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Convergence Rates of Cascade Algorithms with Infinitely Supported Masks

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Abstract. We investigate the solutions of refinement equations of the form

$$\phi(x) = \sum_{\alpha\in\mathbb{Z}^s} a(\alpha)\,\phi(Mx-\alpha),$$

where the function ϕ is in $L_p(\mathbb{R}^s)(1 \le p \le \infty)$, a is an infinitely supported sequence on $\mathbb{Z}^s$ called a refinement mask, and M is an $s \times s$ integer matrix such that $\lim_{n\to\infty} M^{-n} = 0$. Associated with the mask a and M is a linear operator $Q_{a,M}$ defined on $L_p(\mathbb{R}^s)$ by $Q_{a,M}\phi_0 := \sum_{\alpha\in\mathbb{Z}^s} a(\alpha)\phi_0(M\cdot -\alpha)$. Main results of this paper are related to the convergence rates of $(Q^n_{a,M}\phi_0)_{n=1,2,\dots}$ in $L_p(\mathbb{R}^s)$ with mask a being infinitely supported. It is proved that under some appropriate conditions on the initial function ϕ_0, $Q^n_{a,M}\phi_0$ converges in $L_p(\mathbb{R}^s)$ with an exponential rate.

1 Introduction

We are interested in refinement equations of the form

$$\phi(x) = \sum_{\alpha\in\mathbb{Z}^s} a(\alpha)\,\phi(Mx-\alpha), \quad x \in \mathbb{R}^s, \tag{1.1}$$

where ϕ is the unknown function defined on the s-dimensional Euclidean space $\mathbb{R}^s$, a is an infinitely supported sequence on $\mathbb{Z}^s$ called a refinement mask, and M is an $s \times s$ integer matrix such that $\lim_{n\to\infty} M^{-n} = 0$. Any solution of (1.1) is called a refinable function, and the matrix M is called a dilation matrix.

In order to study the refinement equation (1.1), we employ the following iteration scheme. From an initial function $\phi_0 \in L_p(\mathbb{R}^s)$ for $1 \le p \le \infty$, let $\phi_n := Q^n_{a,M}\phi_0$, $n = 1, 2, \dots$, where $Q_{a,M}$ is a linear operator on $L_p(\mathbb{R}^s)$ as follows:

$$Q_{a,M} f := \sum_{\alpha\in\mathbb{Z}^s} a(\alpha) f(M\cdot -\alpha), \quad f \in L_p(\mathbb{R}^s).$$

This iteration scheme is called a *cascade algorithm* or a *subdivision scheme* associated with a and M. We say that the cascade algorithm associated with a and M converges

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in the L_p-norm if there exists a function $f \in L_p(\mathbb{R}^s)$ such that for any compactly supported function ϕ_0 in $L_p(\mathbb{R}^s)$ satisfying the Strang–Fix conditions of order 1,

$$\lim_{n\to\infty} \|Q_{a,M}^n \phi_0 - f\|_p = 0.$$

A typical choice of the initial function ϕ_0 can be chosen by the hat function

$$\phi_0(x) := \prod_{j=1}^{s} \max\{1 - |x_j|, 0\}, \quad x = (x_1, \dots, x_s) \in \mathbb{R}^s.$$

Let us recall the definition of Strang–Fix conditions (see [25]). Suppose that the function g decays fast enough so that the partial derivative $D^\alpha \widehat{g}$ exists and is continuous for $|\alpha| \le k$. We say that g satisfies the Strang–Fix conditions of order k if

$$\widehat{g}(0) = 1 \quad \text{and} \quad D^\alpha \widehat{g}(2\beta\pi) = 0 \qquad |\alpha| \le k-1, \quad \beta \in \mathbb{Z}^s \backslash \{0\}.$$

The convergence of cascade algorithms is fundamental to wavelet theory and subdivision. When mask a is finitely supported, the cascade algorithm has been extensively studied by many authors (see [2, 3, 8, 13, 15, 19, 24] and many references therein). However, due to some desirable properties, infinitely supported masks including masks with exponential decay such as Butterworth filters and masks with polynomial decay such as various types of fractional splines are also of interest in some applications in the area of digital signal processing ([4, 5, 7, 12, 27]). More recently, the convergence of cascade algorithms associated with an infinitely supported mask has been investigated by some authors (see [9–11, 21–23]).

The purpose of this paper is to investigate the convergence rates of cascade algorithms in $L_p(\mathbb{R}^s)$ ($1 \le p \le \infty$) associated with an infinitely supported mask and a dilation matrix. For the case when mask a exhibits polynomial decay, we mean $a \in B_k$ for some $k \in \mathbb{Z}_+$, where B_k denotes the linear space of all sequences u on $\mathbb{Z}^s$ for which

$$\|u\|_{B_k} := \sum_{\alpha\in\mathbb{Z}^s} |u(\alpha)|(1 + |\alpha|)^k < \infty.$$

Equipped with the norm $\|\cdot\|_{B_k}$, B_k becomes a Banach space ([18]).

For $k \in \mathbb{Z}_+$, let $L_{\infty,k}(\mathbb{R}^s)$ denote the linear space of all functions f such that

$$(1 + |\cdot|)^k f(\cdot) \in L_\infty(\mathbb{R}^s).$$

We point out that these $L_{\infty,k}(\mathbb{R}^s)$ spaces are closely related to Wiener Amalgam spaces with polynomial weight, which are important in sampling and shift-invariant spaces theory ([1]). For simplicity, we abbreviate $L_{\infty,k}(\mathbb{R}^s)$ as $L_{\infty,k}$.

When mask a is finitely supported, the convergence rates of cascade algorithms have been considered by several authors. For the case $M = 2$ and the compactly supported solution ϕ of (1.1) lies in $W_\infty^k(\mathbb{R}^s)$, Zhang showed in [28] that if the shifts of ϕ are stable and $D^\mu \phi(2\pi\beta) = D^\mu \phi_0(2\pi\beta)$ for all $\beta \in \mathbb{Z}^s$ and $|\mu| < k$, then $\|Q_a^n \phi_0 - \phi\|_\infty \le C2^{-kn}$, where ϕ_0 is a compactly supported continuous function. In [15], Jia investigated the convergence rates of cascade algorithms in $L_p(\mathbb{R}^s)$ associated with

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an isotropic dilation matrix. Li ([20]) and Sun ([26]) characterized the convergence rates of vector cascade algorithms in $(L_p(\mathbb{R}^s))^r$. Most approaches in these papers were based on the theory of a shift-invariant space whose generator ϕ is compactly supported. However, this technique cannot be applied to the case when the generator has non-compact support, since it essentially relies on the fact that the restriction of the shift-invariant space to a finite cube is finite dimensional.

In this paper, under the assumption that the solutions of (1.1) lie in $L_{\infty,k}$, we characterize the convergence rates of cascade algorithms. Our characterizations extend some main results in [15] with finitely supported masks to the case in which masks are infinitely supported. It is proved that under some appropriate conditions on the initial function ϕ_0, the cascade algorithm $Q^n_{a,M}\phi_0$ converges in $L_p(\mathbb{R}^s)$ with an exponential rate. Furthermore, for the case in which $a \in B_k$, we extend Han's result in [6] with finitely supported masks to the case in which a has polynomial decay.

In fact, for the case $a \in B_k$, the solutions of refinement equation (1.1) in general have noncompact support. In many cases, the solutions belong to $L_{\infty,k}$. For example, Cohen ([4]) characterized the existence of $L_{\infty,k}$-solution of (1.1) with $a \in B_k$. Besides, Unser and Blu ([27, Theorem 3.1]) showed that fractional splines ϕ of degree $\alpha > -1$ satisfy the refinement equation with $a \in B_{\alpha+2}$ and $\phi \in L_{\infty,\alpha+2}$. Therefore, it is interesting to characterize the convergence rates of cascade algorithms associated with an infinitely supported mask under the assumption that the solutions of (1.1) lie in $L_{\infty,k}$.

As usual, let $\mathbb{Z}_+$ denote the set of positive integers and let $\mathbb{N}_0$ denote the set of nonnegative integers. For $j = 1, \dots, s$, let e_j be the j-th coordinate unit vector in $\mathbb{R}^s$. The norm in $\mathbb{R}^s$ is defined by

$$|y| := |y_1| + \cdots + |y_s|, \qquad y = (y_1, \dots, y_s) \in \mathbb{R}^s.$$

Denote by $\ell(\mathbb{Z}^s)$ the linear space of all (complex-valued) sequences on $\mathbb{Z}^s$. Denote by δ the sequence on $\mathbb{Z}^s$ given by $\delta(0) = 1$ and $\delta(k) = 0$ if $k \neq 0$. The difference operator ∇_j on $\ell(\mathbb{Z}^s)$ is defined by $\nabla_j a := a - a(\cdot - e_j), a \in \mathbb{Z}^s$. An element $\mu = (\mu_1, \dots, \mu_s) \in \mathbb{N}_0^s$ is called a multi-index. ∇^μ is the difference operator $\nabla_1^{\mu_1} \cdots \nabla_s^{\mu_s}$. For $j = 1, \dots, s$, D_j denotes the partial derivative with respect to the j-th coordinate. For $\mu = (\mu_1, \dots, \mu_s) \in \mathbb{N}_0^s$, D^μ is the differential operator $D_1^{\mu_1} \cdots D_s^{\mu_s}$. For $k \in N_0$, denote by Π_k the linear space of all polynomials of degree at most k.

For $1 \le p \le \infty$, denote by $L_p(\mathbb{R}^s)$ the Banach space of all (complex-valued) functions f such that $\|f\|_p < \infty$, where

$$\|f\|_p := \left(\int_{\mathbb{R}^s} |f(x)|^p dx \right)^{1/p} \qquad \text{for} \quad 1 \le p < \infty,$$

and $\|f\|_\infty$ is the essential supremum of f on $\mathbb{R}^s$.

Analogously, denote by $\ell_p(\mathbb{Z}^s)$ the Banach space of all complex-valued sequences $a = (a(\alpha))_{\alpha \in \mathbb{Z}^s}$ such that $\|a\|_p < \infty$, where

$$\|a\|_p := \left(\sum_{\alpha \in \mathbb{Z}^s} |a(\alpha)|^p \right)^{1/p} \qquad \text{for} \quad 1 \le p < \infty,$$

and $\|a\|_\infty$ is the supremum of a on $\mathbb{Z}^s$.

The Fourier transform of a function in $L_1(\mathbb{R}^s)$ is defined by

$$\widehat{f}(\xi) := \int_{\mathbb{R}^s} f(x)e^{-ix\cdot\xi}dx, \qquad \xi \in \mathbb{R}^s,$$

where $x \cdot \xi$ is the inner product of two vectors x and ξ in $\mathbb{R}^s$. The domain of the Fourier transform can be naturally extended to functions in $L_2(\mathbb{R}^s)$ and tempered distributions.

We denote the space of all continuous functions on $\mathbb{R}^s$ by $C(\mathbb{R}^s)$. For $k \in N_0$, denote by $C^k(\mathbb{R}^s)$ the space of all functions $f \in C(\mathbb{R}^s)$ for which $D^\mu f \in C(\mathbb{R}^s)$ for all $|\mu| \le k$. Moreover, denote by $C_c^k(\mathbb{R}^s)$ the space of all functions in $C^k(\mathbb{R}^s)$ with compact support. We denote by $W_p^m(\mathbb{R}^s)$ the usual Sobolev space and by $|f|_{m,p}$ the seminorm of a function $f \in W_p^m(\mathbb{R}^s)$.

For $y \in \mathbb{R}^s$, the difference operator is defined by

$$\nabla_y f := f - f(\cdot - y),$$

where f is a function defined on $\mathbb{R}^s$. Let k be a positive integer. The k-th modulus of continuity of f in $L_p(\mathbb{R}^s)$ is defined by

$$\omega_k(f,h)_p := \sup_{|y|\le h} \|\nabla_y^k f\|_p, \qquad h \ge 0.$$

For $1 \le p \le \infty$, $0 < \nu \le 1$, we denote by $\mathrm{Lip}(\nu, L_p(\mathbb{R}^s))$ the Lipschitz space of all functions $f \in L_p(\mathbb{R}^s)$ such that

$$w_1(f,h)_p \le Ch^\nu, \qquad \forall\, h > 0,$$

where C is a positive constant independent of h. For a general $\nu > 0$, write $\nu = r + \eta$, where r is an integer and $0 < \eta \le 1$. Denote by $\mathrm{Lip}(\nu, L_p(\mathbb{R}^s))$ the Lipschitz space of all functions f such that $D^\mu f \in \mathrm{Lip}(\eta, L_p(\mathbb{R}^s))$ for all multi-indices μ with $|\mu| = r$.

Let M be a fixed integer matrix with $m = |\det M|$. Then the coset space $\mathbb{Z}^s/M\mathbb{Z}^s$ consists of m elements. Let $\gamma_k + M\mathbb{Z}^s$, $k = 0, 1, \dots, m-1$ be the m distinct elements of $\mathbb{Z}^s/M\mathbb{Z}^s$ with $\gamma_0 = 0$. We denote $\Gamma = \{\gamma_k, k = 0, 1, \dots, m-1\}$. Thus, each element $\alpha \in \mathbb{Z}^s$ can be uniquely represented as $\gamma + M\varepsilon$, where $\gamma \in \Gamma$ and $\varepsilon \in \mathbb{Z}^s$.

We say that mask a satisfies the basic sum rule if

$$\sum_{\alpha\in\mathbb{Z}^s} a(\gamma + M\alpha) = \sum_{\alpha\in\mathbb{Z}^s} a(M\alpha) \qquad \forall\gamma \in \Gamma.$$

The concept of stability plays an important role in the study of refinable functions. Let $\phi \in L_p(\mathbb{R}^s)$; we say that the shifts of ϕ are stable if there exist positive constants A_p and B_p such that for all finitely supported sequences a,

$$A_p\|a\|_p \le \Big\| \sum_{\alpha\in\mathbb{Z}^s} a(\alpha)\phi(\cdot - \alpha) \Big\|_p \le B_p\|a\|_p.$$

2 Quasi-Projection Operators

When mask a is an infinitely supported sequence, the iterated functions $Q_a^n\phi_0$ are in general not compactly supported. In fact, under some conditions on a, solutions of (1.1) are polynomially decaying [4, 27]. Thus, in order to discuss the cascade algorithm associated with an finitely supported mask and a dilation matrix, we investigate the quasi-projection operator whose generator decays polynomially fast.

Let $\phi \in L_{\infty,k}$ for some $k \in \mathbb{Z}_+$, and let g be a compactly supported function in $L_q(\mathbb{R}^s)$. Let $\mathscr{P}_{g,\phi}$ be a linear operator on $L_p(\mathbb{R}^s)$ defined by

$$\mathscr{P}_{g,\phi} f := \sum_{\alpha\in\mathbb{Z}^s} \big\langle f, g(\cdot - \alpha)\big\rangle \phi(\cdot - \alpha), \qquad f \in L_p(\mathbb{R}^s),$$

where $\langle f, g\rangle := \int_{\mathbb{R}^s} f(x)\overline{g(x)}dx$ and $1/p + 1/q = 1$. Such an operator $\mathscr{P}_{g,\phi}$ is called a quasi-projection operator and is a bounded operator on $L_p(\mathbb{R}^s)$.

Let M be an $s \times s$ matrix with its entries in $\mathbb{C}$. We say that M is isotropic if M is similar to a diagonal matrix diag $\{\lambda_1, \dots, \lambda_s\}$ with $|\lambda_1| = \cdots = |\lambda_s|$.

The following lemma will be needed in our study of convergence rates of cascade algorithms associated with an infinitely supported mask and an isotropic dilation matrix.

Lemma 2.1 *Let $\phi \in L_{\infty,k+s+1}$ for some $k \in \mathbb{Z}_+$, and let g be a compactly supported function in $L_q(\mathbb{R}^s)$ ($1 \le q \le \infty$). Let M be an isotropic dilation matrix with $m := |\det M|$. For $n = 1, 2, \dots$, let $\mathscr{P}^n_{g,\phi}$ be the quasi-projection operator given by*

$$\mathscr{P}^n_{g,\phi} f := \sum_{\alpha\in\mathbb{Z}^s} \big\langle f, m^n g(M^n \cdot - \alpha)\big\rangle \phi(M^n \cdot - \alpha),$$

where $f \in L_p(\mathbb{R}^s)$ and $1/p + 1/q = 1$. If $\mathscr{P}_{g,\phi} q = q$ for all $q \in \Pi_{k-1}$, then there exists a constant C such that

$$\|\mathscr{P}^n_{g,\phi} f - f\|_p \le C\omega_k(f, m^{-n/s})_p$$

for $n = 1, 2, \dots$ and $f \in L_p(\mathbb{R}^s)$ ($f \in C(\mathbb{R}^s)$ in the case $p = \infty$). In particular, if $f \in Lip(\mu, L_p(\mathbb{R}^s))$ with $0 < \mu \le k$, then

$$\|\mathscr{P}^n_{g,\phi} f - f\|_p \le C(m^{-1/s})^{\mu n}.$$

Proof Let $\rho \in C_c^k(\mathbb{R}^s)$ such that $\int_{\mathbb{R}^s} \rho(x)dx = 1$. For $n = 1, 2, \dots$, let $\mathscr{A}_\rho^n$ be the linear operator on $L_p(\mathbb{R}^s)$ ($C(\mathbb{R}^s)$ in the case $p = \infty$) given by

$$\mathscr{A}_\rho^n f(x) := \int_{\mathbb{R}^s} (f - \nabla_y^k f)(x) m^n \rho(M^n y)dy, \qquad f \in L_p(\mathbb{R}^s), \quad x \in \mathbb{R}^s.$$

By [16, Lemma 2.1], $\mathscr{A}_\rho^n f \in C^k(\mathbb{R}^s)$ and there exists a constant C independent of n and f such that

$$\|f - \mathscr{A}_\rho^n f\|_p \le C\omega_k(f, m^{-n/s})_p. \tag{2.1}$$

Since $\phi \in L_{\infty,k+s+1}$, we have

$$ess \sup_{x\in[0,1)^s} \sum_{\alpha\in\mathbb{Z}^s} |\phi(x+\alpha)|\big(1+|x+\alpha|\big)^k < \infty.$$

Let $K(x,y) := \sum_{\alpha\in\mathbb{Z}^s} \phi(x-\alpha)g(y-\alpha)$. Then, $\mathscr{P}_{g,\phi} f = \int_{\mathbb{R}^s} K(x,y)f(y)dy$. It is easy to check that the kernel $K(x,y)$ satisfies the conditions in [18, Theorem 2.1]. Besides, $\mathscr{P}_{g,\phi} q = q$ for all $q \in \Pi_{k-1}$. Thus,

$$\|\mathscr{P}^n_{g,\phi}\mathscr{A}^n_\rho f - \mathscr{A}^n_\rho f\|_p \le Cm^{-\frac{nk}{s}}|\mathscr{A}^n_\rho f|_{k,p}. \tag{2.2}$$

By [16, Lemma 2.2], we have

$$m^{-\frac{nk}{s}}|\mathscr{A}^n_\rho f|_{k,p} \le C\omega_k(f, m^{-n/s})_p. \tag{2.3}$$

Clearly, for a nontrivial function $f \in L_p(\mathbb{R}^s)$, we have

$$\frac{\|\mathscr{P}^n_{g,\phi} f(M^n\cdot)\|_p}{\|f(M^n\cdot)\|_p} = \frac{\|(\mathscr{P}_{g,\phi} f)(M^n\cdot)\|_p}{\|f(M^n\cdot)\|_p} = \frac{\|\mathscr{P}_{g,\phi} f\|_p}{\|f\|_p} \le C.$$

Thus, we obtain

$$\begin{aligned}\|\mathscr{P}^n_{g,\phi}\mathscr{A}^n_\rho f - \mathscr{P}^n_{g,\phi} f\|_p &\le \|\mathscr{P}^n_{g,\phi}\|\|\mathscr{A}^n_\rho f - f\|_p \\ &\le C\|\mathscr{A}^n_\rho f - f\|_p \le C\omega_k(f, m^{-n/s})_p.\end{aligned} \tag{2.4}$$

Combining (2.1)–(2.4), we conclude that

$$\|\mathscr{P}^n_{g,\phi} f - f\|_p \le C\omega_k(f, m^{-n/s})_p.$$

If $f \in \mathrm{Lip}(\mu, L_p(\mathbb{R}^s))$, then $\omega_k(f,h)_p \le Ch^\mu$ for any $h > 0$. This immediately implies that

$$\|\mathscr{P}^n_{g,\phi} f - f\|_p \le C(m^{-1/s})^{\mu n}.$$

The proof of Lemma 2.1 is complete. ■

Remark 2.2 Lemma 2.1 was established by Jia in [15] in the case when $\phi \in L_p(\mathbb{R}^s)$ is compactly supported.

3 Convergence Rates of Cascade Algorithms

In this section, we shall characterize the convergence rates of cascade algorithms associated with an infinitely supported mask and a dilation matrix. It is proved that under some appropriate conditions on the initial function ϕ_0, $Q^n_{a,M}\phi_0$ converges in $L_p(\mathbb{R}^s)$ with an exponential rate.

Theorem 3.1 *Let $\phi \in Lip(\mu, L_p(\mathbb{R}^s)) \cap L_{\infty,k+s+1}$ be the normalized solution of refinement equation* (1.1) *with an infinitely supported mask a and an isotropic dilation matrix M, where $\mu > 0$, $1 \le p \le \infty$ and k is the integer such that $k-1 < \mu \le k$. Let $m := |\det M|$. Suppose that ϕ_0 is a compactly supported function in $L_p(\mathbb{R}^s)$ satisfying the Strang–Fix conditions of order k. If the shifts of ϕ are stable and $D^\nu \widehat{\phi}_0(0) = D^\nu \widehat{\phi}(0)$ for all $|\nu| < k$, then there exists a constant $C > 0$ such that*

$$\|Q_a^n \phi_0 - \phi\|_p \le C(m^{-1/s})^{\mu n} \qquad \forall n \in \mathbb{N}.$$

Proof Our proof of Theorem 3.1 follows [15]. Since $\phi \in \mathrm{Lip}(\mu, L_p(\mathbb{R}^s)) \cap L_{\infty,k+s+1}$, by the Riemann–Lebesgue lemma and the Leibniz formula for differentiation, we conclude that ϕ satisfies the Strang–Fix conditions of order k ([14, Theorem 6.3]).

Then we can find a compactly supported function $g \in L_q(\mathbb{R}^s)$ such that for any $0 < |\nu| < k$, $D^\nu \widehat{g}(0) = 0$ and $\widehat{g}(0) = 1$. Thus, we have $D^\nu(1 - \widehat{g}\widehat{\phi})(0) = 0$. Similar to the proof of [15, Lemma 3.2], we can prove that for any $q \in \Pi_{k-1}$, $\widehat{g}\widehat{\phi}(\widehat{q}(iD)\delta) = \widehat{q}(iD)\delta$, where δ stands for the Dirac distribution. This is equivalent to

$$\widehat{\phi}(-iD)q * g = q. \tag{3.1}$$

By using the Poisson summation formula, we have

$$\sum_{\alpha \in \mathbb{Z}^s} q * g(\alpha)\phi(\,\cdot\, - \alpha) = \widehat{\phi}(-iD)q * g. \tag{3.2}$$

Combining (3.1) and (3.2), we see that

$$\sum_{\alpha \in \mathbb{Z}^s} q * g(\alpha)\phi(\,\cdot\, - \alpha) = q \qquad \forall q \in \Pi_{k-1}.$$

Thus, the quasi-projection operator given by

$$\mathscr{P}_{g,\phi} f := \sum_{\alpha \in \mathbb{Z}^s} \langle f, g(\,\cdot\, - \alpha)\rangle \phi(\,\cdot\, - \alpha)$$

reproduces all polynomials of order at most $k-1$, *i.e.*, $\mathscr{P}_{g,\phi} q = q$ for all $q \in \Pi_{k-1}$.

In addition, since $D^\nu \widehat{\phi}_0(0) = D^\nu \widehat{\phi}(0)$ for all $|\nu| < k$, we have $D^\nu(1 - \widehat{g}\widehat{\phi}_0)(0) = 0$ for all $|\nu| < k$. By using the same method as above, we also have $\mathscr{P}_{g,\phi_0} q = q$ for all $q \in \Pi_{k-1}$.

For $n = 1, 2, \ldots$, let

$$f_n := \sum_{\alpha \in \mathbb{Z}^s} b_n(\alpha)\phi(M^n \cdot - \alpha) \quad \text{and} \quad g_n := \sum_{\alpha \in \mathbb{Z}^s} b_n(\alpha)\phi_0(M^n \cdot - \alpha),$$

where $b_n(\alpha) = \langle \phi, m^n g(M^n \cdot - \alpha)\rangle$. In light of Lemma 2.1, we have

$$\|f_n - \phi\|_p \le C_1(m^{-1/s})^{\mu n} \quad \text{and} \quad \|g_n - \phi\|_p \le C_2(m^{-1/s})^{\mu n}.$$

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Note that $Q^n_{a,M}\phi = \sum_{\alpha\in\mathbb{Z}^s} S^n_{a,M}\delta(\alpha)\phi(M^n\cdot-\alpha) = \phi$, where $S_{a,M}$ is defined by (3.3). Since the shifts of ϕ are stable, we obtain

$$\|f_n-\phi\|_p = \Big\|\sum_{\alpha\in\mathbb{Z}^s}(b_n - S^n_{a,M}\delta)(\alpha)\phi(M^n\cdot-\alpha)\Big\|_p \ge C_3 m^{-\frac{n}{p}}\|b_n - S^n_{a,M}\delta\|_p.$$

On the other hand, applying [17, Theorem 2.1], we find that

$$\begin{aligned}\|Q^n_{a,M}\phi_0 - g_n\|_p &= \Big\|\sum_{\alpha\in\mathbb{Z}^s}(S^n_{a,M}\delta - b_n)(\alpha)\phi_0(M^n\cdot-\alpha)\Big\|_p\\ &\le \Big(\int_{[0,1)^s}\Big(\sum_{\alpha\in\mathbb{Z}^s}|\phi_0(x-\alpha)|\Big)^p dx\Big)^{1/p} m^{-\frac{n}{p}}\|S^n_{a,M}\delta - b_n\|_p.\end{aligned}$$

Since ϕ_0 is a compactly supported function in $L_p(\mathbb{R}^s)$, we have

$$\Big(\int_{[0,1)^s}\Big(\sum_{\alpha\in\mathbb{Z}^s}|\phi_0(x-\alpha)|\Big)^p dx\Big)^{1/p} < +\infty.$$

It follows that $\|Q^n_{a,M}\phi_0 - g_n\|_p \le C_4\|f_n-\phi\|_p$.

Therefore, we conclude that

$$\begin{aligned}\|Q^n_{a,M}\phi_0 - \phi\|_p &\le \|Q^n_{a,M}\phi_0 - g_n\|_p + \|g_n - \phi\|_p\\ &\le C_4\|f_n-\phi\|_p + \|g_n-\phi\|_p \le C(m^{-1/s})^{\mu n}.\end{aligned}$$ ■

Remark 3.2 We point out that under assumptions that the normalized solution ϕ of (1.1) lies in $L_{\infty,k+s+1}$ and the shifts of ϕ are stable, Theorem 3.1 characterizes the convergence rates of cascade algorithms with an infinitely supported mask and an isotropic dilation matrix. In particular, if mask a is finitely supported, then ϕ is compactly supported. In this case, when the shifts of ϕ are linearly independent, Theorem 3.1 was established by Jia in [15].

For the case when mask a is finitely supported, Han [6] also investigated the convergence rates of cascade algorithms in $L_p(\mathbb{R}^s)$ in terms of the joint spectral radius, using a different method. In the following theorem, we shall extend his result to the case in which $a \in B_k$.

Let M be a dilation matrix and $a\in B_k$ for some $k\in\mathbb{Z}_+$. The subdivision operator associated with M and a is a linear operator on $\ell_p(\mathbb{Z}^s)$ defined by

$$S_{a,M}u(\alpha) := \sum_{\beta\in\mathbb{Z}^s} a(\alpha - M\beta)u(\beta), \qquad \alpha\in\mathbb{Z}^s, \qquad u\in\ell_p(\mathbb{Z}^s). \tag{3.3}$$

For $1\le p\le\infty$, we define

$$\rho(a,M;p) := \max\big\{\limsup_{n\to\infty}\|\nabla^\mu S^n_{a,M}\delta\|_p^{\frac{1}{n}} : |\mu| = 1\big\}.$$

The quantity $\rho(a, M; p)$ plays an important role in the study of subdivision schemes and wavelets (see [10] and many references therein for detail).

Similar to the case when mask a is finitely supported, for the case $a \in B_k$, it is easy to prove that if a satisfies the basic sum rule and $\rho(a, M; p) < |\det M|^{1/p}$, then the cascade algorithm associated with a and M converges in $L_p(\mathbb{R}^s)$. Moreover, the following theorem shows that if the normalized solution ϕ of (1.1) lies in $L_{\infty,k}$, then the cascade algorithm converges with an exponential rate.

Theorem 3.3 *Let $a \in B_k$ for some $k > s + 1$ and $\rho(a, M; p) < m^{1/p}$, where M is a dilation matrix with $m := |\det M|$. Let a satisfy the basic sum rule and $\sum_{\alpha\in\mathbb{Z}^s} a(\alpha) = m$. Suppose $\phi \in L_{\infty,k}$ is the normalized solution of* (1.1). *If $\phi_0 \in L_{\infty,k}$ satisfies the Strang–Fix condition of order* 1, *then $r := \rho(a, M; p)m^{-1/p} < 1$ and for any $0 < \epsilon < 1 - r$, there exists a constant $C > 0$ such that*

$$\|Q^n_{a,M}\phi_0 - \phi\|_p \le C(r + \epsilon)^n \qquad \forall n \in \mathbb{N}.$$

Proof Since $\phi_0 \in L_{\infty,k}$, we have

$$\begin{aligned}(1 + |x|)^k\,|Q_{a,M}\phi_0(x)| &\le \sum_{\alpha\in\mathbb{Z}^s} |a(\alpha)\phi_0(Mx - \alpha)|\big(1 + |M^{-1}Mx|\big)^k \\ &\le \sum_{\alpha\in\mathbb{Z}^s} |a(\alpha)\phi_0(Mx - \alpha)|\big(1 + |Mx|\big)^k \\ &\le \sum_{\alpha\in\mathbb{Z}^s} |\phi_0(Mx - \alpha)|\big(1 + |Mx - \alpha|\big)^k |a(\alpha)|(1 + |\alpha|)^k \\ &\le C \sum_{\alpha\in\mathbb{Z}^s} |a(\alpha)|\big(1 + |\alpha|\big)^k = C\|a\|_{B_k}.\end{aligned}$$

It follows that $Q_{a,M}\phi_0 \in L_{\infty,k}$. Similarly, we have $Q_{a,M}\phi \in L_{\infty,k}$.

In addition, since a satisfies the basic sum rule and ϕ_0 satisfies the Strang–Fix conditions of order 1, we obtain

$$\begin{aligned}\sum_{\alpha\in\mathbb{Z}^s} Q_{a,M}\phi_0(\cdot + \alpha) &= \sum_{\alpha\in\mathbb{Z}^s}\sum_{\beta\in\mathbb{Z}^s} a(\beta)\phi_0(M\cdot - M\alpha - \beta) \\ &= \sum_{\beta\in\mathbb{Z}^s}\Big(\sum_{\alpha\in\mathbb{Z}^s} a(\beta - M\alpha)\Big)\phi_0(M\cdot - \beta) = \sum_{\beta\in\mathbb{Z}^s}\phi_0(M\cdot - \beta) = 1.\end{aligned}$$

Thus, $Q_{a,M}\phi_0$ also satisfies the Strang–Fix conditions of order 1. Let $\psi_0 := Q_{a,M}\phi_0 - \phi$; we conclude that

$$\sum_{\alpha\in\mathbb{Z}^s}\psi_0(x + \alpha) = 0 \qquad a.e. \quad x \in \mathbb{R}^s.$$

By virtue of [10, Lemma 2] or [11, Lemma 3.1], there exists a set of functions h_j, $j = 1, \ldots, s$, with each $h_j \in L_{\infty,k-1}$ such that

$$\psi_0 = \sum_{j=1}^s \nabla_j h_j.$$

Observe that $Q_{a,M}\phi = \phi$. It follows that

$$\|Q_{a,M}^{n+1}\phi_0 - \phi\|_p = \|Q_{a,M}^{n+1}\phi_0 - Q_{a,M}^{n}\phi\|_p = \|Q_{a,M}^{n}\psi_0\|_p = \|\sum_{j=1}^{s} \nabla_j Q_{a,M}^{n} h_j\|_p$$
$$\leq m^{-\frac{n}{p}} \sum_{j=1}^{s} \|\nabla_j S_{a,M}^{n}\delta\|_p \Big(\int_{[0,1)^s} \Big(\sum_{\alpha\in\mathbb{Z}^s} |h_j(x-\alpha)| \Big)^p dx \Big)^{1/p}.$$

Since $h_j \in L_{\infty,k-1}$, it is easy to see that

$$\Big(\int_{[0,1)^s} \Big(\sum_{\alpha\in\mathbb{Z}^s} |h_j(x-\alpha)| \Big)^p dx \Big)^{1/p} < \infty.$$

Therefore, for any $0 < \epsilon < 1 - r$, there exists a constant C such that

$$\|Q_{a,M}^{n}\phi_0 - \phi\|_p \leq C m^{-n/p} (\rho(a, M; p) + \epsilon)^n \leq C(r+\epsilon)^n.$$

This completes the proof of Theorem 3.3. ∎

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