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SOLVABLE GROUPS WHOSE NONNORMAL SUBGROUPS HAVE FEW ORDERS

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Abstract

Suppose that G is a finite solvable group. Let $t = n_c(G)$ denote the number of orders of nonnormal subgroups of G. We bound the derived length $dl(G)$ in terms of $n_c(G)$. If G is a finite p-group, we show that $|G'| \leq p^{2t+1}$ and $dl(G) \leq \lceil \log_2(2t + 3) \rceil$. If G is a finite solvable nonnilpotent group, we prove that the sum of the powers of the prime divisors of $|G'|$ is less than t and that $dl(G) \leq \lfloor 2(t + 1)/3 \rfloor + 1$.

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1. Introduction

A finite group is said to be a Dedekind group if all its subgroups are normal. Such groups were precisely classified by Dedekind in [6]. Groups having only a few nonnormal subgroups can be considered close to Dedekind groups. There are many results about such groups that characterise the structure of finite groups with a small number of conjugacy classes of nonnormal subgroups (see [3–5, 7, 9–11]). There are also explorations based on the number of orders of nonnormal subgroups.

Let G be a finite group. For convenience, we introduce the notation,

$$n_c(G) = \text{the number of orders of nonnormal subgroups of } G.$$

Obviously, $n_c(G) = 0$ if and only if G is a Dedekind group. Passman in [12] classified finite p-groups, all of whose nonnormal subgroups are cyclic, including finite p-groups with $n_c(G) = 1$. Later, Berkovich and Zhang in [2, 13] classified finite groups with $n_c(G) = 1$, and An in [1] classified finite p-groups with $n_c(G) = 2$. These results are mainly concerned with the structure of G. In particular, Passman in [12] gave several interesting properties of finite p-groups based on the orders of their nonnormal subgroups, which served as inspiration for this study.

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The aim of this paper is to estimate the derived length of a finite solvable group G in terms of $n_c(G)$. We examine nilpotent groups (Section 2) and solvable nonnilpotent groups (Section 3). In fact, the derived length of a nilpotent group with $n_c(G) = t$ is less than the derived length of p-groups with $n_c(G) = t$. Therefore, we consider finite p-groups instead of nilpotent groups.

In [12], Passman showed that, for a finite p-group G, if the maximal order of nonnormal subgroups of G is p^m, then $|G'| \le p^m$, and hence the nilpotent class $c(G) \le m + 1$. Also, it is trivial that $n_c(G) \le m$. We obtain the following result.

THEOREM 1.1. *Let G be a p-group. If $n_c(G) = t$, then $dl(G) \le \lceil \log_2(2t + 3) \rceil$.*

Assume that G is a finite solvable nonnilpotent group. We establish an upper bound for the derived length $dl(G)$ in terms of $n_c(G)$.

THEOREM 1.2. *Let G be a solvable nonnilpotent group. If $n_c(G) = t$, then the derived length $dl(G) \le \lfloor (2t + 2)/3 \rfloor + 1$.*

Let G be a finite solvable group with $|G| = \prod_{i=1}^{k} p_i^{\alpha_i}$. For convenience, we define

$$s_p(G) = \sum_{i=1}^{k} \alpha_i.$$

For the remainder of this paper, all groups are finite and we refer to [8] for standard notation concerning the theory of finite groups.

2. The p-groups with $n_c(G) = t$

In this section, we bound the order of G' and the derived length $dl(G)$ for a p-group G in terms of the number of orders of nonnormal subgroups $n_c(G)$. We begin with four lemmas.

LEMMA 2.1 [2, Lemma 1.4]. *Let G be a p-group and let $N \trianglelefteq G$. If N has no abelian normal subgroups of G of type (p, p), then N is either cyclic or a 2-group of maximal class.*

LEMMA 2.2 [12, Lemma 1.4]. *Let N be a minimal nonnormal subgroup of a p-group P. Then N is cyclic.*

Suppose that G is a group and $N \trianglelefteq G$. Note that $n_c(G/N)$ is the number of orders of nonnormal subgroups of G containing N. The following lemma is easy but important, and it will frequently be used later in the paper.

LEMMA 2.3. *Let G be a group. Assume that N is a normal subgroup of G. Then $n_c(G/N) \le n_c(G)$. Moreover, if $n_c(G/N) = n_c(G)$, then the orders of all nonnormal subgroups of G are divisible by the order of N.*

PROOF. Obviously, the projection of the nonnormal subgroups of G/N onto G are still nonnormal, and hence $n_c(G/N) \le n_c(G)$. If there exists a nonnormal subgroup of G whose order is not divisible by $|N|$, then $n_c(G/N) < n_c(G)$. This completes the proof. □

Let G be a p-group. We say that $H_1 > H_2 > \cdots > H_k$ is a chain of nonnormal subgroups of G if each $H_i \ntrianglelefteq G$ and if $|H_i : H_{i+1}| = p$ for $1 \leq i \leq k-1$. Passman in [12] used $\mathrm{chn}(G)$ to denote the maximum of the lengths of the chains of nonnormal subgroups of G, and proved that if $\mathrm{chn}(G) = t$, then $s_p(G') \leq 2t + \lfloor 2/p \rfloor$. It is trivial that $\mathrm{chn}(G) \leq n_c(G)$. In the next lemma, we weaken the condition.

LEMMA 2.4. *Let G be a p-group. If $n_c(G) = t$, then $s_p(G') \leq 2t + 1$.*

PROOF. Let G be a p-group and assume that $n_c(G) = t$. If G has no elementary abelian normal subgroup of order p^2, then, by Lemma 2.1, G is either a cyclic group or a 2-group of maximal class. It is easy to see that $s_p(G') \leq n_c(G) + 1$ and the result follows.

Now, suppose that there exists an elementary abelian normal subgroup N of order p^2. In this case, we perform induction on t. If $t = 0$, clearly, G is Dedekind and $s_p(G') \leq 1$, as required. Next, suppose that $t \geq 1$. We consider the factor group G/N. Assume that M is a nonnormal subgroup of minimal order of G. Then M is cyclic by Lemma 2.2. Let $|M| = p^m$. We claim that $n_c(G/N) \leq t - 1$. If $p^m \leq p^2$, it follows from Lemma 2.3 that $n_c(G/N) \leq t - 1$. Conversely, if $p^m > p^2$, then G/N has no nonnormal subgroups of order p^{m-2}. Otherwise, there exists a noncyclic nonnormal subgroup of order p^m of G, which contradicts the minimality of M. Thus, according to Lemma 2.3, we have $n_c(G/N) \leq t - 1$, as claimed. Here, by induction on t, it follows that $s_p((G/N)') \leq 2(t-1) + 1$. Therefore,

$$s_p(G') \leq s_p(N) + s_p((G/N)') \leq 2t + 1.$$

The proof is complete. □

COROLLARY 2.5. *Let G be a nilpotent group. If $n_c(G) = t$, then $s_p(G') \leq 2t + 1$.*

PROOF. Let $P_i \in \mathrm{Syl}_{p_i}(G)$ and assume that $G = P_1 \times P_2 \times \cdots \times P_k$ with $n_c(G) = t$. If $k = 1$, the result is trivial by Lemma 2.4. Now, let $k \geq 1$. We assume that $G = H \times P_k$. Since $n_c(G) = t$, we have $n_c(H) < t/2$ and $n_c(P_k) \leq t/2$. By induction on k, it follows that $s_p(H') < t + 1$ and $s_p(P_k{}') < t + 1$. Therefore, $s_p(G') \leq 2t + 1$. □

We denote by $c(G)$ the nilpotent class and use G_i and $G^{(i)}$ to denote the ith terms of the lower central series and the commutator series for a group G, respectively. We are now ready to prove Theorem 1.1

PROOF OF THEOREM 1.1. Let G be a p-group and assume that $n_c(G) = t$. By Lemma 2.4, we see that $|G'| \leq p^{2t+1}$ and thus $c(G) \leq 2t + 2$. It suffices to show that $G^{(i)} \leq G_{2^i}$ for $i \geq 1$ since, by induction on i,

$$G^{(i)} = [G^{(i-1)}, G^{(i-1)}] \leq [G_{2^{i-1}}, G_{2^{i-1}}] \leq G_{2^i}.$$

Note that $1 = G_{2t+3} = G^{(dl(G))} \le G_{2^{dl(G)}}$. Consequently, $2^{dl(G)} \le 2t+3$, that is, $dl(G) \le \lceil \log_2(2t+3) \rceil$. This completes the proof. □

3. The solvable nonnilpotent groups with $n_c(G) = t$

In this section, we investigate the solvable nonnilpotent groups with $n_c(G) = t$ and prove the main result of this paper.

First, we state the characterisation of finite groups with $n_c(G) = 1$ and provide a basic fact about nilpotent groups.

LEMMA 3.1 [13, Theorem 2.3]. *Let G be a finite group. If all nonnormal subgroups of G possess the same order, then G is a finite p-group or $G = \langle a \rangle \rtimes \langle b \rangle$, where $o(a) = p_2$, $o(b) = {p_1}^{n_1}$, p_1, p_2 are primes with $p_1 < p_2$ and $[a, b^{p_1}] = 1$. Moreover, if $G = \langle a \rangle \rtimes \langle b \rangle$, as stated, then all nonnormal subgroups of G are of order ${p_1}^{n_1}$.*

LEMMA 3.2 [8, Lemma 5.1.2]. *Let G be a group and let $N \le Z(G)$. Then G is nilpotent if and only if G/N is nilpotent.*

For solvable nonnilpotent groups, we have the following further conclusion based on Lemma 2.3.

LEMMA 3.3. *Let G be a solvable nonnilpotent group. Then there exists a minimal normal subgroup N such that $n_c(G/N) \le n_c(G) - s_p(N)$.*

PROOF. By Lemma 2.3, $n_c(G/N) \le n_c(G)$. First, we claim that there exists a minimal normal subgroup N of G such that $n_c(G/N) < n_c(G)$. Let $P_i \in \mathrm{Syl}_{p_i}(G)$. Noting that G is nonnilpotent, we may assume that P_1 is a nonnormal Sylow subgroup of G. If, for $i \ge 2$, there exists a Sylow subgroup P_i such that P_i is nonnormal, we may assume that P_2 is nonnormal. Then $n_c(G/N) < n_c(G)$ is always true for any minimal normal subgroup $N \ne 1$. Otherwise, by Lemma 2.3, the orders of both P_1 and P_2 are divisible by the order of N, so that $N = 1$, which is a contradiction. On the other hand, if $P_i \trianglelefteq G$ for all $i \ge 2$, we may take $N \le P_2$. According to Lemma 2.3 again, $n_c(G/N) < n_c(G)$ since the order of P_1 is not divisible by the order of N. This proves the claim.

Since N is a minimal normal subgroup of G, it follows that N is an elementary abelian p-group and proper subgroups of N are nonnormal subgroups of G. There are $s_p(N) - 1$ nonnormal subgroups of G contained by N. Thus,

$$n_c(G/N) \le n_c(G) - (s_p(N) - 1).$$

Here, if $n_c(G/N) = n_c(G) - s_p(N) + 1$, then, similarly, both the orders of P_1 and P_2 are divisible by p, which is a contradiction. Hence, $n_c(G/N) \le n_c(G) - s_p(N)$ and the proof is complete. □

The next crucial lemma establishes an upper bound on the order of G' in terms of $n_c(G)$ for a solvable nonnilpotent group G.

LEMMA 3.4. *Let G be a solvable nonnilpotent group. If $n_c(G) = t$, then $s_p(G') \le t$.*

PROOF. Assume that $n_c(G) = t$. The proof will be done by induction to t. If $t = 1$, then, by Lemma 3.1,

$$G = \langle a \rangle \rtimes \langle b \rangle,$$

where $o(a) = p_2$, $o(b) = {p_1}^{n_1}$ and p_1, p_2 are different primes. Since $G/\langle a \rangle$ is cyclic, we have $s_p(G') = 1$.

Now, let $t \ge 2$. According to the proof of Lemma 3.3, it suffices to show that there exists a minimal normal subgroup N such that $n_c(G/N) < t$.

Case 1: G/N is nonnilpotent.

In this case, since $n_c(G/N) < t$, it follows that $s_p((G/N)') \le n_c(G/N)$ by induction on t. In addition, $|G'| = |G' \cap N||(G/N)'|$ because $(G/N)' \cong G'/(G' \cap N)$. Hence, $|N||(G/N)'|$ is divisible by $|G'|$. Therefore,

$$s_p(G') \le s_p(N) + s_p((G/N)') \le s_p(N) + n_c(G/N).$$

By Lemma 3.3, $n_c(G/N) \le n_c(G) - s_p(N)$, and hence

$$s_p(G') \le s_p(N) + n_c(G/N) \le n_c(G) = t.$$

This completes the proof in *Case 1*.

Case 2: G/N is nilpotent. In this case, we consider the following two situations.

Case 2a: there exists a minimal normal subgroup M such that $M \ne N$.

Since G is a nonnilpotent group, it follows that G/M is also nonnilpotent. Otherwise, since $G/(M \cap N) \lesssim G/M \times G/N$, we see that $G/(M \cap N)$ is nilpotent. However, $G/(M \cap N) \cong G$ is nonnilpotent, which is a contradiction. Now, assume that $|M| = p^m$ and $|N| = q^n$, where p, q are different primes. We consider two cases, namely, $m \ge 2$ and $m = 1$. If $m \ge 2$, since $N_1M_1 \ntrianglelefteq G$ for all $1 < M_1 < M$ and $1 \le N_1 \le N$, then

$$n_c(G/M) \le n_c(G) - (m-1)(n+1) \le n_c(G) - m.$$

Here, it follows easily by induction that $s_p((G/M)') \le n_c(G/M)$. This condition is similar to *Case 1* and it follows that

$$s_p(G') \le s_p(M) + n_c(G/M) \le n_c(G).$$

Now suppose that $m = 1$, that is, $|M| = p$. If there exists a nonnormal subgroup H such that $|H|$ is not divisible by p, then $n_c(G/M) \le n_c(G) - 1$ from Lemma 2.3, and so $s_p((G/M)') \le n_c(G/M)$ by induction. As before, the result holds. On the other hand, if, for every subgroup H of G whose order is not divisible by p, H is always normal, then we may assume that $G = KP$, where K is a Hall p'-subgroup of G. Obviously, all subgroups of K are normal and P is nonnormal. We consider the following two cases.

(i) If there exists a minimal normal subgroup T of G contained in K satisfying $T \ne N$, then G/T is nonnilpotent. It suffices to show that $n_c(G/T) \le n_c(G) - 1$ by Lemma 2.3, and thus $s_p((G/T)') \le n_c(G/T)$ by induction. As before, the result holds.

(ii) If N is a unique minimal normal subgroup of G contained in K, then K is a group of prime power order. It follows from Lemma 2.1 that K is either a cyclic group or a 2-group of maximal class. In addition, since every subgroup of K is a normal subgroup of G, it follows that K is either a cyclic group or a quaternion group Q_8. We claim that K is cyclic. Otherwise, $K \cong Q_8$. Note that $N \leq Z(G) \cap Q_8$ and G/N is nilpotent. According to Lemma 3.2, G is nilpotent, which is a contradiction. Now, let K be a cyclic group of order q^r with $r \geq 2$. For $1 \leq K_1 \leq K$, it follows that K_1P_1 is nonnormal as $P_1 \leq P$ and $P_1 \ntrianglelefteq G$. Also, there exists a maximal subgroup M of P that is normal in P, but MK_1 is a nonnormal subgroup of G for $1 \leq K_1 < K$. Hence,

$$n_c(G/K)(r+1) + r \leq t.$$

By Lemma 2.4, $s_p((G/K)') \leq 2(t-r)/(r+1) + 1$. Note that $n_c(G) = t \geq 2r+1$ and $r \geq 2$. Therefore,

$$\begin{aligned} s_p(G') \leq s_p(K) + s_p((G/K)') &\leq r + \frac{2(t-r)}{r+1} + 1 \\ &\leq \frac{r(r+1) + r(t-r) + (r+1)}{r+1} \leq \frac{r(t+1) + t - r}{r+1} = t. \end{aligned}$$

Case 2b: N is a unique minimal normal subgroup of G.

In this case, G/H is nilpotent for $1 \neq H \trianglelefteq G$. We can assume that $G/N = P_1 \rtimes P_2$ with $N \leq P_1$. Let $|N| = {p_1}^k$. Then there are $k-1$ nonnormal subgroups of G contained in N. Clearly, if NK is nonnormal in G for $K \leq G$, then $K \ntrianglelefteq G$. Note that $P_2N \trianglelefteq G$ but P_2 is a nonnormal subgroup of G. Moreover, we can always find $gN \in Z(G/N)$ such that $g \in G - N$ and $g^p \in N$ since G/N is nilpotent. Also, $\langle g \rangle N \trianglelefteq G$ but $\langle g \rangle$ is nonnormal in G. Therefore,

$$2n_c(G/N) + (k-1) + 1 + 1 \leq t.$$

It follows that $n_c(G/N) \leq (t-k-1)/2$ and, by Lemma 2.5, $s_p((G/N)') \leq t-k$. Hence,

$$s_p(G') \leq s_p(N) + s_p((G/N)') \leq k + t - k \leq t.$$

The proof is complete. □

Next, we will prove Theorem 1.2. To do this, we need the following lemma.

LEMMA 3.5. *Let G be a solvable group. If $s_p(G) = n$, then $dl(G) \leq \lfloor (2n+2)/3 \rfloor$.*

PROOF. We prove the result by induction on n. If $n = 1$, the result is trivially true. Assume that $n \geq 2$. If $s_p(G/G') \geq 2$, then $s_p(G') \leq n-2$. It follows that $dl(G') \leq \lfloor (2n-2)/3 \rfloor$ by the inductive hypothesis applied to G'. Hence,

$$dl(G) \leq \lfloor (2n-2)/3 \rfloor + 1 \leq \lfloor (2n+2)/3 \rfloor.$$

In this case, the proof is complete.

Now, let $s_p(G/G') = 1$, that is, $s_p(G') = n-1$. We may assume that $dl(G) = k+1$ where $k \geq 2$. Then $G^{(k)} > 1$. Also, suppose that N is a maximal abelian normal subgroup of G containing $G^{(k)}$. If $s_p(N) \geq 2$, we see that $s_p(G/N) \leq n-2$. Application of the inductive hypothesis to G/N yields $dl(G/N) \leq \lfloor (2n-2)/3 \rfloor$. Thus,

$$dl(G) \leq \lfloor (2n-2)/3 \rfloor + 1 \leq \lfloor (2n+2)/3 \rfloor,$$

and the result follows.

The remaining case is where $s_p(N) = 1$, which implies that $N = G^{(k)}$. Since $G/N = N_G(N)/C_G(N) \lesssim \mathrm{Aut}(N)$ is cyclic, it suffices to show that $N = G^{(k)} \leq Z(G')$. Hence,

$$N = G^{(k)} \leq Z(G^{(k-1)}).$$

Now $G^{(k-1)}$ is nonabelian since $G^{(k)} \neq 1$. We claim that $s_p(G^{(k-1)}) \geq 3$. Otherwise, $G^{(k-1)}$ is a nonabelian group of order pq with $p \neq q$. Since $G^{(k-1)}/G^{(k)}$ is cyclic, it suffices to show that $G^{(k-1)}$ is an abelian group, which is a contradiction. Hence, $s_p(G/G^{(k-1)}) \leq n-3$. Apply the inductive hypothesis to $G/G^{(k-1)}$. Then $dl(G/G^{(k-1)}) \leq \lfloor (2n-4)/3 \rfloor$. Therefore,

$$dl(G) \leq \lfloor (2n-4)/3 \rfloor + 2 = \lfloor (2n+2)/3 \rfloor.$$

The proof is complete. □

Finally, we are ready to prove Theorem 1.2.

Proof of Theorem 1.2. Suppose that G is a solvable nonnilpotent group with $n_c(G) = t$. From Lemma 3.4, $s_p(G') \leq t$, and hence, by Lemma 3.5,

$$dl(G') \leq \lfloor (2t+2)/3 \rfloor.$$

Hence, $dl(G) \leq \lfloor (2t+2)/3 \rfloor + 1$. The proof is complete. □

In addition, if G be a solvable nonnilpotent group, the number of prime divisors of $|G|$ can be bounded by $n_c(G)$. For convenience, we use $\pi(G)$ to denote the number of prime divisors of $|G|$.

Corollary 3.6. *Let G be a solvable nonnilpotent group. If $n_c(G) = t$, then $\pi(G) \leq t+1$.*

Proof. Assume that $\pi(G) \geq t+2$. Since G is a solvable group, G possesses a Sylow system $\mathcal{S}$. Suppose that $\mathcal{S} = \{P_1, P_2, \ldots, P_{t+2}, \ldots\}$. Note that G is nonnilpotent and we may assume that P_1 is a nonnormal Sylow subgroup of G. Let

$$\mathcal{T} = \{P_1P_2, P_1P_3, P_1P_4, \ldots, P_1P_{t+2}\}.$$

Obviously, for $1 \leq i \leq t+2$, P_1P_i is a subgroup of G. If, for the set $\mathcal{T}$, there are two or more normal subgroups of G, then P_1 is a normal subgroup, which is a contradiction. Thus, at most one normal subgroup is contained in the set $\mathcal{T}$ and it follows that $n_c(G) \geq t+1$. This contradicts the hypothesis and the proof is complete. □

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