W)}^{-\frac{1}{2}}(a)\sigma(a)^B C_{W,V}(a,b) da. \end{aligned} \tag{4.3}$$

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Hence, if we denote $A \cdot \mathrm{Vol}(K_1)^2$ by A', then for any $\varepsilon_0 = 2\varepsilon > 0$, using equation (2.2) again, we have

(4.4)

$$\int_{O(V)} \left(\int_{\mathrm{Sp}(W)} |(\omega_{W,V}(g,h)\phi,\phi')(\pi(g)v,v')|\,dg \right)^{2(1+\varepsilon)} dh$$
$$\leq A' \int_{O(V)} \left(\int_{A_W^+} \delta_{P,\mathrm{Sp}(W)}^{-\frac{1}{2}}(a)\sigma(a)^B C_{W,V}(a,b)\,da \right)^{2(1+\varepsilon)} dh$$
$$\leq A' \int_{A_V^+} \delta_{P,O(V)}^{-1}(b) \int_{K_2\times K_2} \left(\int_{A_W^+} \delta_{P,\mathrm{Sp}(W)}^{-\frac{1}{2}}(a)\sigma(a)^B C_{W,V}(a,b)\,da \right)^{2(1+\varepsilon)} dk_2\,db\,dk_2'$$
$$= A' \cdot \mathrm{Vol}(K_2)^2 \int_{A_V^+} \delta_{P,O(V)}^{-1}(b) \left(\int_{A_W^+} \delta_{P,\mathrm{Sp}(W)}^{-\frac{1}{2}}(a)\sigma(a)^B C_{W,V}(a,b)\,da \right)^{2(1+\varepsilon)} db.$$

By the formula (2.3), we have

$$\sigma(a) \leq 1 - \sum_{i=1}^{r_W} \log|a_i| \leq 1 - \sum_{i=1}^{r_W} \log|a_i| - \sum_{j=1}^{r_V} \log|b_j|.$$

The modular characters $\delta_{P,\mathrm{Sp}(W)}$ and $\delta_{P,O(V)}$ are given by the following formula:

$$\delta_{P,\mathrm{Sp}(W)}(a) = \prod_{i=1}^{r_W} |a_i|^{2n+2-2i},$$
$$\delta_{P,O(V)}(b) = \begin{cases} \prod_{j=1}^{r_V} |b_j|^{2n+1-2j}, & \text{if } K = \mathbb{R}, \\ \prod_{j=1}^{r_V} |b_j|^{2n-2j}, & \text{if } K = \mathbb{H}. \end{cases}$$

Thus, we have the integral

$$(4.5) \qquad (4.4) = \frac{\int_{A_W^+\times A_V^+} \prod_{i=1}^{n} |a_i|^{(2i-1)(1+\varepsilon)} \prod_{j=1}^{r_V} |b_j|^{2j-2n-1} \prod_{k=1}^{n}\prod_{j=1}^{r_V} \Upsilon(a_k b_j^{-1})^{2+2\varepsilon}}{\left(1 - \sum_{i=1}^{n}\log|a_i| - \sum_{j=1}^{r_V}\log|b_j|\right)^{B(2+2\varepsilon)}}\,da\,db,$$

if $K = \mathbb{R}$, and

(4.6)

$$(4.4) = \frac{\int_{A_W^+\times A_V^+} \prod_{i=1}^{r_W} |a_i|^{2(i-1)(1+\varepsilon)} \prod_{j=1}^{r_V} |b_j|^{(2j-2n)+2(1+\varepsilon)(n-r_W)} \prod_{k=1}^{r_W}\prod_{j=1}^{r_V} \Upsilon(a_k b_j^{-1})^{2+2\varepsilon}}{\left(1 - \sum_{i=1}^{r_W}\log|a_i| - \sum_{j=1}^{r_V}\log|b_j|\right)^{B(2+2\varepsilon)}}\,da\,db,$$

if $K = \mathbb{H}$.

To prove the convergence of the integral (4.1), it suffices to show the integrals (4.5) and (4.6) are convergent.

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4.2 Proof of the convergence of the integral (4.5)

We prove the convergence of the integral (4.5), this method also works for the convergence of the integral (4.6). Let $(p_1, \dots, p_{r_V+1})$ be an (r_V+1)-tuple of nonnegative integers such that

$$p_1 + \cdots + p_{r_V+1} = n.$$

Let $S_{p_1,\dots,p_{r_V+1}}$ be the subset of $A_W^+ \times A_V^+$, defined by the condition

$$\begin{aligned} &|a_1| \le \cdots \le |a_{p_1}| \le |b_1| \\ &\le |a_{p_1+1}| \le \cdots \le |a_{p_1+p_2}| \le |b_2| \le |a_{p_1+p_2+1}| \le \cdots \le |a_{p_1+\cdots+p_{r_V}}| \\ &\le |b_{r_V}| \le |a_{p_1+\cdots+p_{r_V}+1}| \le \cdots \le |a_{p_1+\cdots+p_{r_V+1}}| \le 1. \end{aligned} \tag{4.7}$$

We can break the domain $A_W^+ \times A_V^+$ of the integral (4.5) by $S_{p_1,\dots,p_{r_V+1}}$, and it suffices to show that over each region $S_{p_1,\dots,p_{r_V+1}}$, the integral (4.5) converges. We will use the following simple lemma to conclude its convergence.

Lemma 4.1 *Let N be a natural number. Let $s_1, \dots, s_N$ and B be real numbers. If $s_1 + \cdots + s_i > 0$ for all $1 \le i \le N$, then the integral*

$$\int_{|x_1| \le \cdots \le |x_N| \le 1} |x_1|^{s_1} \cdots |x_N|^{s_N} \left(1 - \sum_{i=1}^{N} \log|x_i|\right)^B dx_1 \cdots dx_N$$

converges.

Note that in a fixed region $S_{p_1,\dots,p_{r_V+1}}$, we have

$$\prod_{i=1}^{n} \prod_{j=1}^{r_V} \Upsilon(a_i b_j^{-1}) = \prod_{j=1}^{r_V} \left(\left| \prod_{i=1}^{p_{j+1}} a_{i+\sum_{k=1}^{j} p_k} \right|^{-j} \cdot |b_j|^{n-(\sum_{k=1}^{j} p_k)} \right).$$

We rearrange the terms in the integral (4.5) with respect to the order given by the condition (4.7). To prove that the integral (4.5) converges, it suffices to prove that the integral (4.5) on region $S_{p_1,\dots,p_{r_V+1}}$ satisfies the condition of Lemma 4.1 with respect to this order.

For $0 \le t \le p_{j+1}, 1 \le j \le r_V$, we check the sum of the exponents in the integral (4.5) up to $a_{p_1+\cdots+p_j+t}$:

(1) The sum of the exponents of $a_i (1 \le i \le p_1 + \cdots + p_j + t)$:

$$\begin{aligned} &(1 + 3 + \cdots + (2(p_1 + \cdots + p_j + t) - 1))(1+\varepsilon) \\ &\quad - (p_2 + 2p_3 + \cdots + (j-1)p_j + jt)(2+2\varepsilon), \end{aligned}$$

(2) The sum of the exponents of $b_i (1 \le i \le j)$:

$$2(1 + \cdots + j) - j(2n+1) + ((n - p_1) + \cdots + (n - p_1 - \cdots - p_j))(2+2\varepsilon).$$

Summing these two terms, we get

$$(p_1 + \cdots + p_j + t)^2 + \varepsilon((p_1 + \cdots + p_j + t)^2 + 2j(n - p_1 - \cdots - p_j - t)) > 0.$$

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The same type of verification shows that the sum of the exponents up to b_j is positive. Hence, the integral (4.5) satisfies the condition of Lemma 4.1. As a consequence, the integral (4.1) converges.

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