1
Stars

In antiquity stars were generally supposed to be bright spots fixed on a sphere that revolves once a day about the Earth. Modern astrophysics began in the early nineteenth century, with the discovery by Joseph von Fraunhofer (1787–1826) of dark lines in the spectra of the Sun and some bright stars, which showed that they all have similar composition, and with the measurement by Friedrich Bessel (1766–1828) and William Wollaston (1784–1826) of the distances of stars like Y Cygni and α Centauri, which showed that their absolute luminosity is not very different from that of the Sun. By the end of the nineteenth century hydrodynamics and thermodynamics had been applied to the structure of the Sun and stars. Only the source of their energy was still mysterious, not to be understood in detail until the development of nuclear physics in the 1930s.

It would be most logical to begin this chapter with an introduction to the physics required to understand modern stellar theory, including calculations of nuclear energy production and opacity, and only then go on to the stars themselves. Logical, but perhaps a bit boring. It is not always possible to maintain one's interest in the details of nuclear and atomic physics without knowing how these results are to be used. So in this chapter we start with the stars.

First in Section 1.1 we derive the equations of hydrostatic equilibrium for stars. This leads to the virial theorem, which illuminates the stars' early history. Then in Sections 1.2 and 1.3 we adopt a simple model in which energy is transported in the star solely by radiation, leaving convection for later sections. In this model we can see how the structure of the star is uniquely determined by the formulas that give pressure, opacity, and nuclear energy production in terms of density and temperature, with just one free stellar parameter, that can be taken to be the star's total mass. With this as motivation, in Sections 1.4 and 1.5 we describe the physics underlying the formulas for opacity and nuclear energy generation. It turns out to be a fair approximation to take the opacity and energy generation as well as the pressure as proportional to products of powers of density and temperature. This approximation is used in Section 1.6 to give formulas for stellar properties, including luminosity, radius, central temperature, etc., in terms of the star's mass. We come to convection in Section 1.7, and

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show that the presence of convective zones does not greatly change the results of Sections 1.3 and 1.6.

We then turn to stars of a more exotic breed. In Section 1.8 we consider the general class of stars in which the pressure is simply proportional to some power of the density. Where this power is close to 4/3, the star is close to instability. The detailed conditions for stellar instability are worked out in Section 1.9. Then we consider white dwarf and neutron stars in Section 1.10 and supermassive stars in Section 1.11, using the results of Section 1.8 to describe their structure and of Section 1.9 to find where they become unstable.

This chapter deals only with isolated single stars. Binary stars and their emission of gravitational radiation will be considered in the following chapter.

1.1 Hydrostatic Equilibrium

Suppose a star is in equilibrium and is spherically symmetric, so that the mass density ρ and pressure p are functions only of the distance r from the center. Consider a thin spherical shell of radius r and thickness dr. Its mass is $4\pi r^2 \rho(r)\, dr$, so it feels a gravitational force

$$F_{\text{gravitational}} = -G\frac{4\pi r^2 \rho(r)\, dr\ \mathcal{M}(r)}{r^2} = -4\pi G\rho(r)\mathcal{M}(r)\, dr, \qquad (1.1.1)$$

where $\mathcal{M}(r)$ is the total mass interior to the radius r:

$$\mathcal{M}(r) = \int_0^r 4\pi r'^2 \rho(r')\, dr'. \qquad (1.1.2)$$

The minus sign in Eq. (1.1.1) indicates that this force points inward. The shell also feels an outward buoyant force, equal to the pressure force on the inner surface of the shell minus the pressure force on its outer surface:

$$F_{\text{buoyant}} = 4\pi r^2\big[p(r) - p(r+dr)\big] = -4\pi r^2\, p'(r)\, dr. \qquad (1.1.3)$$

In equilibrium the sum of these forces vanishes, so

$$\frac{dp(r)}{dr} = -\frac{G\mathcal{M}(r)\rho(r)}{r^2}. \qquad (1.1.4)$$

This is the fundamental equation of hydrostatic equilibrium for stars. For some purposes it is convenient to rewrite Eq. (1.1.2) also as a differential equation

$$\frac{d\mathcal{M}(r)}{dr} = 4\pi r^2 \rho(r), \qquad (1.1.5)$$

with initial condition $\mathcal{M}(0) = 0$.

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Equations (1.1.4) and (1.1.5) lead to a useful inequality for the pressure.[1] We note that

$$\frac{d}{dr}\left[p(r)+\frac{G\mathcal{M}^2(r)}{8\pi r^4}\right]=-\frac{G\mathcal{M}(r)\rho(r)}{r^2}-\frac{G\mathcal{M}^2(r)}{2\pi r^5}+\frac{G\mathcal{M}(r)\mathcal{M}'(r)}{4\pi r^4}.$$

The first and third terms cancel, leaving the negative second term, so

$$\frac{d}{dr}\left[p(r)+\frac{G\mathcal{M}^2(r)}{8\pi r^4}\right]\leq 0. \tag{1.1.6}$$

In particular, assuming that the density is finite at $r=0$, we have $\mathcal{M}(r)\propto r^3$ for $r\to 0$, so $\mathcal{M}^2(r)/r^4\to 0$ for $r\to 0$. Assuming also that the pressure vanishes at some nominal stellar radius R, and taking $\mathcal{M}(R)=M$, the quantity in square brackets in (1.1.6) is $p(0)$ at $r=0$ and $GM^2/8\pi R^4$ at $r=R$, so (1.1.6) yields a useful inequality for the central pressure:

$$p(0)\geq\frac{GM^2}{8\pi R^4}=4.44\times 10^{14}\big(M/M_\odot\big)^2\big(R/R_\odot\big)^{-4}\,\text{dyne/cm}^2. \tag{1.1.7}$$

(The subscript $\odot$ denotes values for the Sun. For comparison, recall that one standard atmosphere equals 1.013×10^6 dyne/cm^2.) Using methods described in this chapter, it has been calculated that the pressure at the center of the Sun is $p_\odot(0)\simeq 2\times 10^{17}$ dyne/cm^2, in accord with the inequality (1.1.7).

Equation (1.1.4) can be used to derive a simple formula for the total gravitational potential energy Ω of the star, related to the virial theorem of celestial mechanics. We define $-\Omega$ as the energy required to remove the mass of the star to infinity, peeling it shell by shell from the outside in. Once all the mass exterior to a radius r has been removed, the energy required to remove the shell at r of thickness dr is the integral over the distance r' between the shell and the star's center of the gravitational force $G\mathcal{M}(r)/r'^2\times 4\pi r^2\rho(r)\,dr$ exerted by a mass $\mathcal{M}(r)$ on the shell's mass:

$$G\mathcal{M}(r)\times 4\pi r^2\rho(r)\,dr\times\int_r^\infty\frac{dr'}{r'^2}=4\pi Gr\mathcal{M}(r)\rho(r)\,dr,$$

so the total gravitational binding energy is

$$-\Omega=4\pi G\int_0^R r\mathcal{M}(r)\rho(r)\,dr, \tag{1.1.8}$$

where R is the radius of the nominal stellar surface, where $p(R)=0$. Using Eq. (1.1.4) for $-G\mathcal{M}\rho$, we have

[1] S. Chandrasekhar, *An Introduction to the Study of Stellar Structure* (University of Chicago Press, Chicago, IL, 1939), Chapter III. This chapter also gives other general theorems derived from Eqs. (1.1.4) and (1.1.5).

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$$\Omega = 4\pi \int_0^R \frac{dp(r)}{dr} r^3 \, dr = -3 \int_0^R p(r) \, 4\pi r^2 \, dr, \qquad (1.1.9)$$

in which we have integrated by parts, using the vanishing of $r^3 p(r)$ at both endpoints of the integral.

Incidentally, the definition of Ω can also be written in terms of the familiar gravitational potential

$$\phi(r) = -G \int_r^\infty \mathcal{M}(r') \, dr'/r'^2. \qquad (1.1.10)$$

(This formula satisfies the defining condition that $-\phi'(r)$ should equal the Newtonian force per mass $-G\mathcal{M}(r)/r^2$. An arbitrary additive constant has been chosen so that $\phi(r) \to 0$ for $r \to \infty$.) Integrating by parts, we have

$$\int_0^\infty \phi(r)\mathcal{M}'(r) \, dr = -\int_0^\infty \phi'(r)\mathcal{M}(r) \, dr = -G \int_0^\infty \mathcal{M}^2(r) dr/r^2.$$

With $-1/r^2 = d/dr(1/r)$ and integrating by parts again, we see that the final expression is 2Ω, so

$$\Omega = \frac{1}{2} \int_0^\infty \phi(r)\mathcal{M}'(r) \, dr. \qquad (1.1.11)$$

The integral here is the sum of the gravitational energies of each bit of stellar matter, due to the gravitational field of each bit of matter, so in the integral each bit of stellar matter is counted twice, a double counting corrected by the factor $1/2$.

The total energy of the star is the sum of Ω and the star's thermal energy, given by

$$\Upsilon \equiv \int_0^R \mathcal{E}(r) \, 4\pi r^2 \, dr, \qquad (1.1.12)$$

where $\mathcal{E}(r)$ is the density of internal thermal energy, not including rest mass energies or gravitational energy. The total non-relativistic energy (not including rest masses) of the star is then

$$E = \Upsilon + \Omega = \int_0^R \left[\mathcal{E}(r) - 3p(r)\right] 4\pi r^2 \, dr. \qquad (1.1.13)$$

We see that the star has negative energy and is therefore stable against dispersal of its matter to infinity if $\mathcal{E}(r) < 3p(r)$.

It is frequently the case that the density $\mathcal{E}$ of internal energy is proportional to the pressure, a relation conventionally written as

$$\mathcal{E} = p/(\Gamma - 1). \qquad (1.1.14)$$

(Such stars are called *polytropes*, and are discussed in detail in Section 1.8.) For instance, for an ideal gas of monatomic particles with number density n we have $p = nk_B T$ and $\mathcal{E} = 3nk_B T/2$ (where k_B is Boltzmann's constant), so here $\Gamma = 5/3$. For radiation $p = \mathcal{E}/3$, so $\Gamma = 4/3$. In such cases, the thermal and gravitational energies of the star are given in terms of its total non-relativistic energy by Eqs. (1.1.9), (1.1.12), and (1.1.14) as

$$\Upsilon = -\frac{E}{3\Gamma - 4}, \quad \Omega = \frac{(\Gamma - 1)E}{\Gamma - 4/3}. \tag{1.1.15}$$

The star will explode if E is positive, so stability requires that $E < 0$, and since Eq. (1.1.9) gives $\Omega < 0$, this means that $\Gamma > 4/3$. Stars whose pressure is dominated by highly relativistic particles (such as very massive ordinary stars and white dwarfs and neutron stars with masses near their upper limit) have Γ only slightly above 4/3 and are therefore trembling on the brink of instability.

Equation (1.1.15) plays a crucial role in governing the early history of stars. A cloud of cold diffuse gas will have little internal or gravitational energy, so its total energy E will be small. Unless the cloud is at absolute zero temperature it will radiate some light, chiefly at infrared wavelengths. If its total energy becomes negative, the cloud will no longer be able to disperse. According to Eq. (1.1.15), as the cloud loses energy then, as long as $\Gamma > 4/3$, Ω will decrease, becoming increasingly negative, but *the internal energy* Υ *will increase.* The star behaves as if it has negative specific heat; the more it loses energy, the hotter it gets. With increasing temperature the star radiates energy more rapidly, and the process accelerates. Eventually the central temperature of the star becomes so high that nuclei can penetrate the Coulomb repulsion that separates them (discussed in Section 1.5); nuclear energy generation begins and increases until it balances the energy lost by radiation; and the star becomes stable, at least until the nuclear fuel at the star's center is exhausted. Paradoxically, the onset of nuclear reactions *stops* the heating of the star.

As a protostar radiates energy and heats up, it also contracts. We can define a mass-weighted mean radius $\bar{r}$, by

$$M^2 \bar{r}^{-1} \equiv \int_0^R r \mathcal{M}(r) \rho(r)\, dr.$$

Then Eq. (1.1.8) may be written $\Omega = -4\pi G M^2/\bar{r}$. As $-\Omega$ increases, $\bar{r}$ must decrease.

Before the discovery of radioactivity, with its implications for the source of heat of stars, William Thomson (1824–1907, a.k.a. Lord Kelvin), estimated the length of time that the Sun could have been shining with its present luminosity, deriving its heat solely from gravitational contraction.[2] As we have seen, the

2 W. Thomson, *Phil. Mag.* **23**, 158 (1862); reprinted in *Mathematical and Physical Papers by Sir William Thomson, Baron Kelvin*, ed. J. Larmor (Cambridge University Press, Cambridge, 1911).

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energy E of a star is related to its gravitational energy Ω by Eq. (1.1.15), which for $\Gamma = 5/3$ gives

$$E = \Omega/2. \tag{1.1.16}$$

We can get a fair estimate of Ω by taking $\rho(r)$ constant in Eq. (1.1.6), so that $\rho(r) = 3M/4\pi R^3$ and $\mathcal{M}(r) = Mr^3/R^3$, in which case

$$E \simeq -\frac{1}{2} \times 4\pi G \times \frac{MR^2}{5} \times \frac{3M}{4\pi R^3} = -\frac{3GM^2}{10R}. \tag{1.1.17}$$

This is minus the energy the star has lost in contracting from a cloud with negligible gravitational and thermal energy, if no internal energy sources have contributed to its heat since the contraction began. For the Sun, $M_\odot = 1.9891 \times 10^{33}$ g and $R = 6.960 \times 10^{10}$ cm, so $E \simeq -1.1 \times 10^{48}$ ergs. The Sun's present luminosity is $L_\odot = 3.9 \times 10^{33}$ erg/sec, so in the absence of internal energy sources it could only have been shining at that rate for roughly $|E|/L_\odot \simeq 10^7$ years.[3] Kelvin's 1862 conclusion was not very different: "It seems therefore most probable that the sun has not illuminated the earth for 100,000,000 years." Already in the nineteenth century it was known that this was too short a time for the evolution of life and of features of the Earth's surface, but the path to a resolution of the problem first appeared with the discovery of nuclear energy in 1897.

(By the way, this calculation is sometimes done setting the energy radiated during the Sun's previous life equal to $|\Omega|$ rather than to $|E|$. This ignores the fraction of the energy of gravitational contraction that goes into heating the Sun. As we have seen, that fraction is given by the virial theorem as 1/2 for $\Gamma = 5/3$, so the Sun's age calculated here is reduced by a factor 1/2. This serves to emphasize the peculiar aspect of gravitation mentioned above, that as a young star condenses under the influence of gravitation without the production of nuclear energy, it heats up, so that the temperature of a gravitationally condensing body increases as it loses energy.)

In some cases, such as zero-temperature white dwarf stars, the pressure p is a known function of the mass density ρ, which otherwise depends only on chemical composition and universal constants such as $\hbar$, c, and m_e. (This is discussed in Section 1.10.) In such cases, Eqs. (1.1.4) and (1.1.5) yield a definite stellar model.

More generally $p(r)$ depends on the temperature at r as well as on $\rho(r)$, and so Eqs. (1.1.4) and (1.1.5) do not in themselves lead to any definite result for the structure of a star. For this, we also need to understand how energy is transported in the star. There are two chief mechanisms for energy transport, radiation and convection, to be studied in the following sections.

[3] See e.g. C. J. Hansen, S. D. Kawaler, and V. Trimble, *Stellar Interiors*, 2nd edn. (Springer, New York, 2004).

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1.2 Radiative Energy Transport

The equations of hydrostatic equilibrium involve the pressure, which depends on the temperature, so in order to use them we need equations of energy transport, that dictate how the temperature varies through the star. There are two chief mechanisms of energy transport: radiation and convection. (Because mean free paths are small in stars, conduction is much less important.) In this section we shall work out the coupled differential equations, Eqs. (1.2.28) and (1.2.30), that govern the r-dependent temperature and luminosity for a star in which energy transport is dominated by radiation. Convection will be considered in Section 1.7.

Let $\ell(\hat{n}, \mathbf{x}, \nu, t)\, d^2\hat{n}\, d\nu$ be the energy per volume at position $\mathbf{x}$ and time t of photons with directions within a solid angle $d^2\hat{n}$ around the unit vector $\hat{n}$ and frequencies between ν and $\nu + d\nu$. Our first task is to calculate various contributions to the rate of change of $\ell(\hat{n}, \mathbf{x}, \nu, t)$. Later we shall assume that the total rate of change of $\ell(\hat{n}, \mathbf{x}, \nu, t)$ vanishes, and use that requirement as the condition of equilibrium when energy transport is dominated by radiation.

There are four contributions to this rate of change.

Transport

If nothing is happening to the radiation, then at time $t+dt$ the energy of photons per volume, per solid angle, and per frequency interval traveling in direction $\hat{n}$ with frequency ν at position $\mathbf{x}$ will be what it was at time t and position $\mathbf{x}-c\hat{n}\, dt$:

$$\ell(\hat{n}, \mathbf{x}, \nu, t + dt) = \ell(\hat{n}, \mathbf{x} - c\hat{n}\, dt, \nu, t).$$

Thus the rate of change of ℓ solely due to the transport of radiation is

$$\left(\frac{\partial}{\partial t}\ell(\hat{n}, \mathbf{x}, \nu, t)\right)_{\text{transport}} = -c\hat{n} \cdot \nabla\ell(\hat{n}, \mathbf{x}, \nu, t). \tag{1.2.1}$$

Absorption

It is important to distinguish here between absorption and scattering. We will understand absorption to be any process in which an incident photon disappears without producing a photon whose direction is correlated with that of the incident photon. For instance, in a so-called *bound–free* transition, a photon gives its energy to raising the energy of a bound electron so that it becomes a free particle. In a *free–free* transition the incident photon is absorbed by a free electron in the Coulomb field of an ion (which allows such a transition to conserve energy and momentum). In either case the final free electron merges with the surrounding medium, increasing its temperature. The medium may then give up this energy by emitting photons, but the directions of these photons will

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be uncorrelated with the initial photon's direction, so these transitions count as absorption. In a *bound–bound* transition the energy of the initial photon goes to raise the atom to a higher energy state. Typically the atom then undergoes collisions, which either drain the excitation energy or change the excited state so that even if it decays radiatively the final photon direction is uncorrelated with the direction of the initial photon. In either case, these bound–bound transitions also count as absorption.

Suppose that the net fraction of radiation of frequency ν absorbed at position $\mathbf{x}$ and time t in a time interval dt is $c\kappa_{\text{abs}}(\mathbf{x},\nu,t)\rho(\mathbf{x},t)\,dt$, where ρ is the mass density and κ_{abs} is a coefficient characterizing the medium, called the *absorption opacity*. (As discussed in Section 1.4, stimulated emission counts here as negative absorption.) A factor of the speed of light is inserted here to give $1/\kappa_{\text{abs}}\rho$ the dimensions of length; it is the average distance that a typical photon travels before being absorbed in a homogeneous medium. Then the rate of change of ℓ due to absorption is

$$\left(\frac{\partial}{\partial t}\ell(\hat{n},\mathbf{x},\nu,t)\right)_{\text{absorption}} = -c\kappa_{\text{abs}}(\mathbf{x},\nu,t)\rho(\mathbf{x},t)\ell(\hat{n},\mathbf{x},\nu,t). \tag{1.2.2}$$

For a two-body absorption process like a bound–free or bound–bound transition $\kappa_{\text{abs}}\rho$ is the absorption cross section times the number density of absorbers, and hence κ_{abs} is the absorption cross section divided by the mean absorber mass. (As we will see in Section 1.4, free–free transitions are more complicated.)

Scattering

These are processes in which the disappearance of an initial photon yields a final photon, whose direction generally differs from the initial direction, but is correlated with it. The leading example is Thomson scattering, the elastic scattering of photons with energies well below $m_e c^2$ on non-relativistic electrons. A bound–bound transition could also be regarded as a scattering, if the excited atom were to decay radiatively before the atom undergoes collisions that wipe out any correlation of the final and initial photons.

The fraction of radiation energy of frequency ν traveling in a direction $\hat{n}$ that in a time interval dt at time t is scattered at position $\mathbf{x}$ into a solid angle $d^2\hat{n}'$ around a final direction $\hat{n}'$ is written as $c\kappa_{\mathcal{S}}(\hat{n}\to\hat{n}';\mathbf{x},\nu,t)\,\rho(\mathbf{x},t)\,d^2\hat{n}'\,dt$, where $\kappa_{\mathcal{S}}$ is a coefficient characterizing the scatterers, independent of the photon distribution function ℓ. In calculating the rate of change of $\ell(\hat{n},\mathbf{x},\nu,t)$, we must now take into account not only the scattering of photons at position $\mathbf{x}$ and time t with initial directions $\hat{n}$ into any other directions $\hat{n}'$, but also the earlier scattering of photons elsewhere with arbitrary initial directions $\hat{n}'$ into the position $\mathbf{x}$ and direction $\hat{n}$. For this purpose, we assume that $1/\kappa_{\mathcal{S}}\rho$ is so much smaller than the distance over which conditions in the star vary that we can assume that any photon that after scattering reaches a given position $\mathbf{x}$ at time t can only have been scattered at a position and time where the photon distribution function ℓ

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and density ρ were essentially the same as at $\mathbf{x}$ and t. (This may not be true near the surface of a star.) Then the contribution of scattering to the rate of change of ℓ is

$$\left(\frac{\partial}{\partial t}\ell(\hat{n},\mathbf{x},\nu,t)\right)_{\text{scattering}} = c\rho(\mathbf{x},t)\int d^2\hat{n}'\left[-\kappa_{\mathcal{S}}(\hat{n}\to\hat{n}';\mathbf{x},\nu,t)\ell(\hat{n},\mathbf{x},\nu,t) + \kappa_{\mathcal{S}}(\hat{n}'\to\hat{n};\mathbf{x},\nu,t)\ell(\hat{n}',\mathbf{x},\nu,t)\right]. \tag{1.2.3}$$

(We are here ignoring any shift in frequency in scattering. Such shifts are small if the photon energy $h\nu$ is much less than the rest mass energy of the particles responsible for scattering, and if the velocity of these particles is much less than the speed of light, though even small frequency shifts can be important when scattering cross sections are very sensitive to frequency, as in resonant scattering.)

If (as is usually the case) the scattering is a two-body process, with photons scattered each time by a single particle of the medium, we have

$$\kappa_{\mathcal{S}}(\hat{n}\to\hat{n}';\mathbf{x},\nu,t) = N_{\text{scat}}(\mathbf{x},t)\sigma(\hat{n}\to\hat{n}',\nu),$$

where $\sigma(\hat{n}\to\hat{n}',\nu)$ is the differential scattering cross section, and $N_{\text{scat}}(\mathbf{x},t)$ is the ratio of the number density of scattering centers to the mass density ρ; in other words, it is the number of scattering centers per gram.

Emission (thermal and nuclear)

We suppose that the radiation energy emitted in any direction per time, per volume, per solid angle, and per frequency interval at position $\mathbf{x}$ and time t is

$$\left(\frac{\partial}{\partial t}\ell(\hat{n},\mathbf{x},\nu,t)\right)_{\text{emission}} = j(\mathbf{x},\nu,t)\rho(\mathbf{x},t)/4\pi, \tag{1.2.4}$$

where j is another coefficient characterizing the medium and the radiation field. Note that j includes any radiation emitted isotropically subsequent to photon absorption, along with the ordinary thermal radiation from the stellar material, which is heated by nuclear processes. (Stimulated emission, which creates a photon with the same momentum and helicity as one already present, will be included as a negative term in the absorption coefficient κ_{abs}.)

Putting together these four terms, we have

$$\begin{aligned}\frac{\partial}{\partial t}\ell(\hat{n},\mathbf{x},\nu,t) = &-c\hat{n}\cdot\nabla\ell(\hat{n},\mathbf{x},\nu,t)\\ &-c\kappa_{\text{abs}}(\mathbf{x},\nu,t)\rho(\mathbf{x},t)\ell(\hat{n},\mathbf{x},\nu,t)\\ &+c\rho(\mathbf{x},t)\int d^2\hat{n}'\big[-\kappa_{\mathcal{S}}(\hat{n}\to\hat{n}';\mathbf{x},\nu,t)\ell(\hat{n},\mathbf{x},\nu,t)\\ &\qquad+\kappa_{\mathcal{S}}(\hat{n}'\to\hat{n};\mathbf{x},\nu,t)\ell(\hat{n}',\mathbf{x},\nu,t)\big]\\ &+j(\mathbf{x},\nu,t)\rho(\mathbf{x},t)/4\pi.\end{aligned} \tag{1.2.5}$$

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If we now require the photon distribution function ℓ and the stellar material to be unchanging, we arrive at the condition of radiative equilibrium

$$
\begin{aligned}
0 = &-c\hat{n}\cdot\nabla\ell(\hat{n},\mathbf{x},\nu) \\
&- c\kappa_{\text{abs}}(\mathbf{x},\nu)\rho(\mathbf{x})\ell(\hat{n},\mathbf{x},\nu) \\
&+ c\rho(\mathbf{x})\int d^2\hat{n}'\,\big[-\kappa_{\mathcal{S}}(\hat{n}\to\hat{n}';\mathbf{x},\nu)\ell(\hat{n},\mathbf{x},\nu) \\
&\qquad\qquad + \kappa_{\mathcal{S}}(\hat{n}'\to\hat{n};\mathbf{x},\nu)\ell(\hat{n}',\mathbf{x},\nu)\big] \\
&+ j(\mathbf{x},\nu)\rho(\mathbf{x})/4\pi, \qquad (1.2.6)
\end{aligned}
$$

in which we assume that κ, j, and ρ as well as ℓ are all independent of time, and so drop the argument t everywhere.

We want to use this result to derive relations between three fundamental quantities, the radiation energy per volume and per frequency interval

$$\mathcal{E}_{\text{rad}}(\mathbf{x},\nu) \equiv \int d^2\hat{n}\;\ell(\hat{n},\mathbf{x},\nu), \qquad (1.2.7)$$

the flux vector of radiation energy per frequency interval

$$\Phi_i(\mathbf{x},\nu) \equiv c\int d^2\hat{n}\;\hat{n}_i\ell(\hat{n},\mathbf{x},\nu), \qquad (1.2.8)$$

and the spatial part of the energy-momentum tensor of radiation per frequency interval

$$\Theta_{ij}(\mathbf{x},\nu) \equiv \int d^2\hat{n}\;\hat{n}_i\hat{n}_j\ell(\hat{n},\mathbf{x},\nu). \qquad (1.2.9)$$

(Here i and j etc. run over the Cartesian coordinate indices 1, 2, 3. Note that $\Phi_i\mathcal{N}_i\,dA\,d\nu$ is the rate at which radiant energy of frequency between ν and $\nu + d\nu$ passes through a small patch with area dA and unit normal $\mathcal{N}_i$.)

To derive our relations, we first integrate Eq. (1.2.6) over the direction of $\hat{n}$, which gives

$$\nabla\cdot\mathbf{\Phi}(\mathbf{x},\nu) = -c\kappa_{\text{abs}}(\mathbf{x},\nu)\rho(\mathbf{x})\mathcal{E}_{\text{rad}}(\mathbf{x},\nu) + j(\mathbf{x},\nu)\rho(\mathbf{x}). \qquad (1.2.10)$$

Note that the scattering term in Eq. (1.2.6) does not contribute here, because the integrand in this term is antisymmetric in $\hat{n}$ and $\hat{n}'$.

Let us pause at this point to note a relation between the quantities $\kappa(\mathbf{x},\nu)$, $j(\mathbf{x},\nu)$, and $\mathcal{E}_{\text{rad}}(\mathbf{x},\nu)$. These quantities depend only on ν and on the density $\rho(\mathbf{x})$, temperature $T(\mathbf{x})$, and chemical composition at $\mathbf{x}$; they vary with position because $\rho(\mathbf{x})$ and $T(\mathbf{x})$ and perhaps the chemical composition vary with position, but they have no independent dependence on position. That is, we can write $\kappa(\mathbf{x},\nu)$, $j(\mathbf{x},\nu)$, and $\mathcal{E}_{\text{rad}}(\mathbf{x},\nu)$ as ν-dependent functions only of $\rho(\mathbf{x})$, $T(\mathbf{x})$, and chemical composition at $\mathbf{x}$. Now, if the energy emission density $j(\mathbf{x},\nu)$ received no contribution from nuclear processes then the medium could come to equilibrium with thermal emission balancing absorption at each point and

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at each frequency, as in a black-body cavity. We could thus imagine a homogeneous medium that everywhere had the same temperature, density, and chemical composition that the real star has at a given position $\mathbf{x}$. For this hypothetical homogeneous medium, Eq. (1.2.10) would require that $j = c\kappa_{\rm abs}\mathcal{E}_{\rm rad}$. Hence in the inhomogeneous real star, we have

$$j(\mathbf{x},\nu) = c\kappa_{\rm abs}(\mathbf{x},\nu)\mathcal{E}_{\rm rad}(\mathbf{x},\nu) + \epsilon(\mathbf{x},\nu), \tag{1.2.11}$$

where $\epsilon(\mathbf{x},\nu)$ is the rate per gram and per frequency interval of energy generation from nuclear reactions. Equation (1.2.10) then reads

$$\nabla\cdot\mathbf{\Phi}(\mathbf{x},\nu) = \epsilon(\mathbf{x},\nu)\rho(\mathbf{x}). \tag{1.2.12}$$

We next multiply Eq. (1.2.6) with $\hat{n}_i$ and then integrate the product over the directions of $\hat{n}$:

$$\begin{aligned}\nabla_j\Theta_{ij}(\mathbf{x},\nu) = &-\kappa_{\rm abs}(\mathbf{x},\nu)\rho(\mathbf{x})\Phi_i(\mathbf{x},\nu)\\ &-c\rho(\mathbf{x})\int d^2\hat{n}'\int d^2\hat{n}\;\hat{n}_i\,\Big[\kappa_{\mathcal{S}}(\hat{n}\to\hat{n}';\mathbf{x},\nu)\ell(\hat{n},\mathbf{x},\nu)\\ &\qquad -\kappa_{\mathcal{S}}(\hat{n}'\to\hat{n};\mathbf{x},\nu)\ell(\hat{n}',\mathbf{x})\Big].\end{aligned}$$

(In accord with the usual summation convention, the index j is here summed over the values 1, 2, 3. The emission term in Eq. (1.2.6) does not contribute here, because $j\rho$ is independent of photon direction.) Under the assumption that $\kappa_{\mathcal{S}}$ is invariant under rotations together of both initial and final photon directions, we may define

$$\int d^2\hat{n}'\;\kappa_{\mathcal{S}}(\hat{n}\to\hat{n}';\mathbf{x},\nu) \equiv \kappa_{\rm out}(\mathbf{x},\nu) \tag{1.2.13}$$

and

$$\int d^2\hat{n}\;\hat{n}_i\kappa_{\mathcal{S}}(\hat{n}'\to\hat{n};\mathbf{x},\nu) \equiv \hat{n}'_i\kappa_{\rm in}(\mathbf{x},\nu). \tag{1.2.14}$$

It follows then that

$$c\int d^2\hat{n}'\int d^2\hat{n}\;\hat{n}_i\kappa_{\mathcal{S}}(\hat{n}\to\hat{n}';\mathbf{x},\nu)\ell(\hat{n},\mathbf{x},\nu) = \kappa_{\rm out}(\mathbf{x},\nu)\Phi_i(\mathbf{x},\nu)$$

and

$$c\int d^2\hat{n}'\int d^2\hat{n}\;\hat{n}_i\kappa_{\mathcal{S}}(\hat{n}'\to\hat{n};\mathbf{x},\nu)\ell(\hat{n}',\mathbf{x},\nu) = \kappa_{\rm in}(\mathbf{x},\nu)\Phi_i(\mathbf{x},\nu),$$

and therefore

$$c\,\nabla_j\Theta_{ij}(\mathbf{x},\nu) = -\kappa(\mathbf{x},\nu)\rho(\mathbf{x})\Phi_i(\mathbf{x},\nu), \tag{1.2.15}$$

where κ is the *total opacity*:

$$\kappa(\mathbf{x},\nu) \equiv \kappa_{\rm abs}(\mathbf{x},\nu) + \kappa_{\rm out}(\mathbf{x},\nu) - \kappa_{\rm in}(\mathbf{x},\nu). \tag{1.2.16}$$

To derive a formula for κ_{in} that clarifies its relation to κ_{out}, we contract Eq. (1.2.14) with $\hat{n}'$. This gives

$$\kappa_{\text{in}}(\mathbf{x},\nu) = \int d^2\hat{n}\ (\hat{n}\cdot\hat{n}')\kappa_{\mathcal{S}}(\hat{n}'\to\hat{n};\mathbf{x},\nu) = \int d^2\hat{n}'\ (\hat{n}\cdot\hat{n}')\kappa_{\mathcal{S}}(\hat{n}\to\hat{n}';\mathbf{x}), \tag{1.2.17}$$

which differs from the definition (1.2.13) of κ_{out} by the factor $\hat{n}\cdot\hat{n}'$. Textbook treatments of opacity often do not distinguish between absorption and scattering, and so do not encounter the term κ_{in}. This is obviously wrong, because κ_{out} would not vanish even if the scattering were restricted to an infinitesimal neighborhood of the forward direction $\hat{n}' = \hat{n}$, in which case the scattering should have no effect. The inclusion of κ_{in} removes this paradox, since

$$\kappa_{\text{out}}(\mathbf{x},\nu) - \kappa_{\text{in}}(\mathbf{x},\nu) = \int d^2\hat{n}'\ [1-\hat{n}\cdot\hat{n}']\kappa_{\mathcal{S}}(\hat{n}\to\hat{n}';\mathbf{x},\nu), \tag{1.2.18}$$

which vanishes for purely forward scattering, as it must. The authors of these treatments can get away with this oversight, because, for reasons described in Section 1.4, κ_{in} happens to vanish for Thomson scattering. But κ_{in} might matter in other scattering, such as bound–bound transitions in which the excited state decays radiatively, with the final photon direction correlated with that of the incoming photon.

So far, this has been exact, aside from the approximations made in deriving Eq. (1.2.3). We will now extend the approximation of short mean free path used there to the rest of our analysis. That is, we assume again that the opacity κ is so large that the mean path $1/\kappa\rho$ of typical photons is much smaller than the distance over which conditions vary. This is appropriate for the interiors of most stars, though not necessarily for their outer layers. It follows that to a good approximation $\ell(\hat{n},\mathbf{x},\nu)$ is independent of the photon direction $\hat{n}$, so that Θ_{ij} is approximately proportional to δ_{ij}. From the trace of Eq. (1.2.9) we have then

$$\Theta_{ij}(\mathbf{x},\nu) \simeq \frac{1}{3}\delta_{ij}\mathcal{E}_{\text{rad}}(\mathbf{x},\nu). \tag{1.2.19}$$

We also note that with $1/\kappa\rho$ very short the radiation is in thermal equilibrium with local matter at a temperature T, so that

$$\mathcal{E}_{\text{rad}}(\mathbf{x},\nu) \simeq B\big(\nu, T(\mathbf{x})\big), \tag{1.2.20}$$

where B is the Planck black-body distribution

$$B(\nu,T) = \frac{8\pi h}{c^3}\frac{\nu^3}{\exp(h\nu/k_{\text{B}}T)-1}. \tag{1.2.21}$$

Using Eqs. (1.2.19) and (1.2.20) in Eq. (1.2.15),

$$c\,\nabla B\big(\nu, T(\mathbf{x})\big) = -3\kappa(\mathbf{x},\nu)\rho(\mathbf{x})\mathbf{\Phi}(\mathbf{x},\nu). \tag{1.2.22}$$

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Of course, $\ell(\hat{n}, \mathbf{x}, \nu)$ does depend somewhat on $\hat{n}$. Even deep in a star, there is some difference between up and down, the directions toward and away from the star's surface. We are neglecting this in Eqs. (1.2.19) and (1.2.20), but since $\kappa\rho$ is assumed large, we may not neglect the quantity $\kappa\rho\Phi_i$ in Eq. (1.2.22), even though perfect isotropy of the photon distribution would make Φ_i vanish.

Now let us take up the special case of greatest interest, a spherically symmetric star in which the only special direction at any point is the radial direction, which distinguishes up and down. We then take the flux vector to point in the direction $\hat{x} \equiv \mathbf{x}/r$, and otherwise to depend only on ν and $r \equiv |\mathbf{x}|$, so that we may write

$$\mathbf{\Phi}(\mathbf{x}, \nu) = \hat{x}\,\frac{\mathcal{L}(r, \nu)}{4\pi r^2}. \tag{1.2.23}$$

Then $\mathcal{L}(r, \nu)$ is the total radiant energy flux, the radiant energy per time and per frequency interval passing outward through a sphere of radius r. In this case, Eqs. (1.2.12) and (1.2.22) take the form

$$\frac{d\mathcal{L}(r, \nu)}{dr} = 4\pi r^2 \epsilon(r, \nu)\rho(r), \tag{1.2.24}$$

and

$$c\,\frac{d\,B\big(\nu, T(r)\big)}{dr} = -3\kappa(r, \nu)\rho(r)\frac{\mathcal{L}(r, \nu)}{4\pi r^2}. \tag{1.2.25}$$

To calculate the temperature distribution in a star, it suffices to consider the total radiant energy for all frequencies. The total radiant energy flux is defined by

$$\mathcal{L}(r) \equiv \int d\nu\ \mathcal{L}(r, \nu), \tag{1.2.26}$$

and the total energy per gram emitted by nuclear processes at all frequencies is

$$\epsilon(r) \equiv \int d\nu\ \epsilon(r, \nu). \tag{1.2.27}$$

Then integrating Eq. (1.2.24) over frequency, we have

$$\frac{d\mathcal{L}(r)}{dr} = 4\pi r^2 \epsilon(r)\rho(r). \tag{1.2.28}$$

In order to write the equation for dT/dr in terms of $\mathcal{L}(r)$, we divide Eq. (1.2.25) by $\kappa(r, \nu)$ and integrate over ν:

$$-3\rho(r)\frac{\mathcal{L}(r)}{4\pi r^2} = c\int d\nu\ \frac{1}{\kappa(r, \nu)}\left(\frac{\partial B(\nu, T)}{\partial T}\right)_{T=T(r)} T'(r).$$

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We define the *Rosseland mean opacity*[4] $\kappa(r)$ as the inverse of the average of the inverse of $\kappa(r,\nu)$, evaluated with a weighting function $(\partial B(\nu,T)/\partial T)_{T=T(r)}$:

$$\int d\nu \, \frac{1}{\kappa(r,\nu)} \left(\frac{\partial B(\nu.T)}{\partial T}\right)_{T=T(r)} \equiv \frac{1}{\kappa(r)} \int d\nu \left(\frac{\partial B(\nu.T)}{\partial T}\right)_{T=T(r)} = \frac{4aT^3(r)}{\kappa(r)}, \tag{1.2.29}$$

where a is the radiation energy constant, $a = 8\pi^5 k_B^4/15h^3c^3 = 7.566 \times 10^{-15}$ erg cm^{-3} K^{-4}. So

$$-3\rho(r)\frac{\mathcal{L}(r)}{4\pi r^2} = \frac{4acT^3(r)T'(r)}{\kappa(r)},$$

or, multiplying by $\kappa(r)/4acT^3(r)$:

$$\frac{dT(r)}{dr} = -\frac{3\rho(r)\kappa(r)}{4acT^3(r)}\frac{\mathcal{L}(r)}{4\pi r^2}. \tag{1.2.30}$$

Equations (1.2.28) and (1.2.30) are the fundamental equations of radiative energy transport in spherical star interiors.

It is convenient for some purposes to introduce an opacity function $\kappa(\rho, T, \nu)$ and its Rosseland mean $\kappa(\rho, T)$ that depend on density and temperature rather than on position, with

$$\kappa(r) = \kappa\big(\rho(r), T(r)\big), \qquad \kappa(r,\nu) = \kappa\big(\rho(r), T(r), \nu\big). \tag{1.2.31}$$

Then the definition (1.2.29) of the Rosseland mean takes the position-independent form

$$\int d\nu \, \frac{1}{\kappa(\rho, T, \nu)} \left(\frac{\partial B(\nu, T)}{\partial T}\right) = \frac{4aT^3}{\kappa(\rho, T)}. \tag{1.2.32}$$

1.3 Radiative Models

In this section we shall describe the differential equations and boundary conditions that govern a star in which energy transport is everywhere dominated by radiation. The most important result here is that for a set of stars of a given age and initial uniform chemical composition (such as the stars in many clusters), any stellar parameter, such as radius, luminosity, etc., may be expressed as a function of stellar mass. In consequence, when any two of these parameters are plotted against one another, the plot is a one-dimensional curve. (One such relation is the plot of luminosity against effective temperature, known as the Hertzsprung–Russell relation, about which more later.) The following two

[4] S. Rosseland, *Mon. Not. Roy. Astron. Soc.* **84**, 525 (1924).

sections will consider the opacity and nuclear energy generation per mass, which appear as ingredients in these differential equations. Then in Section 1.6 we will derive consequences from these equations in the form of power laws for various stellar properties for stars that are on the main sequence of the Hertzsprung–Russell diagram. Section 1.7 considers energy transport by convection, and shows that convection does not affect the main results of this section and Section 1.6.

With the chemical composition fixed and uniform, we can regard the pressure $p(r)$, opacity $\kappa(r)$, and nuclear energy production per mass $\epsilon(r)$ as fixed functions of the density $\rho(r)$ and temperature $T(r)$. The star's structure is then described by four functions of the radial coordinate r: the mass $\mathcal{M}(r)$ contained within a sphere of radius r; the radiant energy per second $\mathcal{L}(r)$ flowing outward through a spherical surface of radius r; and the density $\rho(r)$ and temperature $T(r)$. These four quantities are governed by four first-order differential equations: the equations (1.1.4) and (1.1.5) of hydrostatic equilibrium

$$\frac{dp(r)}{dr} = -\frac{G\mathcal{M}(r)\rho(r)}{r^2} \tag{1.3.1}$$

and

$$\frac{d\mathcal{M}(r)}{dr} = 4\pi r^2 \rho(r), \tag{1.3.2}$$

and the equations (1.2.28) and (1.2.30) of radiative energy transport

$$\frac{d\mathcal{L}(r)}{dr} = 4\pi r^2 \epsilon(r)\rho(r) \tag{1.3.3}$$

and

$$\frac{dT(r)}{dr} = -\frac{3\kappa(r)\rho(r)}{4caT^3(r)}\frac{\mathcal{L}(r)}{4\pi r^2}. \tag{1.3.4}$$

There are also four boundary conditions – two at the center,

$$\mathcal{M}(0) = \mathcal{L}(0) = 0; \tag{1.3.5}$$

and two at the star's nominal radius R,

$$\rho(R) = T(R) = 0. \tag{1.3.6}$$

With the pressure p, Rosseland mean opacity κ, and nuclear energy production per mass ϵ assumed to be given as functions of density and temperature, the differential equations (1.3.1)–(1.3.4) and boundary conditions (1.3.5) and (1.3.6) then govern the four unknown functions $\rho(r)$, $\mathcal{M}(r)$, $T(r)$, and $\mathcal{L}(r)$.

Before considering the implications of these differential equations and boundary conditions, we need to say a bit about the implausible boundary condition that the temperature and density vanish at the star's surface. With four first-order differential equations for four unknown functions, and only two boundary conditions at $r = 0$, there is enough freedom to impose these two

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additional conditions at any radius in the generic case. We call the value of r where these conditions are imposed on the solutions of Eqs. (1.3.1)–(1.3.4) the "nominal radius" R of the star. But of course the surfaces of stars are not actually at absolute zero temperature. Not even close. In fact, the approximation of nearly perfect isotropy that we used in deriving the equations (1.3.3) and (1.3.4) breaks down close to the stellar surface, where there is a big difference between up, down, and sideways. Specifically, this approximation breaks down at values of r for which $R - r$ is no longer large compared with the typical photon free path $1/\rho(r)\kappa(r)$ at r. In this region, known as the stellar atmosphere, we need to use the full equation (1.2.6) of radiative equilibrium, and we do not find a surface with absolute zero temperature. The nominal radius R is where the density and temperature *would* vanish if Eqs. (1.3.1)–(1.3.4) held out to this radius.

In the real world, instead of a surface at which the density and temperature vanish, there is a "true surface" with radius $R_{\rm true}$ beyond which there is essentially empty space, with only outgoing radiation and some gas of very low density, such as the solar corona. But this is not the surface from which comes the light we see. To the extent that the light of a star resembles black-body radiation, we can think of it as coming from an effective surface with radius $R_{\rm eff}$, defined by the condition

$$\sigma T^4(R_{\rm eff}) \times 4\pi R_{\rm eff}^2 = L, \tag{1.3.7}$$

where $\sigma = ac/4$ is the Stefan–Boltzmann constant, and L is the star's luminosity, the value of $\mathcal{L}(r)$ at all values of r outside the stellar core in which nuclear energy production occurs. The depth of the effective surface below the true surface is best described in terms of its optical depth

$$\tau_{\rm eff} = \int_{R_{\rm eff}}^{R_{\rm true}} \kappa(r)\rho(r)\,dr. \tag{1.3.8}$$

Since it is the typical photon free path $1/\kappa\rho$ that sets the scale of variations with radius near the surface, we expect $\tau_{\rm eff}$ to be of order unity. (In fact, there is a time-honored but rather unconvincing calculation[5] that gives the optical depth of the effective surface as $\tau_{\rm eff} = 2/3$.)

The important point for us is that the thickness of the stellar atmosphere is much less than R. As long as we restrict our interest to the star's interior, we can therefore continue to use the differential equations (1.3.1)–(1.3.4), with the boundary conditions (1.3.5) and (1.3.6), with the understanding that the condition (1.3.6) just means that the density and pressure are much less at the star's true surface than deep in the interior. For instance, the central density and temperature of the Sun are (98 ± 15) g/cm^3 and $(13.6 \pm 1.2) \times 10^6$ K, while even deep in the stellar atmosphere, at an optical depth $\tau = 10$, the solar density and

[5] For instance, see J. P. Cox and R. T. Giuli, *Principles of Stellar Structure: Application to Stars*, Vol. 2 (Gordon & Breach, New York, 1968), Chapter 20.

temperature are only about $5 \times 10^{-7}\,\mathrm{g/cm^3}$ and 9,700 K, much less than the central values.

With four first-order equations and four boundary conditions in which there appear only a single parameter R, we expect a one-parameter family of solutions. This result is close to a conclusion that is often called the *Vogt–Russell theorem*,[6] which asserts that for a definite chemical composition there is a unique solution to the equations of stellar structure, that depends on just a single stellar parameter, such as the radius R or the total mass M. In fact, we can't be sure of the existence of a solution, because it is possible that a singularity could be encountered that prevents a solution, though no such case of astronomical relevance is known. Also, assuming a solution exists, it may not be unique.

The possibility of non-uniqueness arises from the peculiar feature, that the boundary conditions refer to two different boundaries, $r = 0$ and $r = R$. Consider how we would actually construct a solution. Starting at $r = 0$, we can adopt various trial values ρ_c and T_c of the central density $\rho(0)$ and central temperature $T(0)$, so that with the original conditions $\mathcal{M}(0) = \mathcal{L}(0) = 0$ we have four initial conditions. Integrating Eqs. (1.3.1)–(1.3.4) with these initial conditions gives a unique solution, depending on ρ_c and T_c. We can then adjust these two initial values so that the other conditions, $\rho(R) = T(R) = 0$, are satisfied at any given R. With two conditions on the two parameters ρ_c and T_c, there is likely to be a solution, but possibly more than one. As long as the number of solutions is finite, they can each depend on only a single free parameter, which so far we have taken as the stellar radius R.

Of course, if all stellar parameters depend on a single parameter R, they can be taken to depend on any one of the other stellar parameters, not necessarily R. In particular, since the stellar mass M is the one thing that remains essentially fixed as a star evolves (until the star in its old age begins to blow off mass), it is more natural to take the single parameter as M rather than R. We can (though we need not) do this directly, by a reinterpretation of the differential equations. We can take the independent variable to be $\mathcal{M}$ rather than r, with the dependent variables taken as $r(\mathcal{M})$ along with $\rho(\mathcal{M})$, $T(\mathcal{M})$, and $\mathcal{L}(\mathcal{M})$. The differential equations are the reciprocal of Eq. (1.3.2),

$$\frac{dr(\mathcal{M})}{d\mathcal{M}} = \frac{1}{4\pi r^2(\mathcal{M})\rho(\mathcal{M})}, \tag{1.3.9}$$

and the ratios of Eqs. (1.3.1), (1.3.3), and (1.3.4) to Eq. (1.3.2):

$$\frac{dp(\mathcal{M})}{d\mathcal{M}} = -\frac{G\mathcal{M}}{4\pi r^4(\mathcal{M})}, \tag{1.3.10}$$

$$\frac{d\mathcal{L}(\mathcal{M})}{d\mathcal{M}} = \epsilon(\mathcal{M}), \tag{1.3.11}$$

6 H. Vogt, *Astron. Nachr.* **226**, 301 (1926); H. N. Russell, *Astronomy (Boston)* **2**, 910 (1927).

and

$$\frac{dT(\mathcal{M})}{d\mathcal{M}} = -\frac{3\kappa(\mathcal{M})\mathcal{L}(\mathcal{M})}{4caT^3(\mathcal{M})\big(4\pi r^2(\mathcal{M})\big)^2}. \tag{1.3.12}$$

Instead of imposing boundary conditions at $r = 0$ and $r = R$, here they are imposed at $\mathcal{M} = 0$,

$$r(\mathcal{M}) = \mathcal{L}(\mathcal{M}) = 0 \text{ at } \mathcal{M} = 0, \tag{1.3.13}$$

and at $\mathcal{M}$ equal to the total stellar mass M,

$$\rho(\mathcal{M}) = T(\mathcal{M}) = 0 \text{ at } \mathcal{M} = M. \tag{1.3.14}$$

With the equations written in this way, there is no need to input any stellar parameter aside from the mass M.

It is the dependence of stellar structure on just a single parameter that explains a remarkable feature of observations of clusters of stars. The dozens or hundreds of stars in an open cluster like the Pleiades generally condensed at about the same time from the same cloud of interstellar material, so they all have pretty much the same initial chemical composition and age as well as distance, though differing widely in their masses. The only thing on which any observable feature of the stars in such a cluster can depend that varies from one star to another will thus be the stars' masses. Hence when any pair of observables for the cluster stars are plotted against each other, these points will fall on a one-dimensional curve, each different point on this curve corresponding to a different stellar mass.

This is less so for the thousands or hundreds of thousands of stars in a globular cluster like M15, where there is a greater spread in age and initial chemical composition. But even here the plot of any pair of observables against each other is a more or less thickened curve.

The most easily observable stellar quantities are the luminosity L (or, if the distance d to the cluster is not known, the apparent luminosity $L/4\pi d^2$) and the effective temperature $T_{\rm eff}$. The effective temperature is defined by the condition that $L = \sigma T_{\rm eff}^4 \times 4\pi R^2$, but it is estimated from observations of the star's color[7] and/or spectrum, as described in the following table:[8]

[7] The color of a star is measured by the differences of its luminosity when the star is observed with several different filters. As seen by an observer without filters, the color depends on the distribution with frequency of the radiant energy emitted by the star, for those frequencies that are visible to the eye. For hot stars with temperatures $T > 30{,}000$ K, these frequencies are all much less than $k_B T/h$, and therefore, according to the black-body formula (1.2.21), the energy emitted between visible frequencies ν and $\nu + d\nu$ is proportional to $\nu^2\, d\nu$. As it happens, this is the same frequency distribution as for the light scattered by molecules and other small particles in the atmosphere, which gives the sky its color. Hence sky blue is the asymptotic visible color of black bodies with very high temperature.

[8] The information here is taken from F. LeBlanc, *Introduction to Stellar Atmospheres* (John Wiley & Sons, Chichester, 2010), with some additions from other sources.

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Typical spectral lines, effective temperatures, colors, and examples of various types of star

Type	*Lines*	T_{eff} (K)	*Color*	*Example*
O	HeII abs	>30,000	Sky blue	λ Ori
B	HeI abs, H	10,000–30,000	Blue–White	Rigel
A	H, CaII	7,500–10,000	White	Sirius A, Vega
F	CaII, H weaker	6,000–7,500	Yellow–White	Procyon
G	CaII, Fe, H weak	5,000–6,000	Yellow	Sun
K	Metals, CH, CN	3,500–5,000	Orange	Arcturus
M	TiO	<3,500	Red	Antares

The graph of observed absolute or apparent luminosity versus effective temperature is known as the Hertzsprung–Russell diagram, which was first constructed a century ago.[9]

In practice, the Hertzsprung–Russell diagram of a cluster is a thick curve, not strictly one-dimensional. This is because the cluster stars did not all begin at precisely the same time with precisely the same chemical composition. There are also observational problems: a star's color and spectrum do not give a precise value for the effective temperature, and it is often difficult to distinguish binary stars from single stars. Even so, one can clearly see in the data that there is a one-dimensional curve of luminosity versus effective temperature, not just points everywhere in the plot.

The Hertzsprung–Russell diagram for a cluster commonly contains a *main sequence*, consisting of stars like the Sun that are still burning hydrogen at their cores. On the main sequence L increases smoothly with T_{eff}, with the most massive stars the hottest and most luminous. (In Section 1.6 we will show how to estimate the shape of the main sequence curve by applying dimensional analysis to Eqs. (1.3.1)–(1.3.4).) As the cluster evolves, the Hertzsprung–Russell diagram develops a red giant branch, consisting of stars that have converted most of the hydrogen at their cores to helium, and are burning hydrogen only in a shell around the inert helium core. On this branch, the effective temperature *decreases* (and radius increases) with increasing luminosity, accounting for the red color of very luminous red giant stars such as Betelgeuse and Antares. The heavier stars on the main sequence have larger L and therefore evolve more quickly, so as time passes more and more of the upper part of the main sequence bends over into the red giant branch. Observations of this main sequence

[9] E. Hertzsprung, *Astron. Nachr.* **179** (24), 373 (1908); H. N. Russell, *Pop. Astron.* **22**, 275 (1914).

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turn-off therefore indicate the age of the cluster.[10] Eventually the more massive stars of the cluster will begin to burn helium, and the Hertzsprung–Russell diagram will develop further complications, but it remains a more-or-less one-dimensional curve, as required by the Vogt–Russell theorem.

There is a general conclusion of some importance, which can be derived immediately from Eqs. (1.3.1)–(1.3.4), without detailed calculation. We note that the pressure p in Eq. (1.3.1) is the sum of the pressures of gas and radiation,

$$p = p_{\rm gas} + p_{\rm rad}, \tag{1.3.15}$$

where, for black-body radiation,

$$p_{\rm rad} = \frac{a}{3}T^4. \tag{1.3.16}$$

For an ideal gas $p_{\rm gas} = \rho k_{\rm B} T / m_1 \mu$, where μ is the molecular weight and m_1 is the nucleon mass, or more precisely, the mass of unit atomic weight. For the present all we need to know about the gas pressure is that it decreases with increasing r. Now, Eq. (1.3.4) may be written

$$\frac{dp_{\rm rad}(r)}{dr} = -\frac{\kappa(r)\rho(r)\mathcal{L}(r)}{4\pi c r^2}.$$

Taking the difference between this and Eq. (1.3.1) gives

$$-\frac{\kappa(r)\rho(r)\mathcal{L}(r)}{4\pi c r^2} + \frac{G\mathcal{M}(r)\rho(r)}{r^2} = -\frac{dp_{\rm gas}(r)}{dr} > 0$$

and therefore, everywhere in the star,

$$\kappa(r)\mathcal{L}(r) < 4\pi G c\, \mathcal{M}(r).$$

In particular, by setting r equal to the nominal stellar radius R, we find an inequality involving the star's luminosity $L = \mathcal{L}(R)$ and mass $M = \mathcal{M}(R)$:

$$\kappa(R)\, L < 4\pi G c M. \tag{1.3.17}$$

If this inequality were violated, then the radiation pressure alone would be strong enough to blow off the outer layers of the star. In the commonly encountered case where the opacity in the star's outer layers is due to Thomson scattering the inequality (1.3.17) is known as the *Eddington limit*. This inequality also limits the luminosity that can be produced by spherically symmetric accretion onto a star or galactic nucleus.

This derivation also shows that if gas pressure were negligible compared with radiation pressure (as it is in only the most massive stars) the inequality would become an equality, $\kappa(R)\, L = 4\pi G c M$.

[10] For a summary of the use of this technique in cosmology, see S. Weinberg, *Cosmology* (Oxford University Press, Oxford, 2008), pp. 62–63.

1.4 Opacity

We saw in Section 1.2 that Eq. (1.2.30), one of the pair of equations that govern the variation of temperature of stars with distance r from the center, involves a quantity $\kappa(r)$, known as the opacity. In general, the opacity is given by Eq. (1.2.16):

$$\kappa \equiv \kappa_{\rm abs} + \kappa_{\rm out} - \kappa_{\rm in}, \tag{1.4.1}$$

with it understood that in Eq. (1.2.30) $\kappa(r)$ is a Rosseland mean value $\kappa(\rho(r), T(r))$, calculated according to Eq. (1.2.32):

$$\int d\nu \, \frac{1}{\kappa(\rho, T, \nu)} \left(\frac{\partial B(\nu, T)}{\partial T} \right) = \frac{4aT^3}{\kappa(\rho, T)},$$

where B is the black-body distribution function

$$B(\nu, T) = \frac{8\pi h}{c^3} \frac{\nu^3}{\exp(h\nu/k_{\rm B}T) - 1}.$$

The first term in Eq. (1.4.1) is defined so that $c\rho\kappa_{\rm abs}$ is the *net* rate of absorption – that is, it is the average rate per photon at which photons are absorbed, less the rate per initial photon at which photons with the same momentum are created by stimulated emission. If $\Gamma_{\rm abs}$ is the rate of absorption alone, then when stimulated emission is taken into account, the net rate of photon absorption is

$$c\rho\kappa_{\rm abs}(\rho, T, \nu) = \Gamma_{\rm abs}(\rho, T, \nu)\left[1 - e^{-h\nu/k_{\rm B}T}\right]. \tag{1.4.2}$$

This can most easily be seen by returning to Eqs. (1.2.11) and (1.2.20), which show that when radiation and matter come to equilibrium in the absence of nuclear energy generation, the absorption opacity is related to the energy $j(\rho, T, \nu)$ emitted by the matter per mass, per time, and per frequency interval, by

$$\kappa_{\rm abs}(\rho, T, \nu) = j(\rho, T, \nu)/cB(\nu, T) = \frac{c^2}{8\pi h\nu^3} j(\rho, T, \nu)\left[\exp(h\nu/k_{\rm B}T) - 1\right].$$

The emission rate j has a familiar factor $\exp(-h\nu/k_{\rm B}T)$, reflecting the probability of excitation by energy $h\nu$ of degrees of freedom in the matter. When combined with the factor $\exp(h\nu/k_{\rm B}T) - 1$ from $1/B$ this gives the correction factor $1 - e^{-h\nu/k_{\rm B}T}$ in Eq. (1.4.2), in which the first and second terms arise from absorption and stimulated emission.[11]

The second and third terms in Eq. (1.4.1) are defined so that $c\rho\kappa_{\rm out}$ and $c\rho\kappa_{\rm in}$ are the rates at which photons are scattered out of or into any given direction.

[11] For a derivation of Eq. (1.4.2) that does not depend on the assumption that the radiation can come into equilibrium with the matter, see R. Flauger and S. Weinberg, *Phys. Rev.* D **99**, 123030 (2019).

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In cases where scattering occurs in a collision with a single particle, such as an electron or atom, these terms are given by Eqs. (1.2.13) and (1.2.17):

$$\kappa_{\rm out} = N_{\rm scat} \int d^2\hat{n}' \; \sigma_{\rm scat}(\hat{n} \to \hat{n}'), \tag{1.4.3}$$

$$\kappa_{\rm in} = N_{\rm scat} \int d^2\hat{n}' \; \left(\hat{n}' \cdot \hat{n}\right) \sigma_{\rm scat}(\hat{n} \to \hat{n}'), \tag{1.4.4}$$

where $\sigma_{\rm scat}(\hat{n} \to \hat{n}')$ is the differential cross section for scattering of a photon traveling in a direction $\hat{n}$ into a direction $\hat{n}'$, and $N_{\rm scat}$ is the number of scatterers per gram. (These integrals are independent of the unit vector $\hat{n}$ because of the invariance of the integrands under simultaneous rotations of $\hat{n}$ and $\hat{n}'$.)

Now let us consider the various contributions to opacity, and the temperature and density dependence of each. It is often a fair approximation to represent the opacity as a simple function of temperature and density, proportional to powers of both:

$$\kappa(\rho, T) = \kappa_1 \rho^\alpha (k_{\rm B} T)^\beta, \tag{1.4.5}$$

where κ_1 as well as α and β are approximately independent of density and temperature. We will estimate α and β below for contributions to opacity of various types, and show in Section 1.6 how these results can be used to relate observable properties of stars.

Thomson Scattering

This is the simplest contribution to opacity. It is the elastic scattering of photons with energies much less than $m_{\rm e}c^2$ on free electrons moving non-relativistically. The differential scattering cross section is

$$\sigma_{\rm Thomson}(\hat{n} \to \hat{n}') = \frac{e^4}{2m_{\rm e}^2 c^4}\left[1 + (\hat{n} \cdot \hat{n}')^2\right]. \tag{1.4.6}$$

(Recall that in this book e is the charge of the electron in unrationalized electrostatic units.) Because this differential cross section is even[12] in $\hat{n}'$, while the factor $\hat{n} \cdot \hat{n}'$ in Eq. (1.4.4) is odd in $\hat{n}'$, here we have $\kappa_{\rm in} = 0$. Hence, where the opacity is dominated by Thomson scattering, the total opacity is

$$\kappa = \kappa_{\rm out} = N_{\rm e} \sigma_{\rm T}, \tag{1.4.7}$$

[12] This forward–backward symmetry can be understood in classical terms. Classically, in Thomson scattering the electron position oscillates under the influence of the electric field of the incoming photon, and this oscillation produces the electromagnetic field of the outgoing photon. This oscillation is in the direction of the polarization vector of the incoming photon, which is normal to the photon's direction, so there is nothing about this oscillation or the field it produces that can distinguish between the forward and backward directions.

where σ_T is the total Thomson scattering cross section, given by the integral of the differential cross section (1.4.6) over solid angle:

$$\sigma_T = \frac{8\pi}{3}\left(\frac{e^2}{\hbar c}\right)^2\left(\frac{\hbar}{m_e c}\right)^2 = 0.66525 \times 10^{-24}\,\text{cm}^2,$$

and N_e is the number of free electrons per gram. For instance, for a medium consisting of completely ionized atoms of atomic number Z and atomic weight A, we have $N_e = Z/Am_1$, where $m_1 = 1.66054 \times 10^{-24}$ g is the mass for unit atomic weight. This gives a Thomson scattering opacity (1.4.7) equal to $0.400 \times Z/A\ \text{cm}^2/\text{g}$.

Since the cross section is constant (aside from a possible dependence of the degree of ionization on temperature and density) the opacity for Thomson scattering has

$$\alpha = \beta = 0. \tag{1.4.8}$$

No averaging over photon frequency is necessary if Thomson scattering dominates the opacity.

Free–Free Absorption

In the absence of external fields, the conservation of energy and momentum forbids the absorption of a photon by a free electron. If the photon has momentum $\mathbf{q}$ then it has energy $c|\mathbf{q}|$, so the conservation of energy and momentum requires that

$$0 = (E' - E)^2 - c^2(\mathbf{p}' - \mathbf{p})^2 = 2m_e^2c^4 - 2E'E + 2c^2\mathbf{p}'\cdot\mathbf{p}.$$

where $\mathbf{p}$ and $\mathbf{p}'$ are the initial and final electron momenta, and $E = [c^2\mathbf{p}^2 + m_e^2c^4]^{1/2}$ and $E' = [c^2\mathbf{p}'^2 + m_e^2c^4]^{1/2}$ are the initial and final electron energies. This is not possible if any energy is absorbed by the electron, for in the frame in which the electron is initially at rest, this requires that $E' = m_ec^2$, so the final electron would have to be also at rest in the same frame.

But in the Coulomb field of an atomic nucleus, the nucleus can take up momentum without carrying away appreciable energy because it is so massive. So absorption is possible on a free electron near a nucleus, with the energy but not the momentum of electron and photon conserved, in the same way that a dropped ball can bounce upward without losing energy, its momentum being taken up by the Earth. This is the inverse of the familiar process of *bremsstrahlung*, in which a photon is emitted when a charged particle is slowed in a collision. (The cooling of interstellar matter by bremsstrahlung is discussed at the end of Section 3.3, and the emission of detectable radiation by bremsstrahlung is considered in Section 3.7.) The absorption of photons by free electrons in the Coulomb field of a nucleus leads to what is known as *Kramers opacity*, named for Hendrik Kramers (1894–1952) who, using classical physics, first attempted a calculation.[13] Kramers' classical result was in effect that the rate of absorption of a photon of frequency ν (averaged over photon directions and helicities) is[14]

$$\Gamma_{\text{Kramers}}(\rho, T, \nu) = \int n_{\text{e}}(\mathbf{v}, T)\, d^3v\, \frac{4\pi Z^2 e^6 n_N}{3\sqrt{3} h m_{\text{e}}^2 v \nu^3}$$

where the integral is over initial electron velocities $\mathbf{v}$; $n_{\text{e}}(\mathbf{v}, T)$ is the number of electrons per spatial volume and per velocity-space volume; n_N is the number density of ions, taken to have charge Ze; e is the magnitude of the electron charge in unrationalized electrostatic units; and $h = 2\pi\hbar$.

Depending on the electron velocity and photon frequency, this can be significantly modified by quantum and other corrections. With or without these corrections, the net rate $c\rho\kappa$ of photon absorption in free–free transitions is quadratic in particle densities, so $\alpha = 1$, but the temperature dependence is more complicated. It was first calculated by John Arthur Gaunt[15] (1904–1944). It has become traditional to express the rate per electron as the Kramers result multiplied by a correction factor, known as the free–free Gaunt factor:

$$\Gamma_{\text{ff abs}}(\rho, T, \nu) = \int n_{\text{e}}(\mathbf{v}, T)\, d^3v\, \frac{4\pi Z^2 e^6 n_N}{3\sqrt{3} h m_{\text{e}}^2 v \nu^3} g_{\text{ff}}(\nu, v). \tag{1.4.9}$$

This absorption rate is quite complicated, given by an integral of the matrix element of the momentum operator of the electron between initial and final electron wave functions, which in a Coulomb potential are Kummer functions. But it is not so difficult to carry out the calculation in Born approximation – that is, to first order in the Coulomb potential. As shown in the appendix to this section, in this order the rate at which a photon of frequency ν is absorbed is[16]

$$\Gamma_{\text{ff abs}}(\rho, T, \nu) = \int n_N n_{\text{e}}(\mathbf{v}, T)\, d^3v\, \frac{4Z^2 e^6}{3h m_{\text{e}}^2 v \nu^3} \ln\left(\frac{v' + v}{v' - v}\right), \tag{1.4.10}$$

where v' is the final electron velocity, given by the energy conservation condition

$$\frac{m_{\text{e}} v'^2}{2} = \frac{m_{\text{e}} v^2}{2} + h\nu. \tag{1.4.11}$$

13 H. Kramers, *Phil. Mag.* **46**, 836 (1923).
14 The fractional rate of decrease of energy in a light ray of frequency ν is $h\nu\Gamma(\nu)$, which for the Kramers formula is independent of Planck's constant. It is this rate that emerges from a purely classical calculation.
15 J. A. Gaunt, *Proc. Roy. Soc.* **126**, 654 (1930).
16 For a different derivation of this formula, using "old-fashioned" second-order perturbation theory, see H.-Y. Chiu, *Stellar Physics* (Blaisdell, Waltham, MA, 1968). The factor v in the denominator of Eq. (1.4.10) appears in Chiu's book as v'; presumably this is a typographical error.

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That is, the Gaunt factor is

$$g_{\rm ff}(\nu, v) = \frac{\sqrt{3}}{\pi} \ln\left(\frac{v' + v}{v' - v}\right), \tag{1.4.12}$$

with v' again given by Eq. (1.4.11). This is a good approximation for non-relativistic electrons if the Coulomb potential at an electron scattered by an atom or ion is typically much less than electron kinetic energies, which is the case if $Ze^2/\hbar v \ll 1$ and $Ze^2/\hbar v' \ll 1$.

In thermal equilibrium at temperature T, far from degeneracy, the electron velocity distribution is given by the Maxwell–Boltzmann formula

$$n_{\rm e}(\mathbf{v}, T) = n_{\rm e} \left(\frac{m_{\rm e}}{2\pi k_{\rm B} T}\right)^{3/2} \exp\left(-\frac{m_{\rm e} v^2}{2k_{\rm B} T}\right), \tag{1.4.13}$$

where $n_{\rm e}$ is the total electron number density. We can find the temperature dependence of the integral (1.4.10) by introducing a re-scaled variable of integration

$$x \equiv v\sqrt{m_{\rm e}/2k_{\rm B}T}.$$

Then Eq. (1.4.10) can be written

$$\Gamma_{\rm ff\,abs}(\rho, T, \nu) = n_{\rm e} n_N \frac{16Z^2e^6}{3hcm_{\rm e}^2\nu^3} \sqrt{\frac{m_{\rm e}}{2\pi k_{\rm B} T}} \int_0^\infty x e^{-x^2}\, dx \times \ln\left(\frac{x' + x}{x' - x}\right), \tag{1.4.14}$$

where $n_{\rm e}$ and n_N are the total number densities of electrons and ions, respectively. If we supply the correction factor $1 - \exp(-h\nu/k_{\rm B}T)$ for stimulated emission, and as usual write the result as $c\rho\kappa_{\rm ff}$, then

$$\kappa_{\rm ff}(\rho, T, \nu) = \rho N_{\rm e} N_N \frac{16Z^2e^6}{3hcm_{\rm e}^2\nu^3} \sqrt{\frac{m_{\rm e}}{2\pi k_{\rm B} T}} \int_0^\infty x e^{-x^2}\, dx \times \ln\left(\frac{x' + x}{x' - x}\right) \big(1 - \exp(-h\nu/k_{\rm B}T)\big), \tag{1.4.15}$$

where $N_{\rm e} \equiv n_{\rm e}/\rho$ is the number of electrons per gram, $N_N \equiv n_N/\rho$ is the number of nuclei per gram, and $x' \equiv v'\sqrt{m_{\rm e}/2k_{\rm B}T}$ is given by the energy conservation equation (1.4.11) as

$$x'^2 = x^2 + y, \qquad y \equiv h\nu/k_{\rm B}T. \tag{1.4.16}$$

The Rosseland mean opacity (1.2.32) is here

$$\kappa(\rho, T) = \frac{8\rho(k_{\rm B}T)^{-7/2} N_{\rm e} N_{\rm N} Z^2 e^6 h^6 (a/k_{\rm B}^4) m_{\rm e}^{-3/2}}{3\sqrt{2}\pi^{3/2} \int_0^\infty dy \frac{y^6 e^y}{(e^y - 1)} \left[\int_0^\infty x\, e^{-x^2}\, dx \times \ln\left(\frac{x' + x}{x' - x}\right)\right]^{-1}}, \tag{1.4.17}$$

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with x' related to the integration variables x and y by the energy conservation condition (1.4.16). The important result is that in Eq. (1.4.5) the Kramers opacity has

$$\alpha = 1, \qquad \beta = -7/2. \tag{1.4.18}$$

The mean opacity has a factor $T^{-7/2}$ because of the factor $1/\sqrt{T}$ in Eq. (1.4.15), and because the factor $1/\nu^3$ in Eq. (1.4.15) is converted into a factor proportional to $1/T^3$ in the Rosseland mean.

It should not be thought that the $T^{-7/2}$ dependence of the free–free opacity continues to arbitrary low temperatures. Obviously, for sufficiently low temperatures, there are very few free electrons, and the free–free and Thomson scattering contributions to the opacity both become negligible.

High-Energy Bound–Free Absorption

When a photon is absorbed by a bound electron whose binding energy is much less than the photon energy, it hardly matters that the electron is initially bound. Thus the temperature dependence in this case is the same as for free–free absorption, with $\beta = -7/2$. The difference is that the relevant density of electrons is not the ambient density of free electrons, but an average square of the bound electron wave function, so the absorption rate $c\rho\kappa$ is proportional just to the density of atoms, and hence $\alpha = 0$ rather than $\alpha = 1$. The contribution to opacity of this sort of photon absorption is often lumped in with free–free absorption in what is called Kramers opacity.

Bound–Bound Absorption and Low-Energy Bound–Free Absorption

In these cases the photon is absorbed by a bound electron whose binding energy is at least comparable to the photon energy. This contribution to opacity involves complications of atomic physics not present for other contributions, and will not be examined further here. The heating of interstellar hydrogen by low-energy bound–free absorption of photons from hot stars is discussed in Section 3.2.

Appendix: Calculation of Free–Free Opacity

We consider a process in which a photon of momentum $\mathbf{q}$ and helicity λ is absorbed by a non-relativistic free electron of momentum $\mathbf{p}$ in the neighborhood of an atomic nucleus, giving the electron a non-relativistic momentum $\mathbf{p}'$. The nucleus serves to provide a potential $V(\mathbf{x})$, but is supposed to be so heavy that it can carry away momentum without receiving appreciable energy, so that $p'^2/2m_e = p^2/2m_e + qc$ (where $q \equiv |\mathbf{q}|$, $p \equiv |\mathbf{p}|$, and $p' \equiv |\mathbf{p}'|$) but $\mathbf{p}' \neq \mathbf{p} + \mathbf{q}$. For the present we will consider a general potential, but will later specialize to a screened Coulomb potential with $V(\mathbf{x}) = -Ze^2 \exp(-r/\ell)/r$ where $r \equiv |\mathbf{x}|$, including the unscreened case where the screening radius ℓ is taken to be infinite.

According to the general rules of quantum mechanics,[17] the differential rate for this process is given by

$$d\Gamma(\mathbf{p} + (\mathbf{q}, \lambda) \to \mathbf{p}') = (2\pi\hbar)^5 n_N \left|M\big(\mathbf{p} + (\mathbf{q}, \lambda) \to \mathbf{p}'\big)\right|^2 \times \delta(p'^2/2m_e - p^2/2m_e - qc)\, d^3p', \qquad (1.4.\text{A1})$$

and so the rate of photon absorption is

$$\Gamma_{\text{abs}}(\mathbf{q}, \lambda) = (2\pi\hbar)^5 n_N \int n_e(\mathbf{p})\, d^3p \times \int d^3p' \left|M\big(\mathbf{p} + (\mathbf{q}, \lambda) \to \mathbf{p}'\big)\right|^2 \delta(p'^2/2m_e - p^2/2m_e - qc), \qquad (1.4.\text{A2})$$

where $n_e(\mathbf{p})\, d^3p$ is the number density of initial electrons with momenta in a range d^3p around $\mathbf{p}$; n_N is the number density of nuclei; M is the coefficient of the energy and momentum conservation delta functions in the S-matrix element for this process; and we have used the momentum conservation delta function in the rate to eliminate the integral over the final nucleus momentum.

We are only concerned with single-photon absorption processes, and will ignore all quantum electrodynamic radiative corrections, so the matrix element M is of first order in the interaction between the electron and the quantized electromagnetic field. It therefore takes the form[18]

$$M = \frac{-2\pi i}{\sqrt{2qc}(2\pi\hbar)^{3/2}} \times \frac{-\sqrt{4\pi}e\hbar^2}{m_e} \int d^3x\, \psi'^*(\mathbf{x})\mathbf{e}(\hat{q}, \lambda) \cdot \nabla\psi(\mathbf{x}). \qquad (1.4.\text{A3})$$

Here ψ and ψ' are "in" and "out" solutions of the Schrödinger equations for the initial and final electrons

$$-\frac{\hbar^2}{2m_e}\nabla^2\psi + V\psi = \frac{p^2}{2m_e}\psi, \quad -\frac{\hbar^2}{2m_e}\nabla^2\psi' + V\psi' = \frac{p'^2}{2m_e}\psi', \qquad (1.4.\text{A4})$$

17 For the general relation between S-matrix elements and rates, see e.g. S. Weinberg, *The Quantum Theory of Fields*, Vol. I (Cambridge University Press, Cambridge, 1995), Section 3.4. Note that in this reference $2\pi M$ was defined as the coefficient of the delta function in the S-matrix, while here this coefficient is just M.

18 For a textbook derivation of this interaction, see e.g. S. Weinberg, *Lectures on Quantum Mechanics*, 2nd edn. (Cambridge University Press, Cambridge, 2015), Eq. 11.7.6. In Eq. (1.4.A3) we are using the electric dipole approximation, in which the photon wavelength is much larger than the de Broglie wavelengths of the initial and final electrons. With photon and electron energies of order $k_B T$, this is a good approximation if $k_B T \ll m_e c^2$, as we shall assume is the case.

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normalized so that for $r \to \infty$

$$\psi(\mathbf{x}) \to \frac{\exp(i\mathbf{p}\cdot\mathbf{x}/\hbar)}{(2\pi\hbar)^{3/2}} + O(1/r), \quad \psi'(\mathbf{x}) \to \frac{\exp(i\mathbf{p}'\cdot\mathbf{x}/\hbar)}{(2\pi\hbar)^{3/2}} + O(1/r), \tag{1.4.A5}$$

where the $O(1/r)$ term is an outgoing wave for ψ and an incoming wave for ψ'. (For an unscreened Coulomb potential the arguments of the exponentials contain additional imaginary terms of order $\ln r$.) Also $e(\hat{q},\lambda)$ is the polarization vector for a photon with direction $\hat{q}$ and helicity λ, normalized so that $\mathbf{e}^*\cdot\mathbf{e} = 1$. We will use the results for M obtained here also in the discussions of bremsstrahlung in Sections 3.3 and 3.7.

Eventually we will be moving on to the Born approximation, in which M is calculated only to first order in V, but it is useful for several reasons to work for a while with Eq. (1.4.A3), which is derived in what is called the *distorted wave Born approximation*;[19] it is valid to all orders in V but only to first order in the interaction of the electron with the annihilation part of the quantized electromagnetic field.

Multiplying Eq. (1.4.A3) with $qc = p'^2/2m_e - p^2/2m_e$ and using the Schrödinger equations (1.4.A4), we have

$$qcM = \frac{-2\pi i}{\sqrt{2qc}(2\pi\hbar)^{3/2}} \times \frac{-\sqrt{4\pi}e\hbar^2}{m_e} \times \int d^3x \left[\left(-\frac{\hbar^2}{2m_e}\nabla^2\psi' + V\psi'\right)^* \mathbf{e}(\hat{q},\lambda)\cdot\nabla\psi - \psi'^*\mathbf{e}(\hat{q},\lambda)\cdot\nabla\left(-\frac{\hbar^2}{2m_e}\nabla^2\psi + V\psi\right)\right].$$

Integration by parts shows that the kinetic energy terms cancel,[20] while the potential terms cancel except for a term proportional to the gradient of the potential:

$$M = \frac{-ie\sqrt{\hbar}}{(qc)^{3/2}m_e}\int d^3x\ \psi'^*(\mathbf{x})\mathbf{e}(\hat{q},\lambda)\cdot\left[\nabla V(\mathbf{x})\right]\psi(\mathbf{x}). \tag{1.4.A6}$$

We now go over to the Born approximation, keeping only terms of first order in the potential V. Since Eq. (1.4.A6) already has an explicit factor V, in the Born approximation we can ignore V in the wave functions, and use for ψ and ψ' just the plane waves

19 For a general textbook account of this approximation, see Weinberg, *op. cit.* Section 8.6.

20 The surface term in the integration by parts may be neglected because of its rapid oscillation as $r \to \infty$ when $p' \neq p$.

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$$\psi(\mathbf{x}) = \frac{\exp(i\mathbf{p}\cdot\mathbf{x}/\hbar)}{(2\pi\hbar)^{3/2}}, \quad \psi'(\mathbf{x}) = \frac{\exp(i\mathbf{p}'\cdot\mathbf{x}/\hbar)}{(2\pi\hbar)^{3/2}}. \tag{1.4.A7}$$

Equation (1.4.A6) then reads

$$\begin{aligned} M &= \frac{-ie\sqrt{\hbar}}{(2\pi\hbar)^3(qc)^{3/2}m_e}\int d^3x\;\mathbf{e}(\hat{q},\lambda)\cdot\big[\nabla V(\mathbf{x})\big]\exp\big(i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{x}/\hbar\big) \\ &= \frac{-e}{(2\pi\hbar)^3(qc)^{3/2}m_e\sqrt{\hbar}}\mathbf{e}(\hat{q},\lambda)\cdot(\mathbf{p}-\mathbf{p}') \\ &\quad\times\int d^3x\;V(\mathbf{x})\exp\big(i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{x}/\hbar\big). \end{aligned} \tag{1.4.A8}$$

For the screened Coulomb potential $V(\mathbf{x}) = -Ze^2e^{-r/\ell}/r$, this reads

$$M = \frac{Ze^3}{(2\pi\hbar)^3(qc)^{3/2}m_e\sqrt{\hbar}}\,\frac{4\pi\,\mathbf{e}(\hat{q},\lambda)\cdot(\mathbf{p}-\mathbf{p}')}{(\mathbf{p}-\mathbf{p}')^2/\hbar^2+1/\ell^2}. \tag{1.4.A9}$$

Orbital electrons in singly ionized atoms obviously produce a partial screening with ℓ of the order of atomic dimensions. But even where ionization is complete, there is a screening due to mobile electrons attracted to the vicinity of the atomic nucleus. This is known as Debye screening, and is discussed in Section 3.7. For the present, we will consider the unscreened case, with ℓ infinite, in which case

$$M = \frac{Ze^3}{2\pi^2(qc\hbar)^{3/2}m_e}\,\frac{\mathbf{e}(\hat{q},\lambda)\cdot(\mathbf{p}-\mathbf{p}')}{(\mathbf{p}-\mathbf{p}')^2}. \tag{1.4.A10}$$

The absorption rate per photon is then given by Eqs. (1.4.A2) and (1.4.A10) as

$$\Gamma_{\rm abs}(\mathbf{q},\lambda) = \int d^3p\,n_e(\mathbf{p})\int d^2\hat{p}'\;\frac{8\pi e^6Z^2\hbar^2p'n_N}{m_ec^3q^3}\left[\frac{(\mathbf{p}-\mathbf{p}')\cdot\mathbf{e}(\lambda,\mathbf{q})}{(\mathbf{p}'-\mathbf{p})^2}\right]^2. \tag{1.4.A11}$$

We average over photon helicity and direction, using

$$\frac{1}{2}\sum_{\lambda=\pm1}\frac{1}{4\pi}\int d^2\hat{q}\;e_i(\lambda,\mathbf{q})e_j^*(\lambda,\mathbf{q}) = \frac{1}{8\pi}\int d^2\hat{q}\;\left[\delta_{ij}-q_iq_j/q^2\right] = \frac{1}{3}\delta_{ij}. \tag{1.4.A12}$$

The integral over the direction of the outgoing electron is then

$$\int d^2\hat{p}'\;\frac{1}{(\mathbf{p}-\mathbf{p}')^2} = \frac{2\pi}{pp'}\ln\left(\frac{p'+p}{p'-p}\right). \tag{1.4.A13}$$

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Equation (1.4.A11) now gives the average photon absorption rate

$$\Gamma_{\rm abs}(q) = \int d^3p\, n_{\rm e}(\mathbf{p}) \frac{16\pi^2 e^6 Z^2 \hbar^2 n_N}{3pm_{\rm e}c^3q^3} \ln\left(\frac{p'+p}{p'-p}\right). \qquad (1.4.A14)$$

This can be rewritten for the purposes of comparison with the main text, setting $v = p/m_{\rm e}$, $v' = p'/m_{\rm e}$, $\nu = qc/h$, and $h = 2\pi\hbar$. Equation (1.4.A14) then becomes Eq. (1.4.10).

In this derivation we have treated the Coulomb interaction between electrons and nuclei only to first order in the Coulomb potential. This is justified if Ze^2/r for typical values of r is much less than the electron kinetic energies. Taking the typical value of r as the de Broglie wavelength $\hbar/m_{\rm e}v$, the ratio of potential to kinetic energy is of order

$$\frac{Ze^2/r}{m_{\rm e}v^2/2} \approx \frac{Ze^2}{\hbar v} \simeq Zc/137v,$$

so this calculation is reliable only if $v/c \gg Z/137$. Our non-relativistic treatment also requires that $v/c \ll 1$. For nuclei like C, N, and O, with $Z \geq 6$, this does not leave much of a range for the electron velocity in which the above calculation is reliable, beyond just giving the order of magnitude of the absorption rate. The contribution to M of terms of higher order in the Coulomb potential is considered in the context of bremsstrahlung in Section 3.7.

1.5 Nuclear Energy Generation

We now consider the nuclear energy production per mass $\epsilon(\rho, T)$. As with opacity in the previous section, one of our aims here will be to estimate the exponents when $\epsilon(\rho, T)$ is approximated by a power-law expression

$$\epsilon(\rho, T) \simeq \epsilon_1 \rho^\lambda (k_{\rm B}T)^\nu, \qquad (1.5.1)$$

with ϵ_1 as well as λ and ν independent of ρ and T.

The nuclear material left over from the first three minutes of the big bang was chiefly ^{1}H (that is, protons), plus about 25% by mass ^{4}He, and only a trace of ^{2}H, ^{3}He, and ^{7}Li. These light nuclei have less binding energy per nucleon than nuclei of medium atomic weight like iron and nickel, so energy can be gained by fusion of hydrogen and helium into heavier elements. But there are no stable nuclei with five or eight nucleons, so it is difficult (though, as we shall see, not impossible) to gain energy from helium in ^{1}H–^{4}He or ^{4}He–^{4}He collisions. Thus, as long as hydrogen lasts in the center of a star, the dominant source of nuclear energy will be the fusion of ^{1}H into ^{4}He, which has by far the greatest binding energy of any of these light elements.

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There are two chief routes by which hydrogen can fuse into helium. One is the proton–proton chain,[21] of which the simplest version is[22]

$$\begin{aligned} &\text{I}: && {}^1\text{H} + {}^1\text{H} \to {}^2\text{H} + e^+ + \nu_e + 1.18\ \text{MeV} \\ &\text{II}: && {}^1\text{H} + {}^2\text{H} \to {}^3\text{He} + \gamma + 5.49\ \text{MeV} \\ &\text{III}: && {}^3\text{He} + {}^3\text{He} \to {}^4\text{He} + {}^1\text{H} + {}^1\text{H} + 12.85\ \text{MeV}. \end{aligned} \tag{1.5.2}$$

The other route is the CNO cycle,[23] which in its simplest variant is

$$\begin{aligned} &\text{i}: && {}^1\text{H} + {}^{12}\text{C} \to {}^{13}\text{N} + \gamma + 1.95\ \text{MeV} \\ &\text{ii}: && {}^{13}\text{N} \to {}^{13}\text{C} + e^+ + \nu_e + 1.50\ \text{MeV} \\ &\text{iii}: && {}^1\text{H} + {}^{13}\text{C} \to {}^{14}\text{N} + \gamma + 7.54\ \text{MeV} \\ &\text{iv}: && {}^1\text{H} + {}^{14}\text{N} \to {}^{15}\text{O} + \gamma + 7.35\ \text{MeV} \\ &\text{v}: && {}^{15}\text{O} \to {}^{15}\text{N} + e^+ + \nu_e + 1.73\ \text{MeV} \\ &\text{vi}: && {}^1\text{H} + {}^{15}\text{N} \to {}^{12}\text{C} + {}^4\text{He} + 4.96\ \text{MeV}, \end{aligned} \tag{1.5.3}$$

where carbon, nitrogen, and oxygen nuclei are understood to be present in the interstellar matter from which stars like the Sun are formed, left over from nuclear processes in an earlier generation of stars. They are catalysts, neither created nor destroyed in a complete cycle. In both cases there are side branches and extensions to which we will return below, but these simple versions will provide us with sufficient examples to illustrate how $\epsilon(\rho, T)$ is estimated.

The detailed calculation of the rates of these various nuclear reactions is beyond the scope of this book. However, we can usefully identify various suppression factors in the rates that tell us a good deal about which reactions are dominant, and about their temperature dependence.

Electromagnetic Coupling

The rate of any reaction in which a single photon is emitted (such as ${}^1\text{H}+{}^2\text{H} \to {}^3\text{He} + \gamma$ in the proton–proton cycle or ${}^1\text{H} + {}^{12}\text{C} \to {}^{13}\text{N} + \gamma$ in the CNO cycle) is suppressed by a factor of order $e^2/\hbar c \simeq 1/137$.

Weak Coupling

The rate of any reaction in which a proton turns into a neutron with the emission of a positron and neutrino (such as the first step ${}^1\text{H} + {}^1\text{H} \to {}^2\text{H} + e^+ + \nu_e$ in

[21] H. A. Bethe and C. H. Critchfield, *Phys. Rev.* **54**, 248 (1938).

[22] The energies listed here for the proton–proton chain and below for the CNO cycle are the energies for each reaction actually deposited in the stellar material. Thus, where a positron is emitted, these energies include not only the rest energy $m_e c^2$ of the emitted positron but also the rest energy of the electron with which that positron inevitably annihilates. On the other hand, the mean energy of the accompanying neutrino is subtracted from the energy released, since virtually all neutrinos leave the star.

[23] C. F. von Weizsäcker, *Phys. Z.* **38**, 176 (1938); H. A. Bethe, *Phys. Rev.* **55**, 434 (1939).

the proton–proton cycle or the beta decays of ^{13}N and ^{15}O in the CNO cycle) is suppressed by two factors of the weak coupling constant $G_{\rm wk} = 1.1664 \times 10^{-11}$ MeV^{-2}. Since the typical energy involved in these nuclear reactions is about 1 MeV, weak interaction processes are typically suppressed by a dimensionless factor of order 10^{-22}.

Coulomb Barrier

The temperature dependence of nuclear reaction rates is chiefly due to the necessity for colliding nuclei to leak through the Coulomb barrier, the field of electrostatic repulsion between positively charged atomic nuclei.[24] The calculation of the effect of the Coulomb barrier on reaction rates requires use of quantum mechanics, but only at a quite elementary level, and will be presented in an appendix at the end of this section. The result is that a reaction involving two nuclei of atomic numbers Z_1 and Z_2 and an energy of relative motion E is suppressed by a factor of order

$$B(E) = \exp\left[-\pi Z_1 Z_2 e^2 \sqrt{\frac{2\mu}{\hbar^2 E}}\right], \tag{1.5.4}$$

where $\mu = m_1 m_2/(m_1 + m_2)$ is the reduced mass.

The nuclei colliding in a star of course do not have any definite value for the energy E of relative motion, but rather a range of values, with probabilities governed by the requirements of kinetic theory at temperature T. Assuming that nuclei spend most of their time sufficiently far from other nuclei that their energy is mostly kinetic, the probability of finding a pair of nuclei in a range of momenta $d^3p_1\, d^3p_2$ is proportional to

$$\exp\left(-\frac{{\mathbf{p}_1}^2}{2m_1 k_{\rm B}T} - \frac{{\mathbf{p}_2}^2}{2m_2 k_{\rm B}T}\right) d^3p_1\, d^3p_2 = \exp\left(-\frac{E}{k_{\rm B}T}\right)\, d^3p \times \exp\left(-\frac{\mathbf{P}^2}{2(m_1+m_2)k_{\rm B}T}\right) d^3P, \tag{1.5.5}$$

where $\mathbf{P} \equiv \mathbf{p}_1 + \mathbf{p}_2$ is the total momentum, and $E = \mathbf{p}^2/2\mu$ is the energy of relative motion, with $\mathbf{p} \equiv \mu(\mathbf{p}_1/m_1 - \mathbf{p}_2/m_2)$ the relative momentum. The rate ϵ of nuclear reactions per gram is then of the form

$$\begin{aligned}\epsilon(\rho, T) &= \int_0^\infty dE\, f(E, \rho, T) \exp(-E/k_{\rm B}T) B(E) \\ &= \int_0^\infty dE\, f(E, \rho, T) \exp\left(-\frac{E}{k_{\rm B}T} - \frac{C}{\sqrt{E}}\right),\end{aligned} \tag{1.5.6}$$

[24] Barrier penetration was first calculated in the context of nuclear α-decay; G. Gamow, *Z. Phys.* **52**, 510 (1928).

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where $f(E, \rho, T)$ arises from power-law factors in the thermal distribution of E and $\mathbf{P}$ and in the probability of the nuclear reaction occurring when the nuclei reach zero separation, and C is the constant in the exponent in Eq. (1.5.4):

$$C = \pi Z_1 Z_2 e^2 \sqrt{\frac{2\mu}{\hbar^2}}. \tag{1.5.7}$$

In practice, $k_B T$ is always much less than C^2, so the exponential $\exp(-C/\sqrt{E})$ will be very small unless E is much greater than $k_B T$, in which case $\exp(-E/k_B T)$ will be very small. The exponential in Eq. (1.5.6) is therefore very sharply peaked at the energy E_T where its argument is a maximum:

$$0 = \left.\frac{d}{dE}\right|_{E=E_T} \left(-\frac{E}{k_B T} - \frac{C}{\sqrt{E}}\right) = -\frac{1}{k_B T} + \frac{C}{2E_T^{3/2}} \tag{1.5.8}$$

so

$$E_T = (C k_B T/2)^{2/3}. \tag{1.5.9}$$

The dominant factor B_T in the temperature dependence of the reaction rate (1.5.6) is simply the exponential function, evaluated at $E = E_T$:

$$B_T = \exp\left(-\frac{E_T}{k_B T} - \frac{C}{\sqrt{E_T}}\right) = \exp\left(-3\left(\frac{\pi Z_1 Z_2 e^2 \sqrt{\mu}}{\hbar\sqrt{2 k_B T}}\right)^{2/3}\right). \tag{1.5.10}$$

Numerically this is

$$B_T = \exp\left[-\left(Z_1^2 Z_2^2 (\mu/m_p) \times \frac{7.726 \times 10^{10}\,\mathrm{K}}{T}\right)^{1/3}\right], \tag{1.5.11}$$

where m_p is the proton mass.

The values of reaction rates depend on a number of other factors besides the barrier penetration factor, but it is the barrier that chiefly governs their temperature dependence. Thus we can use the above calculation of the Coulomb barrier to estimate the exponent ν in the power law $\epsilon \propto (k_B T)^\nu$ that is used to estimate the temperature dependence of the energy generation rate ϵ. We take

$$\nu = T\frac{d}{dT}\ln B_T \simeq \frac{1}{3}\left(Z_1^2 Z_2^2 (\mu/m_p) \times \frac{7.726 \times 10^{10}\,\mathrm{K}}{T}\right)^{1/3}. \tag{1.5.12}$$

(The $T^{-1/3}$ temperature dependence here is sufficiently weak to justify approximating ϵ as proportional to a constant power of temperature.) From Eqs. (1.5.11) and (1.5.12) we infer the general rule that ν is one-third the absolute value of the exponent in whatever barrier penetration factor dominates the temperature dependence of the energy generation rate.

Let us now apply these general remarks to stars that derive their nuclear energy either from the proton–proton chain or from the CNO cycle.

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Proton–Proton Chain

For the first reaction $p + p \to d + e^+ + \nu$ in the proton–proton chain we take $\mu = m_p/2$ and $Z_1 = Z_2 = 1$, so, according to Eq. (1.5.11), if $T = 10^7$K (roughly the temperature at the center of the Sun), the Coulomb barrier suppresses the reaction by a factor $\exp(-15.7) = 1.5 \times 10^{-7}$.

But the reaction $p+p \to d+e^+ +\nu$ is not the end of the story; it is just the first step in a chain of reactions. The Coulomb barrier suppression of the second step, ${}^1H+{}^2H \to {}^3He+\gamma$, is only slightly more severe than that of reaction I, because the charges of the nuclei are the same, and their reduced mass is larger only by a factor 4/3. Taking $\mu = 2m_p/3$, $Z_1 = Z_2 = 1$, and $T \simeq 10^7$ K in Eq. (1.5.11) gives $B_T \approx \exp(-4/3 \times 15.7) = 8 \times 10^{-10}$. Apart from Coulomb suppression, since step I involves a weak interaction it is suppressed by an additional factor of order 10^{-22} and since step II involves an electromagnetic interaction it is suppressed by an additional factor of order 1/137, so the ratio of the rate per proton of step I and the rate per deuteron of step II is expected to be of order

$$\frac{\text{rate/p of } p + p \to d + e^+ + \nu}{\text{rate/d of } p + d \to {}^3He + \gamma} \approx \frac{10^{-22} \times (1.5 \times 10^{-7})}{(1/137) \times (8 \times 10^{-10})} \simeq 3 \times 10^{-18}.$$

(The actual ratio is about 10^{-17}.) Reaction III has a more formidable Coulomb barrier, with $Z_1 Z_2 = 4$. All three reactions release substantial amounts of energy. So which do we need to calculate in order to find ϵ. And in particular, which is the relevant Coulomb barrier?

The answer relies on an assumption of time-independence: The abundances of the intermediate participants in these reactions rapidly evolve to stable values, for which these abundances change little over times in which a very large number of reactions take place in the star's core. Thus, in order that the abundance of deuterons should not change, the rates per volume of reactions I and II, in which deuterons are respectively created and destroyed, should be the same, and in order that the abundance of ^{3}He nuclei should not change, the rate per volume of reaction II should be twice that of reaction III, in which two ^{3}He nuclei are destroyed:

$$\Gamma \equiv \Gamma(\text{I}) = \Gamma(\text{II}) = 2\Gamma(\text{III}), \tag{1.5.13}$$

where the Γs denote the rates per volume of various reactions. It is like the law of economics that supply equals demand. If demand exceeds supply prices will go up, damping demand and providing an incentive for increased supply, until supply and demand approach each other. (Or so they say.) In the same way, if the rate per volume of reaction II were less than that of reaction I the abundance of ^{2}H nuclei would rise until these rates were equal, and just as many ^{2}H nuclei were being destroyed as created. According to the above estimate of the ratio of the rate per proton of reaction I and the rate per deuteron of reaction II, we therefore expect the number density of deuterons to be smaller than the number density of protons by a factor of order 3×10^{-18}.

Though $\Gamma(\text{I})$, $\Gamma(\text{II})$, and $2\Gamma(\text{III})$ must all be equal, their calculation differs in one important respect. The rate of reaction I does not depend on the abundance of the intermediate nuclei ^{2}H and ^{3}He, and in particular is not suppressed by their low abundance, so it can be calculated without knowing anything about the other reactions. Thus it is the Coulomb barrier in reaction I that governs the rate at which hydrogen is converted to helium and energy is produced, and its temperature dependence. In particular, in accordance with the general rule (1.5.12), for the proton–proton cycle the exponent ν in the temperature dependence of ϵ is one-third of the value 15.7 that we previously calculated for the exponent in the barrier penetration factor for reaction I, so $\nu \simeq 5$ at $T \approx 10^7$ K. Fortunately ν has only a mild dependence on temperature, going as $T^{-1/3}$, so this estimate of ν is a fair approximation for a wide range of temperatures.

But although we only need to calculate the rate Γ of reaction I, all of reactions I, II, and III release energy, say an energy E_{I}, E_{II}, and E_{III} per reaction, so the rate $\epsilon\rho$ of total energy production per volume is not just $E_{\text{I}}\Gamma$, but

$$\epsilon\rho = \left(E_{\text{I}} + E_{\text{II}} + \frac{1}{2}E_{\text{III}}\right)\Gamma = 13.1\ \text{MeV} \times \Gamma. \tag{1.5.14}$$

The crucial first step in the proton–proton chain is a collision of two protons. Its rate, and hence the rate per volume $\epsilon\rho$ of energy generation due to the proton–proton chain, is proportional to ρ^2. Hence, if the proton–proton chain dominates nuclear energy generation, we have $\lambda = 1$ as well as $\nu \approx 5$.

The reactions (1.5.2) dominate the energy production in the proton–proton chain, but there are alternative finales to this chain, one of which is of historical importance. In one alternative, instead of a pair of ^{3}He nuclei combining in reaction III, individual ^{3}He nuclei undergo the reaction

$$\text{III}': \quad {}^3\text{He} + {}^4\text{He} \to {}^7\text{Be} + \gamma$$

followed by either

$$\begin{aligned} \text{IV}: &\quad {}^7\text{Be} + e^- \to {}^7\text{Li} + \nu_e \\ \text{V}: &\quad {}^7\text{Li} + {}^1\text{H} \to {}^4\text{He} + {}^4\text{He} \end{aligned} \tag{1.5.15}$$

or else

$$\begin{aligned} \text{IV}': &\quad {}^7\text{Be} + {}^1\text{H} \to {}^8\text{B} + \gamma \\ \text{V}': &\quad {}^8\text{B} \to {}^8\text{Be} + e^+ + \nu_e \\ \text{VI}': &\quad {}^8\text{Be} \to {}^4\text{He} + {}^4\text{He}. \end{aligned} \tag{1.5.16}$$

The probability of a ^{3}He nucleus undergoing the reaction III$'$ rather than III is small, so these alternatives have little effect on the energy generation rate ϵ

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and its density and temperature dependence, but the high energy of the neutrino from the ^{8}B beta decay in reaction V′, extending up to over 10 MeV, offered an early opportunity of observing neutrinos from the Sun.

The reaction $^{37}\text{Cl} + \nu_e \rightarrow {}^{37}\text{Ar} + e^-$ that was used to search for solar neutrinos in the experiments of Davis *et al.*[25] on solar neutrinos is sensitive only to these high-energy neutrinos, not to the much lower-energy neutrinos emitted in the other reactions of the proton–proton chain. The high Coulomb barriers in reactions III′ and IV′ make the flux of high-energy neutrinos extremely sensitive to the temperature profile in the Sun. Detailed calculations by John Bahcall[26] (1934–2005) showed that the high-energy neutrinos should be observable in Davis's experiments, but decades of searching did not find them. Finally solar neutrinos were detected[27] using the reaction $\text{Ga}^{71} + \nu_e \rightarrow \text{Ge}^{71} + e^-$, but the observed rate was substantially less than predicted by Bahcall. Either Bahcall's calculations were inaccurate, or something was happening to neutrinos on the way to the Earth.

In particular, it was speculated by Bruno Pontecorvo (1913–1993) that neutrinos have mass, and that the states of definite mass are not the neutrinos of electron type emitted in the Sun, but superpositions of neutrinos of electron type with neutrinos of muon and tauon type, so that on the way to Earth electron-type neutrinos become an oscillating superposition of types, with only the electron-type fraction observable in reactions like $\text{Cl}^{37}+\nu_e \rightarrow \text{Ar}^{37}+e^-$ or $\text{Ga}^{71}+\nu_e \rightarrow \text{Ge}^{71} + e^-$. The issue was settled by experiments at the Sudbury Neutrino Observatory.[28] By monitoring a large tank of heavy water, experimenters could detect high-energy ^{8}B neutrinos not only in the reaction $\nu_e + d \rightarrow p + p + e^-$, which is sensitive only to electron-type neutrinos, but also in the neutral current process $\nu + d \rightarrow p + n + \nu$, which is equally sensitive to neutrinos of all types, electron, muon, and tauon. It turned out that the total flux of neutrinos of all types agreed with Bahcall's calculations, providing a decisive vote in favor of neutrino oscillations. Since then the existence of neutrino oscillations has been confirmed and neutrino masses and mixing angles measured in numerous terrestrial experiments.

CNO Cycle

Matters are more complicated for the CNO cycle. Here too we assume that the abundances of the intermediate CNO nuclei settle down to constant values. The constancy of the abundance of ^{13}N requires that reactions i and ii have the same rate per volume; the constancy of the abundance of ^{13}C requires that reactions

[25] R. Davis, D. S. Harmer, and K. C. Hoffman, *Phys. Rev. Lett.* **20**, 1205 (1968).
[26] J. N. Bahcall, *Current Science* **77**, 1487 (1999), and earlier references quoted therein.
[27] P. Anselmann *et al.*, *Phys. Lett.* **B342**, 440 (1995); J. N. Abdurashitov *et al.*, *Phys. Rev. Lett.* **77**, 3708 (1996).
[28] Q. R. Ahmad *et al.*, *Phys. Rev. Lett.* **89**, 11301 (2002).

ii and iii have the same rate per volume; and so on, so that all these rates per volume are equal:

$$\Gamma(\mathrm{i}) = \Gamma(\mathrm{ii}) = \Gamma(\mathrm{iii}) = \Gamma(\mathrm{iv}) = \Gamma(\mathrm{v}) = \Gamma(\mathrm{vi}) \equiv \Gamma. \tag{1.5.17}$$

This determines the *ratios* of the abundances. Each of the rates here is proportional to the number density n of the CNO nucleus in the initial state of the reaction

$$\Gamma(\mathrm{i}) = n\left({}^{12}\mathrm{C}\right)R(\mathrm{i}), \quad \Gamma(\mathrm{ii}) = n\left({}^{13}\mathrm{N}\right)R(\mathrm{ii}), \text{ etc.,} \tag{1.5.18}$$

with the rate factors R independent of the densities of anything but hydrogen. For each reaction, R is the rate at which the CNO nucleus in the initial state undergoes that reaction. For instance, $R(\mathrm{i})$ is the rate at which any individual ${}^{12}\mathrm{C}$ nucleus undergoes the reaction ${}^{1}\mathrm{H} + {}^{12}\mathrm{C} \to {}^{13}\mathrm{N} + \gamma$. Then the equality of rates (1.5.17) gives

$$\frac{n\left({}^{13}\mathrm{N}\right)}{n\left({}^{12}\mathrm{C}\right)} = \frac{R(\mathrm{i})}{R(\mathrm{ii})}, \quad \frac{n\left({}^{13}\mathrm{C}\right)}{n\left({}^{12}\mathrm{C}\right)} = \frac{R(\mathrm{i})}{R(\mathrm{iii})}, \quad \text{etc.} \tag{1.5.19}$$

But we cannot in this way find the overall number density of the CNO nuclei

$$n(\mathrm{CNO}) \equiv n\left({}^{12}\mathrm{C}\right) + n\left({}^{13}\mathrm{N}\right) + n\left({}^{13}\mathrm{C}\right) + n\left({}^{14}\mathrm{N}\right) + n\left({}^{15}\mathrm{O}\right) + n\left({}^{15}\mathrm{N}\right), \tag{1.5.20}$$

which does not change in the reactions i through vi, and is determined by the abundances in the interstellar medium from which the star formed. We can, however, express the common rate Γ in terms of $n(\mathrm{CNO})$: Using Eqs. (1.5.20) and (1.5.17) and then (1.5.18) we note that

$$\begin{aligned}\frac{n(\mathrm{CNO})}{\Gamma} &= \frac{n\left({}^{12}\mathrm{C}\right)}{\Gamma(\mathrm{i})} + \frac{n\left({}^{13}\mathrm{N}\right)}{\Gamma(\mathrm{ii})} + \frac{n\left({}^{13}\mathrm{C}\right)}{\Gamma(\mathrm{iii})} + \frac{n\left({}^{14}\mathrm{N}\right)}{\Gamma(\mathrm{iv})} + \frac{n\left({}^{15}\mathrm{O}\right)}{\Gamma(\mathrm{v})} + \frac{n\left({}^{15}\mathrm{N}\right)}{\Gamma(\mathrm{vi})} \\ &= \frac{1}{R(\mathrm{i})} + \frac{1}{R(\mathrm{ii})} + \frac{1}{R(\mathrm{iii})} + \frac{1}{R(\mathrm{iv})} + \frac{1}{R(\mathrm{v})} + \frac{1}{R(\mathrm{vi})},\end{aligned}$$

so the common rate is

$$\Gamma = n(\mathrm{CNO}) \Big/ \left(\frac{1}{R(\mathrm{i})} + \frac{1}{R(\mathrm{ii})} + \frac{1}{R(\mathrm{iii})} + \frac{1}{R(\mathrm{iv})} + \frac{1}{R(\mathrm{v})} + \frac{1}{R(\mathrm{vi})}\right). \tag{1.5.21}$$

That is, the common rate of the reactions equals the harmonic mean of what the individual rates would be if the density of the CNO nucleus in each initial state equaled the total density $n(\mathrm{CNO})$. The rate per volume $\epsilon\rho$ of energy generation in the CNO cycle is Γ times the sum of the energies in Eq. (1.5.3):

$$\epsilon\rho = \Gamma \times 25.03 \text{ MeV}. \tag{1.5.22}$$

Because of the absence of a Coulomb barrier in the beta decays ii and v, these reactions have relatively rapid rates R per CNO nucleus, with mean lives $1/R$

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of 7 minutes and 82 seconds, respectively, while $1/R$ for all the other reactions in the CNO cycle is at least 10^5 years. Thus the terms $1/R(\text{ii})$ and $1/R(\text{v})$ can be neglected in the denominator in Eq. (1.5.21). Also, for the same reason, the number density of the CNO nucleus in the initial states of the beta decay reactions is much smaller than the number densities of the other CNO nuclei, and can be neglected in $n(\text{CNO})$. Thus Eq. (1.5.21) for the rate Γ of the various reactions in the CNO channel is dominated by the two-body reactions i, iii, iv, and vi. As two-body reactions, they all have $\lambda = 1$. Also, these reactions all have about the same value of the reduced mass, ranging from $12m_p/13$ to $15m_p/16$, while Z_1Z_2 only ranges from 6 for reaction i to 7 for reaction vi, so the Coulomb suppression factor and hence the rate factor R is smallest for reaction vi, but not overwhelmingly so. We will take the Coulomb barriers of these reactions to be a compromise, calculated by taking $Z_1Z_2 = 6.5$ and $\mu = m_p$. At any given temperature, the exponent in Eq. (1.5.10) for the effective Coulomb barrier is thus larger than for the proton–proton chain by a factor $6.5^{2/3}2^{1/3} = 4.4$. At the nominal temperature of 10^7 K, the Coulomb barrier in the CNO cycle produces a suppression factor $\exp(-4.4 \times 15.7) \simeq 10^{-30}$. It is only because of the extreme slowness of weak interaction processes such as the first step in the proton–proton chain that the CNO cycle can compete with the proton–proton chain at any temperature.

The power of temperature in Eq. (1.5.1) is larger than for the proton–proton chain by the same factor 4.4, so at $T \approx 10^7$ K we have $\nu \approx 22$, and somewhat less at higher temperatures. As already mentioned, the power of density is $\lambda = 1$.

Here too there are alternative finales. Instead of step vi, the ^{15}N nucleus can undergo the reaction $^1\text{H} + {}^{15}\text{N} \to {}^{16}\text{O} + \gamma$, followed by $^1\text{H} + {}^{16}\text{O} \to {}^{17}\text{F} + \gamma$ and $^{17}\text{F} \to {}^{17}\text{O} + e^+ + \nu$. After that, there are again two possibilities: either $^1\text{H} + {}^{17}\text{O} \to {}^{14}\text{N} + {}^4\text{He}$, or else $^1\text{H} + {}^{17}\text{O} \to {}^{18}\text{F} + \gamma$ followed by $^{18}\text{F} \to {}^{18}\text{O} + e^+ + \nu$ and $^1\text{H} + {}^{18}\text{O} \to {}^{15}\text{N} + {}^4\text{He}$. In all cases the net effect is that four protons turn into a ^{4}He nucleus plus two positrons and two neutrinos, with the CNO catalysts always returned to their original abundances.

Crossover

We can now estimate the crossover temperature at which the rates of energy production in the CNO cycle and proton–proton chain would be equal. We have seen that the rate of the reactions in the proton–proton chain is suppressed by the Coulomb barrier by a factor $\exp\left(-15.7(T\,[10^7\,\text{K}])^{-1/3}\right)$, so the rate of the reactions in the CNO cycle is suppressed by a factor $\exp\left(-4.4 \times 15.7(T\,[10^7\,\text{K}])^{-1/3}\right)$. It is further suppressed relative to the proton–proton chain by the ratio of the number of CNO nuclei to hydrogen nuclei, which for the Sun is about 10^{-3}, and since a photon is emitted, also by a factor $e^2/\hbar c \simeq 10^{-2}$. On the other hand, the reaction $p + p \to d + e^+ + \nu$ in

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the proton–proton chain is a weak interaction, so its rate is proportional to the square of the weak coupling constant, and is therefore suppressed by a dimensionless factor $(G_{\rm wk}E^2)^2$, which for $E \approx 1$ MeV is about 10^{-22}. So, very roughly, the crossover temperature at which the CNO cycle and the proton–proton chain have competitive rates is given by

$$10^{-3} \times 10^{-2} \times \exp\big(-4.4 \times 15.7(T\,[10^7\,\text{K}])^{-1/3}\big) \approx 10^{-22} \times \exp\big(-15.7(T\,[10^7\,\text{K}])^{-1/3}\big),$$

or $T \approx 2.5 \times 10^7$ K. This is not very different from the value given by more detailed calculations,[29] which is not much greater than the temperature 1.36×10^7 K at the center of the Sun. For stars that are considerably more or less massive than the Sun the central temperature is higher or lower, and it is respectively the CNO cycle or the proton–proton chain that dominates energy production.

Beyond Hydrogen Burning

As mentioned in Section 1.3, when the hydrogen has been converted to helium in a star's center, the star leaves the main sequence and becomes a red giant, in which the conversion of hydrogen to helium continues in a shell surrounding the helium core. The core temperature continues to grow, and when it becomes sufficiently high it becomes the turn of helium to undergo nuclear reactions. Although there is no stable nucleus that can be formed in a collision of a proton and a ^{4}He nucleus or in the collision of two ^{4}He nuclei, the latter collision can produce an unstable state of the nucleus ^{8}Be that lives long enough before it undergoes fission back into two ^{4}He nuclei, so that it can serve as an intermediary in the carbon production reactions

$$\begin{aligned} \text{a}: &\quad {}^4\text{He} + {}^4\text{He} \to {}^8\text{Be} + \gamma \\ \text{b}: &\quad {}^4\text{He} + {}^8\text{Be} \to {}^{12}\text{C} + \gamma. \end{aligned} \tag{1.5.23}$$

Although this is a sequence of two-body reactions, it does not lead to an energy production rate per volume $\epsilon\rho$ proportional to ρ^2, as in the proton–proton chain and the CNO cycle. The reason is that there is only a small probability $\mathcal{P}$ for the ^{8}Be nucleus to absorb another ^{4}He nucleus before it fissions. Thus $\epsilon\rho$ is proportional to $\rho^2\mathcal{P}$, and since $\mathcal{P}$ when small is proportional to ρ, $\epsilon\rho$ is proportional to ρ^3, and therefore the exponent λ in Eq. (1.5.1) is $\lambda = 2$.

As usual, the temperature dependence of ϵ is harder to estimate. Reaction a is endothermic, requiring an energy E of relative motion of the two ^{4}He nuclei of at least 92 keV. In order for ^{4}He nuclei to have any chance of having energies this large, the temperature must be at least 10^8 K. Even at such relatively high

[29] R. J. Tayler, *The Stars: Their Structure and Evolution* (Wykeham Publications, London, 1970), Figure 39m gives the crossover temperature as 1.7×10^7 K, while F. LeBlanc, *An Introduction to Stellar Astrophysics* (John Wiley & Sons, Winchester, 2010), Figure 6.7 gives 1.9×10^7 K.

temperatures, there are sizable Coulomb barriers both in the rate for reaction a and in the probability $\mathcal{P}$ that a ^{8}Be nucleus will experience reaction b instead of fissioning. The only reason[30] why carbon production is non-negligible at temperatures of order 10^8 K to 10^9 K is that there is an unstable state of ^{12}C that provides a resonance in the $^4\text{He}+^8\text{Be}$ channel at an accessible excitation energy of 310 keV. This unstable state has an appreciable chance of decaying into the stable ground state of carbon, with the emission of a 7.4 MeV photon. Because of the pair of Coulomb barriers plus the exothermic nature of reaction a, the exponent ν in Eq. (1.5.1) for the temperature dependence of carbon production is quite large, estimated to be of order 30 to 40, depending on the temperature.

Once ^{12}C is formed in this way, it is possible to produce heavier nuclei in various reactions that are suppressed mostly by Coulomb barriers: $^4\text{He}+^{12}\text{C} \to {}^{16}\text{O}+\gamma$, $^4\text{He}+^{16}\text{O} \to {}^{24}\text{Mg}+\gamma$, $^{12}\text{C}+^{12}\text{C} \to {}^{24}\text{Mg}+\gamma$, and so on. There are also reactions that destroy but do not produce various light nuclei with relatively small binding energies, including ^{2}H, ^{3}He, ^{6}Li, ^{7}Li, ^{9}Be, ^{10}B, and ^{11}B. Where these nuclei are found spectroscopically in interstellar clouds, their measured abundance provides a valuable lower bound on the cosmological abundance of light elements left over from the beginning of the big bang.

Appendix: Calculation of Suppression by Coulomb Barriers

Classically, the total energy of a pair of nuclei interacting through a central potential $V(r)$ is

$$E_{\text{tot}} = \frac{\mathbf{p}_1^2}{2m_1} + \frac{\mathbf{p}_2^2}{2m_2} + V(r) = \frac{\mathbf{P}^2}{2(m_1+m_2)} + \frac{\mathbf{p}^2}{2\mu} + V(r), \tag{1.5.A1}$$

where $\mathbf{p}$ and $\mathbf{P}$ are the relative and total momenta, where

$$\mathbf{p} = \mu\left(\frac{\mathbf{p}_1}{m_1} - \frac{\mathbf{p}_2}{m_2}\right), \qquad \mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2, \tag{1.5.A2}$$

and μ again is the reduced mass

$$\mu = \frac{m_1 m_2}{m_1 + m_2}.$$

Since both E_{tot} and $\mathbf{P}$ are time-independent, they can be expressed at any time in terms of the relative and total momenta $\mathbf{p}_0$ and $\mathbf{P}_0$ at a time t_0 early enough that the nuclei are so far apart that $V(r)$ is negligible:

$$E_{\text{tot}} = \frac{\mathbf{P}_0^2}{2(m_1+m_2)} + \frac{\mathbf{p}_0^2}{2\mu}, \qquad \mathbf{P} = \mathbf{P}_0. \tag{1.5.A3}$$

[30] E. E. Salpeter, *Astrophys. J.* **115**, 326 (1952).

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We are assuming here that the potential depends only on the separation $r \equiv |\mathbf{x}|$, $\mathbf{x} = \mathbf{x}_1 - \mathbf{x}_2$.

Quantum mechanically, the probability of finding the nuclei with separation vector $\mathbf{x}_1 - \mathbf{x}_2$ in a small volume $d^3\mathbf{x}$ around $\mathbf{x}$ and center-of-mass position $(m_1\mathbf{x}_1 + m_2\mathbf{x}_2)/(m_1 + m_2)$ in a small volume $d^3\mathbf{X}$ around $\mathbf{X}$ is given in terms of a wave function $\psi(\mathbf{x}, \mathbf{X})$ by $|\psi(\mathbf{x}, \mathbf{X})|^2 \, d^3\mathbf{x} \, d^3\mathbf{X}$. The wave function satisfies the Schrödinger equation $H\psi = E_{\text{tot}}\psi$, where E_{tot} is the numerical quantity given by Eq. (1.5.A3), and H is the Hamiltonian operator, given by replacing $\mathbf{p}$ and $\mathbf{P}$ on the right-hand side of Eq. (1.5.A1) with $-i\hbar$ times gradients with respect to the separation $\mathbf{x}$ and the center-of-mass position $\mathbf{X} = (m_1\mathbf{x}_1 + m_2\mathbf{x}_2)/(m_1 + m_2)$. The Schrödinger equation is then

$$E_{\text{tot}}\psi(\mathbf{x}, \mathbf{X}) = \left[-\frac{\hbar^2}{2(m_1 + m_2)} \nabla^2_{\mathbf{X}} - \frac{\hbar^2}{2\mu} \nabla^2_{\mathbf{x}} + V(r) \right] \psi(\mathbf{x}, \mathbf{X}). \quad (1.5.\text{A4})$$

We can always find a solution of the form

$$\psi(\mathbf{x}, \mathbf{X}) = e^{i\mathbf{P}\cdot\mathbf{X}/\hbar} \psi_E(\mathbf{x}), \quad (1.5.\text{A5})$$

where E is the energy of relative motion, defined by

$$E_{\text{tot}} = \frac{\mathbf{P}^2}{2(m_1 + m_2)} + E, \quad (1.5.\text{A6})$$

and

$$E\psi_E(\mathbf{x}) = \left[-\frac{\hbar^2}{2\mu} \nabla^2_{\mathbf{x}} + V(r) \right] \psi_E(\mathbf{x}). \quad (1.5.\text{A7})$$

This is supposed to hold only outside some very small radius r_0, within which nuclear reactions occur.

To solve this equation, we can often employ the WKB approximation. We suppose that for a range of radii $r > r_0$, $V(r) - E$ is positive and sufficiently large that $V(r)$ changes little in a distance $1/\kappa_E(r)$, where

$$\kappa_E(r) = \left[\frac{2\mu}{\hbar^2} \big(V(r) - E \big) \right]^{1/2}.$$

Then, in this range of r,

$$\psi_E(r) \simeq C_+ \exp\left(+ \int_{r_0}^{r} \kappa_E(r') \, dr' \right) + C_- \exp\left(- \int_{r_0}^{r} \kappa_E(r') \, dr' \right). \quad (1.5.\text{A8})$$

The nuclear reactions that occur within the radius r_0 fix the ratio C_+/C_- to take some value of order unity, which we will not need to calculate. We suppose further that $V(r)$ eventually decreases to zero for $r \to \infty$. Equation (1.5.A8) must break down when r approaches a radius r_E where $V(r_E) = E$, at which

$\kappa(r_E) = 0$. We take the potential barrier between r_0 and r_E to be sufficiently high and thick that

$$\int_{r_0}^{r_E} \kappa_E(r)\, dr \gg 1.$$

Then, for $r = r_E$, Eq. (1.5.A8) reads

$$\psi_E(r_E) \simeq C_+ \exp\left(+\int_{r_0}^{r_E} \kappa_E(r')\, dr'\right),$$

the other term in Eq. (1.5.A8) being negligible. For $r > r_E$ the function $\psi_E(r)$ oscillates, with little change in amplitude, so $|\psi_E(r_E)|$ is determined by the wave function representing the approach of the nuclei from a large separation. Thus the rate of nuclear reactions is suppressed by a barrier penetration factor

$$B(E) \simeq \left|\frac{C_+}{\psi_E(r_E)}\right|^2 = \exp\left(-2\int_{r_0}^{r_E} \kappa_E(r')\, dr'\right). \tag{1.5.A9}$$

For a Coulomb barrier, we have $V(r) = Z_1 Z_2 e^2/r$, so, taking $r_0 \ll r_E$, we have

$$B(E) \simeq \exp\left[-2\int_0^{r_E} dr \sqrt{\frac{2\mu Z_1 Z_2 e^2}{\hbar^2}\left(\frac{1}{r} - \frac{1}{r_E}\right)}\right], \tag{1.5.A10}$$

where $r_E = Z_1 Z_2 e^2/E$. To do this integral, we set $r = r_E u^2$, and use $\int_0^1 du\, \sqrt{1-u^2} = \pi/4$, so that

$$B(E) \simeq \exp\left[-\pi\sqrt{\frac{2\mu Z_1 Z_2 e^2 r_E}{\hbar^2}}\right] = \exp\left[-\pi Z_1 Z_2 e^2 \sqrt{\frac{2\mu}{\hbar^2 E}}\right], \tag{1.5.A11}$$

as was to be shown.

1.6 Relations among Observables: The Main Sequence

As we have seen in Section 1.3, we expect on very general grounds that stellar parameters such as radius, luminosity, central temperature, effective surface temperature, etc. all depend only on the star's mass, age, and initial chemical composition. This is why, when any pair of these parameters for a sample of stars in a cluster that all began at the same time with the same uniform chemical composition are plotted against each other, the values of these parameters will fall close to a one-dimensional curve, such as the Hertzsprung–Russell diagram comparing luminosity and effective surface temperature. But to find the *form* of these curves requires detailed physical assumptions and numerical calculation.

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We shall see in this section that for stars that are still on the main sequence, burning hydrogen at their cores, it is possible to make a good estimate of the form of these curves using dimensional analysis, together with the assumption of power-law behavior for the rate per mass ϵ of nuclear energy generation and for the opacity κ:

$$\epsilon = \epsilon_1 \rho^{\lambda} (k_B T)^{\nu}, \qquad \kappa = \kappa_1 \rho^{\alpha} (k_B T)^{\beta}, \tag{1.6.1}$$

with κ_1 and ϵ_1, α, β, λ, and ν all constants assumed to depend only on chemical composition. (Section 1.4 found $\alpha = \beta = 0$ for Thomson scattering, and $\alpha = 1$ and $\beta = -7/2$ for free–free absorption. Section 1.5 found $\lambda = 1$ for the proton–proton chain and CNO cycle; $\nu \approx 5$ for the proton–proton chain and larger for the CNO cycle, and ν weakly dependent on temperature, with $\nu \propto T^{-1/3}$.) Our discussion in this section will be limited to stars in which thermal energy is transported only by radiation. In the following section we shall show that the presence of convective energy transport does not change our main conclusions.

With these assumptions, each stellar parameter will turn out to be dependent only on the star's mass M and a pair of quantities N_1 and N_2 that depend on chemical composition and fundamental physical constants. Since there are no dimensionless ratios among M, N_1, and N_2, any stellar parameter will be proportional to a product of powers of M, N_1, and N_2, with exponents fixed by dimensional analysis. This only works for stars on the main sequence whose chemical composition (on which κ_1, α, etc. depend) is still approximately uniform. For red giant stars whose stellar parameters also depend on the radius of the helium core, dimensional analysis is not enough. It is also not enough even if we assume that non-uniformities evolve from an initially uniform composition, because then stellar parameters depend on the *age* of the star, as well as on M, N_1, and N_2.

To carry out our dimensional analysis, we write Eqs. (1.3.3) and (1.3.4) in terms of ρ, $k_B T$, and $\mathcal{L}^* \equiv \mathcal{L}/\epsilon_1$:

$$\frac{d\mathcal{L}^*(r)}{dr} = 4\pi r^2 \rho^{\lambda+1}(r) \big(k_B T(r)\big)^{\nu}, \tag{1.6.2}$$

$$\frac{d\big(k_B T(r)\big)^4}{dr} = -3N_1 \rho^{\alpha+1}(r) \big(k_B T(r)\big)^{\beta} \frac{\mathcal{L}^*(r)}{4\pi r^2}, \tag{1.6.3}$$

where

$$N_1 \equiv \frac{\kappa_1 \epsilon_1 k_B^4}{ca}. \tag{1.6.4}$$

We will begin by assuming that the pressure p is dominated by gas pressure, as is the case for all but the most massive stars. (We will return at the end of this section to stars in which p is dominated by radiation pressure.) The pressure then is well approximated by the ideal gas law, $p = k_B T \rho / m_1 \mu$, where μ is the

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molecular weight and m_1 is the mass of unit atomic weight. Then Eqs. (1.3.1) and (1.3.2) are

$$\frac{d\big(\rho(r)k_{\rm B}T(r)\big)}{dr} = -N_2\frac{\mathcal{M}(r)\rho(r)}{4\pi r^2}, \tag{1.6.5}$$

$$\frac{d\mathcal{M}(r)}{dr} = 4\pi r^2\rho(r), \tag{1.6.6}$$

where

$$N_2 \equiv 4\pi G m_1 \mu. \tag{1.6.7}$$

For uniform chemical composition, the stellar parameters R, $L^* \equiv L/\epsilon_1$, $\rho(0)$, $k_{\rm B}T(0)$, etc. can depend only on N_1, N_2, and M.

Next we must work out the dimensionalities of N_1 and N_2 in powers of length, time, and mass. We note that the energy production rate per mass has dimensions

$$[\epsilon] = [\text{energy}][\text{mass}]^{-1}[\text{time}]^{-1} = [\text{velocity}]^2[\text{time}]^{-1} = [\text{length}]^2[\text{time}]^{-3},$$

so

$$\begin{aligned}[\epsilon_1] &= [\text{length}]^2[\text{time}]^{-3}[\text{mass/length}^3]^{-\lambda}[\text{energy}]^{-\nu} \\ &= [\text{length}]^{2+3\lambda-2\nu}[\text{time}]^{-3+2\nu}[\text{mass}]^{-\lambda-\nu}.\end{aligned}$$

Also, since $1/\kappa\rho$ is the mean free path, the opacity has dimensions $[\kappa] = [\text{length}]^{-1}/[\text{mass/length}^3]$, so

$$\begin{aligned}[\kappa_1] &= [\text{length}]^{-1}[\text{mass/length}^3]^{-1-\alpha}[\text{energy}]^{-\beta} \\ &= [\text{length}]^{2+3\alpha-2\beta}[\text{time}]^{2\beta}[\text{mass}]^{-1-\alpha-\beta}.\end{aligned}$$

Finally,

$$\begin{aligned}[ca/k_{\rm B}^4] &= [\text{energy}][\text{time}]^{-1}[\text{area}]^{-1}[\text{energy}]^{-4} \\ &= [\text{energy}]^{-3}[\text{length}]^{-2}[\text{time}]^{-1} \\ &= [\text{length}]^{-8}[\text{time}]^{5}[\text{mass}]^{-3}.\end{aligned}$$

Thus

$$[N_1] = [\text{length}]^{12+3\lambda-2\nu+3\alpha-2\beta}[\text{time}]^{-8+2\nu+2\beta}[\text{mass}]^{2-\lambda-\nu-\alpha-\beta} \tag{1.6.8}$$

and

$$[N_2] = [G][\text{mass}] = [\text{velocity}]^2[\text{length}] = [\text{length}]^3[\text{time}]^{-2}. \tag{1.6.9}$$

To calculate the stellar radius R, we ask what product of form $M^A N_1^{A_1} N_2^{A_2}$ has the dimensions of length. Setting the numbers of powers of length, time, and mass in this product respectively equal to $+1$, 0, and 0, we find

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$$\text{powers of length}: 1 = (12 + 3\lambda - 2\nu + 3\alpha - 2\beta)A_1 + 3A_2, \tag{1.6.10}$$
$$\text{powers of time}: \ 0 = (-8 + 2\nu + 2\beta)A_1 - 2A_2, \tag{1.6.11}$$
$$\text{powers of mass}: \ 0 = A + (2 - \lambda - \nu - \alpha - \beta)A_1. \tag{1.6.12}$$

Using Eq. (1.6.11) to eliminate A_2 in Eq. (1.6.10) gives A_1; Eq. (1.6.11) then gives A_2; and using this in Eq. (1.6.12) gives A. In this way we find

$$A = \frac{-2 + \lambda + \nu + \alpha + \beta}{3\lambda + \nu + 3\alpha + \beta}, \tag{1.6.13}$$

$$A_1 = \frac{1}{3\lambda + \nu + 3\alpha + \beta}, \tag{1.6.14}$$

$$A_2 = \frac{-4 + \nu + \beta}{3\lambda + \nu + 3\alpha + \beta}, \tag{1.6.15}$$

and so

$$R \cong M^A N_1^{A_1} N_2^{A_2}, \tag{1.6.16}$$

with A, A_1, and A_2 given by Eqs. (1.6.13)–(1.6.15). (Here we use $\cong$ to mean "proportional to, and since there are no very large or very small dimensionless constants in the differential equations, also roughly equal to.")

Likewise, the luminosity has dimensions

$$[L] = [\text{energy}]/[\text{time}] = [\text{length}]^2[\text{time}]^{-3}[\text{mass}],$$

so $L^* \equiv L/\epsilon_1$ has dimensions

$$[L^*] = [\text{length}]^{-3\lambda+2\nu}[\text{time}]^{-2\nu}[\text{mass}]^{1+\lambda+\nu}.$$

Following the same procedure as above for R, we find that the unique product of powers of M, N_1, and N_2 that has the same dimensionality as L^* is

$$L^* \cong M^B N_1^{B_1} N_2^{B_2},$$

where

$$B = \frac{(1 + \lambda + \nu)(3\alpha + \beta) + (3 - \alpha - \beta)(3\lambda + \nu)}{3\lambda + \nu + 3\alpha + \beta}, \tag{1.6.17}$$

$$B_1 = -\frac{3\lambda + \nu}{3\lambda + \nu + 3\alpha + \beta}, \tag{1.6.18}$$

$$B_2 = \frac{\nu(3\alpha + \beta) + (4 - \beta)(3\lambda + \nu)}{3\lambda + \nu + 3\alpha + \beta}. \tag{1.6.19}$$

We conclude then that

$$L = \epsilon_1 L^* \cong \epsilon_1 M^B N_1^{B_1} N_2^{B_2}. \tag{1.6.20}$$

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The same reasoning can be applied to other quantities, such as the temperature at the center of the star. The only combination of N_1, N_2, and M that has the same dimensions as $k_B T$ is $M^C N_1^{C_1} N_2^{C_2}$, where

$$C = 2\frac{\lambda + \alpha + 1}{3\lambda + \nu + 3\alpha + \beta}, \tag{1.6.21}$$

$$C_1 = -\frac{1}{3\lambda + \nu + 3\alpha + \beta}, \tag{1.6.22}$$

$$C_2 = 1 + 4C_1, \tag{1.6.23}$$

so we conclude that the central temperature is

$$k_B T(0) \cong M^C N_1^{C_1} N_2^{C_2}. \tag{1.6.24}$$

At this point the reader may be wondering why the central temperatures of stars are so different from their effective surface temperatures, despite their having the same dimensionality. For instance, the effective surface temperature of the Sun is measured as $T_{\text{eff},\odot} = 5,800\,\text{K}$, while detailed solar models give the central temperature of the Sun as $T_\odot(0) = 1.36 \times 10^7\,\text{K}$ etc., larger by a factor 2,340. The answer is that, while the central temperature depends only on M, N_1, and N_2, this is not true of the effective surface temperature, which is defined by the requirement $L = 4\pi R^2 \sigma T_{\text{eff}}^4$, or in other words,

$$k_B T_{\text{eff}} \equiv [k_B^4 L/4\pi\sigma R^2]^{1/4} = [Lk_B^4/\pi acR^2]^{1/4} = [N_1 L^*/\pi R^2 \kappa_1]^{1/4}. \tag{1.6.25}$$

This can be written as the product

$$T_{\text{eff}} = \tau_0^{-1/4} T_0, \tag{1.6.26}$$

where τ_0 is the dimensionless quantity

$$\tau_0 = R\kappa_1 [M/R^3]^{1+\alpha} [k_B T(0)]^\beta,$$

and T_0 has the dimensions of temperature,

$$k_B T_0 = \left[R[M/R^3]^{1+\alpha}[k_B T(0)]^\beta N_1 L^*/\pi R^2\right]^{1/4}.$$

Since T_0 and $T(0)$ depend only on M, N_1, and N_2, and have the same dimensionality, we expect them to be equal, up to factors of order unity. So from Eq. (1.6.26) we expect that

$$T(0)/T_{\text{eff}} \approx \tau_0^{1/4}. \tag{1.6.27}$$

On the other hand, τ_0 is the value that the optical depth of the center of the star would have if the density and temperature had the uniform values M/R^3 and $T(0)$, which is much greater than unity because the star is optically thick. For instance, if we take the Sun to be completely ionized hydrogen and take its opacity to be entirely due to Thomson scattering, then, as shown in Section 1.4,

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$\kappa \simeq 0.4\ \text{cm}^2/\text{g}$, so for a uniform density $\approx M/R_\odot^3$ the optical depth of the center of the Sun is $\tau_0 \approx R_\odot \kappa M_\odot / R_\odot^3 = 1.6 \times 10^{11}$. Our estimate (1.6.27) then suggests that $T(0)/T_{\text{eff}} \simeq 630$, not very different from the actual ratio 2,340 cited above.

We are now in a position to find the shape of the famous Hertzsprung–Russell relation between effective surface temperature and luminosity for stars on the main sequence. From the definition (1.6.25) and our results that $R \propto M^A$ and $L \propto M^B$, we find the mass dependence of the effective surface temperature

$$T_{\text{eff}} \propto M^{[B-2A]/4}. \tag{1.6.28}$$

Therefore, eliminating M from our results for L and T_{eff}, we can express the Hertzsprung–Russell relation as a power law:

$$L \propto T_{\text{eff}}^H \tag{1.6.29}$$

with exponent

$$H = \frac{4B}{B-2A} = 4\left[1 - 2\frac{-2+\lambda+\nu+\alpha+\beta}{(1+\lambda+\nu)(3\alpha+\beta)+(3-\alpha-\beta)(3\lambda+\nu)}\right]^{-1}. \tag{1.6.30}$$

The estimate of H is simplest for stars on the upper part of the main sequence, whose high temperature means that opacity is dominated by Thomson scattering, for which $\alpha = \beta = 0$. For both the proton–proton chain and the CNO cycle $\lambda = 1$, so leaving ν as a free parameter, the Hertzsprung–Russell exponent is

$$H = \frac{12(3+\nu)}{11+\nu}. \tag{1.6.31}$$

In all cases ν is positive-definite and $3.27 < H < 12$. More specifically, for the proton–proton chain and CNO cycle we have roughly $\nu \simeq 5$ and $\nu \simeq 15$, for which respectively $H \simeq 6$ and $H \simeq 8.3$. The comparison with observation is complicated by the fact that, although it is straightforward to measure L for any star whose distance is known (or to measure ratios of values of L for a cluster of stars that are all at the same distance), it is difficult to obtain a precise value for T_{eff} from observations of colors or spectral lines. From one graph[31] of L versus T_{eff} for a large sample of stars with masses between 2 and 10 solar masses, I estimate that $H \simeq 7$.

The problems associated with the measurement of effective surface temperature can be avoided by considering the class of eclipsing binary stars, for which accurate values of R and M can be found from the analysis of the time-dependence of luminosities and Doppler shifts.[32] It is particularly revealing to consider the relation between luminosity and mass for stars, such as those on the

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31 F. LeBlanc, *Introduction to Stellar Astrophysics* (John Wiley & Sons, Chichester, 2010), p. 27.
32 J. Andersen, *Astron. Astrophys. Rev.* **3**, 91 (1991).

upper part of the main sequence, whose opacity is due to Thomson scattering, for which $\alpha = \beta = 0$. For these stars Eqs. (1.6.17)–(1.6.19) give $B = 3$, $B_1 = -1$, and $B_2 = 4$, so here Eq. (1.6.20) reads

$$L \cong \epsilon_1 M^3 N_1^{-1} N_2^4 = \frac{ca(4\pi G m_1 \mu)^4}{\kappa_1 k_B^4} M^3. \tag{1.6.32}$$

It is striking that this result is entirely independent of the parameters ϵ_1, λ, and ν characterizing the mechanism for nuclear energy generation. One suspects that for $\alpha = \beta = 0$ this result is even independent of the assumption that the rate of energy generation per mass is proportional to a product of powers of density and temperature, but I have not been able to prove this.

The data on eclipsing binaries cited by Andersen shows that for $2 \leq M/M_\odot \leq 20$, binaries have $L \propto M^{3.6}$. Another survey[33] shows that bright stars have $L \propto M^{4.0}$, while dimmer stars have $L \propto M^{2.76}$. Stars on the upper part of the main sequence have[34] $L \propto M^{3.5}$. Given the limited statistics from eclipsing binaries and the oversimplification in our assumption of an opacity entirely due to Thomson scattering, the discrepancies among these measured exponents – 3.6, 4.0, 2.76, and 3.5 – and with our result that $L \propto M^3$ – are not surprising.

The luminosity–mass relation provides insight regarding the scale of time over which stars of various mass evolve. The fusion $4\,^1\text{H} \to {}^4\text{He}$ yields 6.5 MeV per proton, so the energy per mass available from hydrogen burning is

$$6.5 \text{ MeV/p} \times 1.602 \times 10^{-5} \text{ erg/MeV}/1.672 \times 10^{-24} \text{ g/p} = 6.23 \times 10^{19} \text{ erg/g}.$$

The Sun has mass $M_\odot = 1.939 \times 10^{33}$ g, but initially only 75% was hydrogen, so the energy available is

$$\begin{aligned} E_H &= 0.75 f \times 1.939 \times 10^{33} \left(\frac{M}{M_\odot}\right) \text{g} \times 6.23 \times 10^{19} \text{ erg/g} \\ &= 0.93 \times 10^{53} f \left(\frac{M}{M_\odot}\right) \text{erg}, \end{aligned}$$

where f is the fraction of the Sun's hydrogen that becomes sufficiently hot to initiate nuclear reactions. The Sun has luminosity $L_\odot = 3.845 \times 10^{33}$ erg/sec, so a star of mass M and luminosity L could go on burning hydrogen for a time

$$E_H/L \simeq 7.6 \times 10^{11} f \frac{M/M_\odot}{L/L_\odot} \text{ years.}$$

The main sequence duration of the Sun is commonly estimated as 10^{10} years, corresponding to $f \simeq 0.013$, a not unreasonable value. Even with an efficiency

33 Cited by J. P. Cox and R. T. Giuli, *Principles of Stellar Structure* (Gordon and Breach, New York, 1968), p. 15.

34 C. J. Hansen, S. D. Kawaler, and V. Trimble, *Stellar Interiors: Physical Principles, Structure and Evolution*, 2nd edn. (Springer, New York, 2004), p. 28.

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this small, the solar main sequence lifetime is much longer than the Kelvin time 10^7 years over which the Sun could go on shining without nuclear reactions, and it is not much less than the present age 1.37×10^{10} years of the big bang. But with our analytic estimate $L \propto M^3$ and the same hydrogen burning efficiency, for $M = 100M_\odot$ the main sequence duration would be only 10^6 years, while with the empirical relation $L \propto M^{3.5}$ the main sequence lifetime would be 10^5 years.

Finally, consider the relation between stellar radii and masses. Recall that $R \propto M^A$, and for $\alpha = \beta = 0$ and $\lambda = 1$, Eq. (1.6.13) gives

$$A = \frac{-1+\nu}{3+\nu}.$$

If for the CNO cycle we take $\nu = 15$, then $R \propto M^{0.78}$. Data[35] for stars with masses between 5 and 20 solar masses give $R \propto M^{0.78}$, while other data[36] for stars on the upper part of the main sequence indicate that $R \propto M^{0.75}$. This is a very satisfactory confirmation of the results of dimensional analysis.

* * * * *

In closing, we return to the case in which the pressure is dominated by radiation rather than hot gas. Here $p = aT^4/3$, so Eqs. (1.6.5) and (1.6.7) are replaced with

$$\frac{d\big(k_B T(r)\big)^4}{dr} = -3N_2' \frac{\mathcal{M}(r)\,\rho(r)}{4\pi r^2},$$

and

$$N_2' \equiv 4\pi G k_B^4/a,$$

while there is no change in Eqs. (1.6.1)–(1.6.4) or (1.6.6). Now stellar parameters R, $L^* \equiv L/\epsilon_1$, $\rho(0)$, $k_B T(0)$, etc. depend only on N_1, N_2', and M. Note that N_2' has the dimensions of G[energy]^4/[energy/volume], or

$$\begin{aligned}[N_2'] &= [G][\text{energy}]^4/[\text{energy/volume}] = [G][\text{energy}]^3[\text{volume}] \\ &= [\text{length}]^{12}[\text{time}]^{-8}[\text{mass}]^2. \end{aligned} \tag{1.6.33}$$

Here again there is a remarkably general simple relation between luminosity and mass in the case where opacity is dominated by Thomson scattering. Recall that L has dimensions

$$[L] = [\text{energy}]/[\text{time}] = [\text{length}]^2[\text{time}]^{-3}[\text{mass}],$$

35 Cited by A. Weiss, W. Hillebrandt, H.-C. Thomas, and H. Ritter, *Cox and Giuli's Principles of Stellar Structure*, 2nd edn. (Cambridge Scientific Publishers, Cambridge, 2004), p. 10.

36 Cited by C. J. Hansen, S. D. Kawaler, and V. Trimble, *Stellar Interiors: Physical Principles, Structure and Evolution*, 2nd edn. (Springer, New York, 2004), p. 28.

so $L^* \equiv L/\epsilon_1$ has dimensions

$$[L^*] = [\text{length}]^{-3\lambda+2\nu}[\text{time}]^{-2\nu}[\text{mass}]^{1+\lambda+\nu}.$$

The only combination of N_1, N'_2, and M that has the same dimensions as L^* is $M^{B'} N_1^{B'_1} {N'}_2^{B'_2}$, where

$$B' = 1 + \frac{(\lambda + \nu/2)(3\alpha + \beta) - (\alpha + \beta/2)(3\lambda + \nu)}{3\lambda + \nu + 3\alpha + \beta}, \tag{1.6.34}$$

$$B'_1 = -\frac{3\lambda + \nu}{3\lambda + \nu + 3\alpha + \beta}, \tag{1.6.35}$$

$$B'_2 = \frac{\nu}{4} + B'_1\left(-1 + \frac{\nu}{4} + \frac{\beta}{4}\right). \tag{1.6.36}$$

Hence

$$L = \epsilon_1 L^* \cong \epsilon_1 M^{B'} N_1^{B'_1} {N'}_2^{B'_2}. \tag{1.6.37}$$

Equations (1.6.25)–(1.6.27) with $\alpha = \beta = 0$ give $B' = 1$, $B'_1 = -1$, $B'_2 = 1$, so Eq. (1.6.37) gives

$$L \cong \epsilon_1 M N_1^{-1} {N'}_2^{1} = \frac{4\pi Gc}{\kappa_1} M, \tag{1.6.38}$$

for any values of λ and ν. This may be compared with the result given at the end of Section 1.3, that in the absence of gas pressure

$$L = \frac{4\pi GcM}{\kappa(R)}. \tag{1.6.39}$$

This result was derived with no assumptions regarding the dependence of opacity or nuclear energy generation on temperature and density. If we assume that the opacity is independent of temperature and density, as it is for Thomson scattering, then $\kappa(R)$ is the same as what in this section we have called κ_1, so Eq. (1.6.38) is the same as Eq. (1.6.39) and dimensional analysis is not needed. Here $\cong$ actually means $=$. But, for a more general dependence of opacity on temperature and density, of the form (1.6.1), though Eq. (1.6.39) is still valid in the absence of gas pressure, the opacity $\kappa(R)$ at the surface is different from κ_1, the relation depending on the profile of density and pressure throughout the star and hence on the stellar mass, so here dimensional analysis comes in handy in finding the luminosity–mass relation.

1.7 Convection

The regime of radiative energy transport discussed in the previous sections, and more generally any smooth model of stellar structure, may not be stable against the onset of convection. Bits of stellar material may separate from their surroundings, and rise or fall, like eddies in a heated pot of water.

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Suppose that a small element of stellar fluid happens to move upward from r to $r + dr$. The balance of forces at the surface of the element will cause the pressure inside to change, from $p(r)$ to the ambient pressure $p(r) + p'(r)\,dr$ at its new location. The density and temperature will also change, but not to the new ambient density and temperature. Since heat conduction is generally very slow in stars, it is reasonable to suppose that the process is adiabatic, with no heat flowing into or out of the fluid element. Then the density and temperature will be some definite function of the pressure (in general depending on initial conditions), and in particular the new density will be

$$\rho(r) + \left[\frac{\partial \rho(p)}{\partial p}\right]_{p=p(r)} p'(r)\,dr, \tag{1.7.1}$$

in which we adopt the convention that a partial derivative in square brackets is to be calculated assuming that variations are adiabatic – that is, with changes in pressure, the temperature and density vary in such a way that no heat flows into or out of the fluid element. If this new density is greater than the ambient density $\rho(r)+\rho'(r)\,dr$ at the new position, then the fluid element will sink back toward its original position, and the initial configuration will be stable. Thus the condition for stability against upward motion is

$$\left[\frac{\partial \rho(p)}{\partial p}\right]_{p=p(r)} p'(r) > \rho'(r). \tag{1.7.2}$$

Similarly, if the blob density (1.7.1) is less than the new ambient density $\rho(r) + \rho'(r)\,dr$ then the fluid element will float upward, so we then have stability against downward motion. Since for downward motion dr is negative, the stability condition is again (1.7.2).

On the other hand, if the left-hand side of Eq. (1.7.2) is less than the right-hand side we have an exponentially growing instability, whereas if the two sides are equal we have instability against a steady drift upwards or downwards.

Under conditions of convective stability, the r-derivative of the temperature is given by the equation (1.3.4) of radiative energy transport, while the r-derivative of the pressure is given by the equation (1.3.1) of hydrostatic equilibrium, so it is convenient (and conventional) to rewrite the equation (1.7.2) of convective stability in terms of temperature and pressure rather than density and pressure. For this purpose, we need the ideal gas law

$$\rho = mp/k_{\rm B}T,$$

where m is the mass of the gas particles, whose value will not concern us. It follows then that

$$\left[\frac{\partial \rho}{\partial p}\right] = \frac{\rho}{p} - \frac{\rho}{T}\left[\frac{\partial T}{\partial p}\right] = \frac{\rho}{p} - \left[\frac{\partial \ln T}{\partial \ln p}\right]$$

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(the square brackets again indicating adiabatic variations), and

$$\rho' = \frac{\rho p'}{p} - \frac{T'\rho}{T},$$

so the quantity appearing in the stability condition (1.7.2) can be written

$$\left[\frac{\partial\rho}{\partial p}\right]_{p=p(r)} p'(r) - \rho'(r) = -\frac{p'(r)\rho(r)}{p(r)}\big(\nabla_{\rm ad}(r) - \nabla(r)\big),$$

where $\nabla_{\rm ad}$ is the value of $\partial \ln T/\partial \ln p$ for adiabatic variations,

$$\nabla_{\rm ad}(r) \equiv \left[\frac{\partial \ln T(p)}{\partial \ln p}\right]_{p=p(r)}, \tag{1.7.3}$$

and $\nabla(r)$ is the actual value of this derivative in the star:

$$\nabla(r) \equiv \frac{T'(r)/T(r)}{p'(r)/p(r)}. \tag{1.7.4}$$

Since the quantity $p'\rho/p$ is everywhere negative, the condition (1.7.2) for convective stability is just

$$\nabla(r) < \nabla_{\rm ad}(r). \tag{1.7.5}$$

Using Eqs. (1.3.1) and (1.3.4), we have

$$\nabla(r) = \frac{3\kappa(r)\mathcal{L}(r)p(r)}{16\pi ca\,T^4(r)G\mathcal{M}(r)}. \tag{1.7.6}$$

It is instructive to write the stability condition $\nabla(r) < \nabla_{\rm ad}(r)$ as a limit on the rate of energy flow through a sphere of radius r that can be carried stably by radiation:

$$\mathcal{L}(r) < 4\,\nabla_{\rm ad}(r)\left(\frac{p_{\rm rad}(r)}{p(r)}\right)\mathcal{L}_{\rm Edd}(r), \tag{1.7.7}$$

where $p_{\rm rad}(r)$ is the radiation pressure $aT(r)^4/3$ and $\mathcal{L}_{\rm Edd}(r)$ is the Eddington limit $4\pi Gc\mathcal{M}(r)/\kappa(r)$. As we saw at the end of Section 1.3, $\mathcal{L}(r)$ must in any case be less than $\mathcal{L}_{\rm Edd}(r)$ in order for radiation not to overcome gravitational attraction and tear the star apart. We will see that $4\,\nabla_{\rm ad}$ is never very different from unity, so for ordinary stars, for which radiation pressure is much less than gas pressure, stability against convection requires that $\mathcal{L}(r)$ must be not just less but very much less than the Eddington limit $\mathcal{L}_{\rm Edd}(r)$.

To calculate $\nabla_{\rm ad}$ we make use of the conservation of energy and mass. (For relativistic theories, in which mass is not conserved, we use baryon number instead.) We take $\mathcal{E}$ as the thermal energy density, excluding the energy associated with rest masses, so the thermal energy per gram is $\mathcal{E}/\rho$. When the volume per gram $1/\rho$ of stellar material increases by a small amount $\delta(1/\rho)$ (which of course is negative for decreasing volume), the work per gram that is done

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against the ambient pressure p is $p\,\delta(1/\rho)$, so in the absence of heat flow the conservation of energy requires that

$$\delta(\mathcal{E}/\rho) + p\,\delta(1/\rho) = 0. \tag{1.7.8}$$

As already mentioned in Section 1.1, for a wide variety of stellar material $\mathcal{E}$ is proportional to p, a relation conventionally written as

$$\mathcal{E} = \frac{p}{\Gamma - 1}. \tag{1.7.9}$$

(This is sometimes written as $\mathcal{E} = np$, where $n \equiv 1/(\Gamma - 1)$.) Using Eq. (1.7.9) in Eq. (1.7.8), the adiabatic energy conservation condition becomes

$$\Gamma p\,\delta(1/\rho) + (1/\rho)\,\delta p = 0,$$

or in other words

$$\delta\left(p/\rho^{\Gamma}\right) = 0. \tag{1.7.10}$$

The adiabatic partial derivative in Eq. (1.7.2) is then

$$\left[\frac{\partial\rho(p)}{\partial p}\right] = \frac{\rho}{\Gamma p}, \tag{1.7.11}$$

and the stability condition (1.7.2) is then just the condition that

$$\frac{\rho(r)\,p'(r)}{\Gamma p(r)} > \rho'(r),$$

or, multiplying by the positive quantity $\Gamma/\rho(r)$,

$$\frac{p'(r)}{p(r)} > \frac{\Gamma\rho'(r)}{\rho(r)}. \tag{1.7.12}$$

(The difference between the left-hand and right-hand sides of this inequality is a quantity known as the *Schwarzschild discriminant*.) Hence stability requires that $p(r)/\rho^{\Gamma}(r)$ increases with r. Where this is not the case, convection occurs.

For an ideal gas, with p proportional to ρT, we have p/ρ^{Γ} proportional to $T^{\Gamma}/p^{(\Gamma-1)}$, so for adiabatic variations $T \propto p^{(\Gamma-1)/\Gamma}$, and the quantity (1.7.3) is the constant

$$\nabla_{\rm ad} = 1 - 1/\Gamma. \tag{1.7.13}$$

This is the value we must use in the stability criterion (1.7.5).

For a monatomic ideal gas of atoms at temperature T the equipartition of energy gives a thermal energy per atom $3k_{\rm B}T/2$, so with $\rho/m_1\mu$ atoms per volume (where μ is the atomic weight and m_1 the mass for unit atomic weight), the thermal energy per volume is $\mathcal{E} = 3k_{\rm B}T\rho/2\mu m_1$, as compared with a pressure given by the ideal gas law as $p = k_{\rm B}T\rho/\mu m_1$, so here Eq. (1.7.9) is satisfied with $\Gamma = 1 + 2/3 = 5/3$, and $\nabla_{\rm ad} = 2/5$.

Matters are not always so simple, even in ordinary stars. For instance in the Sun, as we go inwards from just below the surface to $r \simeq 0.8R_\odot$, the increasing

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temperature goes first to ionizing atomic hydrogen (which takes 13.6 eV per atom), then to singly ionizing atomic helium (24.6 eV per atom), and then to completely ionizing singly ionized helium (54.4 eV per ion), rather than to increasing thermal velocities and pressure. Since $\partial\mathcal{E}/\partial p$ is thus effectively greater than 3/2, the effective value of Γ is less than $5/3$, and $\nabla_{\rm ad}$ is less than 2/5. In the outer layers of the Sun, from just below the surface down to $r \simeq 0.8R_\odot$, the effective value of $\nabla_{\rm ad}$ is approximately 0.15.[37] Elsewhere in the Sun, $\nabla_{\rm ad}$ is close to the nominal value $2/5$.

Energy density is proportional to pressure also if the thermal energy and pressure are both dominated by relativistic particles, such as fast electrons in high-mass white dwarfs or photons in supermassive stars. In such cases we have $p = \mathcal{E}/3$, so Eqs. (1.7.9) and (1.7.13) are satisfied with $\Gamma = 4/3$ and $\nabla_{\rm ad} = 1/4$.

Now suppose that, in some part of a star, the condition (1.7.2) for convective stability is not satisfied, but rather

$$\left[\frac{\partial\rho(p)}{\partial p}\right]_{p=p(r)} p'(r) < \rho'(r), \tag{1.7.14}$$

or equivalently,

$$\nabla(r) > \nabla_{\rm ad}(r). \tag{1.7.15}$$

(This is the case in the Sun from just below the surface, at a depth where $p \approx 10^5$ dyne/cm^2, down to $r \simeq 0.7R_\odot$, where $p \approx 10^{13.5}$ dyne/cm^2.) As we have seen, in this case a blob of stellar fluid that happens to move upwards or downwards will become respectively lighter or heavier than the same volume of ambient fluid along its path, and hence will tend to keep moving in the same direction. The pressure in the blob remains the same as the ambient pressure along its path, so if the energy per volume $\mathcal{E}$ depends only on the pressure, it too remains the same in the blob as in the fluid along its path, but since the mass density ρ in the blob becomes less or greater than in the fluid along its path for a blob going upwards or downwards, the energy per mass $\mathcal{E}(p)/\rho$ becomes respectively greater or less than in the fluid along its path. Specifically, after the blob travels a distance δr, the difference between its density and the density of the surrounding material will be

$$\delta\rho = \left[\left[\frac{\partial\rho(p)}{\partial p}\right]_{p=p(r)} p'(r) - \rho'(r)\right]\delta r,$$

so the difference between the thermal energy per mass of the blob and of the surrounding material will be

[37] Numerical results for ∇ and $\nabla_{\rm ad}$ here and below are taken from Figure 29.4 of R. Kippenhahn and A. Weigert, *Stellar Structure and Evolution* (Springer-Verlag, Berlin, 1990). Other solar parameters are taken from C. W. Allen, *Astrophysical Quantities* (Athlone Press, London, 1955).

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$$\delta\left(\frac{\mathcal{E}}{\rho}\right) = -\mathcal{E}\delta\rho/\rho^2 = \frac{\mathcal{E}(r)}{\rho^2(r)}\left[\rho'(r) - \left[\frac{\partial\rho(p)}{\partial p}\right]_{p=p(r)} p'(r)\right]\delta r. \quad (1.7.16)$$

According to the condition (1.7.14) for convection to occur, the change (1.7.16) in energy per mass of the blob will be positive or negative for outward or inward motion, respectively. Eventually the blob will dissolve into the ambient material, heating the ambient matter above if the blob has gone upward and cooling the matter below if the blob has gone downward. The succession of blobs going up and down thus leads to a flow of heat energy outward through the star.

The convective transport of energy forces a clarification of notation. From now on, we refer to the rate of energy transport outward through a sphere of radius r by radiation and convection as $\mathcal{L}_{\rm rad}(r)$ and $\mathcal{L}_{\rm conv}(r)$, respectively, while the total rate of energy transport is

$$\mathcal{L}_{\rm rad}(r) + \mathcal{L}_{\rm conv}(r) \equiv \mathcal{L}_{\rm tot}(r).$$

The equation (1.3.3) of energy conservation refers of course to the total energy transport rate

$$\frac{d\mathcal{L}_{\rm tot}(r)}{dr} = 4\pi r^2 \epsilon(r)\rho(r), \quad (1.7.17)$$

while it is $\mathcal{L}_{\rm rad}(r)$ that controls variations in temperature through Eq. (1.3.4), which we now write as

$$\frac{dT(r)}{dr} = -\frac{3\kappa(r)\rho(r)}{4caT^3(r)}\frac{\mathcal{L}_{\rm rad}(r)}{4\pi r^2}. \quad (1.7.18)$$

Thus, in the presence of convection, Eq. (1.7.6) refers to the radiative energy transport rate, not the total rate:

$$\nabla(r) = \frac{3\kappa(r)\mathcal{L}_{\rm rad}(r)p(r)}{16\pi ca\, T^4(r)G\mathcal{M}(r)}. \quad (1.7.19)$$

Often one defines a quantity $\nabla_{\rm rad}(r)$ as what $\nabla(r)$ *would be* if energy were transported entirely by radiation:

$$\nabla_{\rm rad}(r) \equiv \frac{3\kappa(r)\mathcal{L}_{\rm tot}(r)p(r)}{16\pi ca\, T^4(r)G\mathcal{M}(r)}. \quad (1.7.20)$$

Since convection carries some energy, the presence of convection means that $\nabla(r)$ is less than $\nabla_{\rm rad}(r)$ (often much less), as well as greater than $\nabla_{\rm ad}$.

Finding $\mathcal{L}_{\rm tot}(r)$ is relatively easy. Equation (1.7.17) tells us that outside a central core where nuclear reactions occur, $\mathcal{L}_{\rm tot}(r)$ is a constant, and hence is equal to the star's luminosity L. But in order to use Eq. (1.7.18) to calculate the variation in the star's temperature, we need to find $\mathcal{L}_{\rm rad}(r)$, which is not so easy. Instead we can often simply assume that in convective zones $\nabla(r) \simeq \nabla_{\rm ad}(r)$, so that the temperature varies in such a way as to keep the pressure simply proportional to ρ^Γ.

To see when this is likely to be the case, it is usual to calculate the convective energy flux employing a radical approximation. One assumes that the dissolution of each blob occurs after it has traveled a distance $\ell(r)$, known as the *mixing length*. (The mixing length at radius r is usually taken to be of the same order of magnitude as the scale height of the stellar fluid at that position, the radial distance in which density, pressure, etc. change appreciably, but it is difficult to justify this guess, and even more difficult to do better.) We assume that the whole mass of the star is involved in this convection, so the energy per time transported by convection through a sphere of radius r is the quantity (1.7.16) (with δr replaced with ℓ) times the mass $4\pi r^2\rho(r)\ell(r)$ in a shell of thickness $\ell(r)$ divided by the time $\approx \ell(r)/u(r)$ that it takes blobs to pass through this shell,

$$\mathcal{L}_{\rm conv}(r) \approx 4\pi r^2 u(r)\frac{\mathcal{E}(r)}{\rho(r)}\left(\rho'(r) - \left[\frac{\partial\rho(p)}{\partial p}\right]_{p=p(r)} p'(r)\right)\ell(r), \quad (1.7.21)$$

where $u(r)$ is a typical blob velocity. To estimate $u(r)$, we note that the buoyant force on a blob of volume V is the acceleration of gravity $g = G\mathcal{M}/r^2 = |p'/\rho|$ times the mass ρV of the ambient material with the same volume V minus the mass $(\rho + \delta\rho)V$ of the blob.[38] To first order the acceleration of the blob is this force divided by ρV, which after traveling a distance $\ell(r)$ is

$$a = \left|\frac{p'(r)}{\rho^2(r)}\right|\left(\rho'(r) - \left[\frac{\partial\rho(p)}{\partial p}\right]_{p=p(r)} p'(r)\right)\ell(r).$$

The average velocity over this time is then of the order $u \approx \sqrt{a\ell}$, or

$$u(r) \approx \left|\frac{p'(r)}{\rho^2(r)}\right|^{1/2}\left(\rho'(r) - \left[\frac{\partial\rho(p)}{\partial p}\right]_{p=p(r)} p'(r)\right)^{1/2}\ell(r). \quad (1.7.22)$$

Together with Eq. (1.7.21), this gives the energy per time transported by convection through a sphere of radius r as

38 A proof of the classic result that the buoyant force on a submerged body equals the weight of the fluid the body displaces was given by Archimedes, "On Floating Bodies," in *The Complete Works of Archimedes*, trans. T. L. Heath (Cambridge University Press, Cambridge, 1897). He compared two columns of fluids. In one, the submerged body is held down by a piston, while in the other, with the same horizontal cross section as the submerged body, the fluid is undisturbed. In order for the fluid to be at rest the force pressing down at the base of the two columns must be the same, so the buoyancy, which equals the force exerted by the piston, plus the weight of the submerged body, plus the weight of the column of fluid less the weight of the fluid displaced by the body, must equal the weight of the fluid in the undisturbed column, which does include the weight of the fluid displaced by the body. The same result can be derived more directly by modern methods. The integral of the pressure force on the surface of the displaced body is related by Gauss's theorem to the integral of the pressure gradient over the displaced volume, which according to the equation of hydrostatic equilibrium equals the weight of the displaced fluid.

$$\mathcal{L}_{\text{conv}}(r) \approx 4\pi r^2 \left| \frac{p'(r)}{\rho^2(r)} \right|^{1/2} \frac{\mathcal{E}(r)}{\rho(r)} \left(\rho'(r) - \left[\frac{\partial \rho(p)}{\partial p} \right]_{p=p(r)} p'(r) \right)^{3/2} \ell^2(r). \tag{1.7.23}$$

In the same way as in our derivation of the condition (1.7.5) for convective stability, for ideal gases we can rewrite Eq. (1.7.23) as

$$\mathcal{L}_{\text{conv}}(r) \approx \mathcal{L}_0(r) \big(\nabla(r) - \nabla_{\text{ad}}(r) \big)^{3/2}, \tag{1.7.24}$$

where

$$\mathcal{L}_0(r) \equiv 4\pi r^2 \frac{p'^2(r)\mathcal{E}(r)\ell^2(r)}{p^{3/2}(r)\rho^{1/2}(r)}. \tag{1.7.25}$$

We say that convection is efficient at r if the coefficient $\mathcal{L}_0(r)$ is much larger than the luminosity L. This is often the case. Where the mixing length $\ell(r)$ is half the pressure scale height, and $\mathcal{E} = 3p/2$, we have

$$\mathcal{L}_0 \approx (3/2)\pi r^2 p^{3/2}(r)/\rho^{1/2}(r).$$

In the Sun at $r = 0.8R_\odot$ we have $r = 5.6 \times 10^{10}$ cm, $p = 1.6 \times 10^{12}$ dyne/cm^2, and $\rho = 0.018$ g/cm^3, so $\mathcal{L}_0 \approx 2 \times 10^{41}$ erg/sec, as compared with the solar luminosity $L = 3.9 \times 10^{33}$ erg/sec. By a wide margin, this is a case of efficient convection. In general cases of efficient convection, Eq. (1.7.24) requires that $\nabla(r)$ is very close to the adiabatic value $\nabla_{\text{ad}}(r)$. In particular, where $\mathcal{E}$ is related to the pressure by Eq. (1.7.9), ∇_{ad} is given by Eq. (1.7.13), and so in the case of efficient convection we have

$$p(r) = K\rho^\Gamma(r), \tag{1.7.26}$$

where K is a constant that depends on conditions at the boundary of this region. This is the case throughout the convective region of the Sun, aside from a thin shell near the surface, where the pressure drops from 10^6 dyne/cm^2 to 10^5 dyne/cm^2. (But, as already mentioned, due to the effect of ionization, Γ is not constant in the outer parts of the convective region.) Where Eq. (1.7.26) holds throughout a star's interior, the star is known as a *polytrope*. Such stars are discussed further in Section 1.8.

There is another way of expressing this. The second law of thermodynamics tells us that there is a function s of ρ, p, etc. known as the *specific entropy*, or entropy per gram, for which[39]

$$T\,ds = d(\mathcal{E}/\rho) + p\,d(1/\rho). \tag{1.7.27}$$

Hence Eq. (1.7.8) can be interpreted as the statement that convection does not change the specific entropy of the convected fluid elements:

$$\delta s = 0. \tag{1.7.28}$$

[39] We are using δ to denote a change in a fluid element as it rises or falls in the star, while d stands for an arbitrary variation, not necessarily related to any actual motion.

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This is because heat conduction is neglected here, which is generally a good approximation in stars. In regions where convection is efficient the specific entropy tends to a nearly uniform value to keep the convective energy transport consistent with the actual luminosity of the star. Stars with a uniform entropy per gram are said to be *isentropic*.

Though not strictly necessary for our purposes, it is instructive to work out a formula for the specific entropy for gases. With the internal energy given by Eq. (1.7.9), Eq. (1.7.27) reads

$$T\,ds = \frac{1}{\Gamma - 1}\left(\frac{dp}{\rho} + \Gamma p\, d\left(\frac{1}{\rho}\right)\right) = \frac{\rho^{\Gamma-1}}{\Gamma - 1}\, d\left(\frac{p}{\rho^{\Gamma}}\right). \tag{1.7.29}$$

For an ideal gas, $T = p/R\rho$, with R constant, so

$$ds = \frac{R}{\Gamma - 1}\left(\frac{\rho^{\Gamma}}{p}\right) d\left(\frac{p}{\rho^{\Gamma}}\right) = \frac{R}{\Gamma - 1}\, d\ln\left(\frac{p}{\rho^{\Gamma}}\right). \tag{1.7.30}$$

Hence

$$s = \frac{R}{\Gamma - 1}\ln\left(\frac{p}{\rho^{\Gamma}}\right) + \text{constant}. \tag{1.7.31}$$

We see again that p/ρ^{Γ} is constant in an isentropic star.

In typical stars there are regions stable against convection, in which energy transport is by radiation and p/ρ^{Γ} increases with r, and others with effective convection, in which p/ρ^{Γ} is constant. For instance, in the Sun there is a core with radiative energy transport, extending from the center where $p \simeq 2 \times 10^{17}$ dyne/cm^2, out to a radius about $0.65 R_{\odot}$ where the pressure has dropped to about 3×10^{13} dyne/cm^2. This is surrounded by an outer convective layer, and (since convection cannot carry energy into empty space) a relatively thin surface layer dominated by radiative energy transport. In more massive stars, there typically is a convective core, and an outer layer dominated by radiative transport that is stable against convection.

None of this affects the general results of Section 1.3 because the radii where regions of convective energy transport begin or end, and the values of p/ρ^{Γ} in these regions, are set by the conditions in the adjacent regions of radiative energy transport, and so are ultimately determined in terms of physical constants and the value of the nominal stellar radius R where the boundary conditions $\rho(R) = p(R) = 0$ are imposed. Also, the general results of Section 1.6 for the main sequence are unchanged, because nothing regarding convection involves new dimensionful constants.

* * * * *

For isentropic stars, whether or not satisfying the conditions for a polytrope, the equations of hydrostatic equilibrium can be expressed as a variational principle, which will prove useful when we come to stellar instability in Section 1.9. Let us consider the variation in the total energy

$$E = \int_0^R 4\pi r^2 \left(\mathcal{E}(r) - \frac{G\mathcal{M}(r)\rho(r)}{r} \right) dr. \tag{1.7.32}$$

Changes $\delta\rho$ and $\delta\mathcal{E}$ in the mass and energy densities produce a change in the total energy

$$\delta E = \int_0^R 4\pi r^2 \left(\delta\mathcal{E}(r) - \frac{G\mathcal{M}(r)\,\delta\rho(r)}{r} - \frac{G\rho(r)}{r} \int_0^r 4\pi r'^2 \delta\rho(r')\,dr' \right) dr. \tag{1.7.33}$$

In the first term, we use Eq. (1.7.8), which gives $\delta\mathcal{E} = (\mathcal{E} + p)\,\delta\rho/\rho$. In the third term, we interchange the order of integration, and also interchange the coordinate labels r and r'. This gives

$$\delta E = \int_0^R 4\pi r^2\, \mathcal{F}(r)\,\delta\rho(r)\,dr, \tag{1.7.34}$$

where

$$\mathcal{F}(r) = \frac{\mathcal{E}(r) + p(r)}{\rho(r)} - \frac{G\mathcal{M}(r)}{r} - G\int_r^R 4\pi r' \rho(r')\,dr'. \tag{1.7.35}$$

A straightforward calculation using Eq. (1.7.8) gives

$$\frac{d\mathcal{F}(r)}{dr} = \frac{1}{\rho(r)}\frac{dp(r)}{dr} + \frac{G\mathcal{M}(r)}{r^2}. \tag{1.7.36}$$

This vanishes according to the equation (1.1.4) of hydrostatic equilibrium, so $\mathcal{F}(r)$ is a constant $\mathcal{F}_0$, and therefore Eq. (1.7.32) reads

$$\delta E = \mathcal{F}_0 \int_0^R 4\pi r^2\,\delta\rho(r)\,dr = \mathcal{F}_0\,\delta M. \tag{1.7.37}$$

Thus, although the equation of hydrostatic equilibrium does not tell us that either E or M is stationary, it does tell us that E is stationary if M is. (The same result applies in general relativity,[40] with the total baryon number N_B times the baryon rest mass m_B taking the place of M and $M - m_B N_B$ taking the place of E.)

1.8 Polytropes

There are several classes of stars for which the pressure is simply proportional to a power of density, at least away from the surface:

$$p = K\rho^\Gamma, \tag{1.8.1}$$

[40] For a textbook demonstration, see Section 11.2 of S. Weinberg, *Gravitation and Cosmology* (Wiley, New York, 1972).

with K and Γ constant throughout the star. Such stars are known as *polytropes* with index Γ. These include the following types,

- Ordinary stars with efficient convective energy transport. As shown in the previous section, these stars obey Eq. (1.8.1), with Γ typically close to $5/3$, and K depending on boundary conditions, such as the values of the central density and pressure.
- As we shall see in Section 1.10, exceptionally light white dwarf stars obey Eq. (1.8.1) with Γ usually close to 5/3, and exceptionally heavy white dwarf stars obey Eq. (1.8.1) with $\Gamma \simeq 4/3$. In both cases K depends only on the chemical composition, as well as on fundamental physical constants.
- Supermassive stars. As discussed in Section 1.11, these stars obey Eq. (1.8.1) with $\Gamma \simeq 4/3$ and with K depending on the molecular weight and on the ratio of matter to radiation pressure, as well as on fundamental physical constants.

In this section we will treat all polytropes in common, not inquiring into the reason for Eq. (1.8.1).

Since the temperature does not enter in Eq. (1.8.1), we can work out the properties of the star using only the hydrostatic equations (1.1.4) and (1.1.5). It will be convenient now to rewrite these two first-order differential equations as a single second-order equation for the density:

$$\frac{d}{dr}\left(\frac{r^2}{\rho(r)}\frac{d}{dr}\rho^\Gamma(r)\right) + \frac{4\pi G}{K}r^2\rho(r) = 0. \tag{1.8.2}$$

As boundary conditions, we can take the central density to have some assumed value $\rho(0)$ and, since the analyticity of ρ as a function of $\mathbf{x}$ requires $\rho(r)$ to be a power series in r^2 near $r = 0$, we also take $\rho'(0) = 0$. With two initial conditions, we have a unique solution, depending only on Γ and on the free parameters K/G and $\rho(0)$.

There is an apparent paradox in the case of stars with efficient convective energy transport. Here there is not just one free stellar parameter, such as the star's mass or radius, but *two* free parameters, which can be taken as $\rho(0)$ and $K = p(0)/\rho(0)^\Gamma$. Thus the Vogt–Russell theorem mentioned in Section 1.3 does not apply to such polytropes. This may seem surprising, because we can think of the star as described by three first-order differential equations: Eqs. (1.1.4) and (1.1.5), together with

$$\frac{d}{dr}\left(\frac{p(r)}{\rho^\Gamma(r)}\right) = 0,$$

together with three parameter-free boundary conditions: $\mathcal{M}(0) = 0$, $\rho(R) = 0$, and $p(R) = 0$. So why, with an equal number of first-order differential equations and parameter-free boundary conditions, do we have any free parameters beyond the radius R at which some of the boundary conditions are imposed? The reason why this counting does not work here, though it may seem the

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same as the sort we used in Section 1.3, is that we are really imposing only one boundary condition at the surface. With $p(r)/\rho^\Gamma(r)$ constant, the condition $\rho(R) = 0$ *implies* that $p(R) = 0$. Having three first-order differential equations and only two independent parameter-free boundary conditions depending on R, there is an additional free parameter, which can be taken as K or $\rho(0)$, in addition to the radius R at which one of the boundary conditions is imposed.

Returning now to general polytropes, the free parameters in Eq. (1.8.2) can be eliminated by re-scaling the independent and dependent variables. First, define

$$\Theta \equiv \left(\frac{\rho(r)}{\rho(0)}\right)^{\Gamma-1}. \tag{1.8.3}$$

Then Eq. (1.8.2) gives

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\Theta\right) + \frac{4\pi G(\Gamma-1)}{K\Gamma}\rho(0)^{(2-\Gamma)}\Theta^{1/(\Gamma-1)} = 0.$$

We can get rid of the constant in the second term by introducing

$$\xi \equiv \left(\frac{4\pi G(\Gamma-1)}{K\Gamma}\right)^{1/2}\rho(0)^{(2-\Gamma)/2}r. \tag{1.8.4}$$

The differential equation (1.8.1) then becomes

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d}{d\xi}\Theta(\xi)\right) + \Theta(\xi)^{1/(\Gamma-1)} = 0, \tag{1.8.5}$$

and the boundary conditions are

$$\Theta(0) = 1, \qquad \Theta'(0) = 0. \tag{1.8.6}$$

(The requirement $\Theta'(0) = 0$ like the requirement $\rho'(0) = 0$ is needed for the analyticity of $\rho(r)$ at $r = 0$ as a function of the Cartesian components of $\mathbf{x}$.)

Equation (1.8.5) is known as the *Lane–Emden equation*,[41] and was much studied in the early years of the twentieth century. It was shown that, for $\Gamma > 6/5$, its solution vanishes at a finite value ξ_1 of ξ, so the radius of the star is

$$R = \left(\frac{4\pi G(\Gamma-1)}{K\Gamma}\right)^{-1/2}\rho(0)^{-(2-\Gamma)/2}\xi_1. \tag{1.8.7}$$

The star's mass is

$$\begin{aligned} M &= \int_0^R 4\pi r^2\rho(r)\,dr \\ &= 4\pi\rho(0)^{(3\Gamma-4)/2}\left(\frac{K\Gamma}{4\pi G(\Gamma-1)}\right)^{3/2}\int_0^{\xi_1}\xi^2\Theta^{1/(\Gamma-1)}(\xi)\,d\xi. \end{aligned}$$

41 The classic discussion of the Lane–Emden equation is by S. Chandrasekhar, *An Introduction to the Study of Stellar Structure* (University of Chicago Press, Chicago, IL, 1939).

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By using Eq. (1.8.5), we easily see that

$$\int_0^{\xi_1} \xi^2 \Theta^{1/(\Gamma-1)}(\xi)\, d\xi = -\xi_1^2 \Theta'(\xi_1),$$

so

$$M = 4\pi\rho(0)^{(3\Gamma-4)/2} \left(\frac{K\Gamma}{4\pi G(\Gamma-1)}\right)^{3/2} \xi_1^2 |\Theta'(\xi_1)|. \tag{1.8.8}$$

There are just three values of $\Gamma > 1$ for which exact non-singular solutions of the Lane–Emden equation are known.

- For $\Gamma = \infty$, Eq. (1.8.5) is linear and inhomogeneous. The general solution is $-\xi^2/6$ plus any linear combination of $1/\xi$ and 1. The condition $\Theta(0) = 1$ fixes the solution to be simply $\Theta(\xi) = 1 - \xi^2/6$. This gives $\xi_1 = \sqrt{6}$ and $\xi_1^2 \Theta'(\xi_1) = -2\sqrt{6}$.
- For $\Gamma = 2$, Eq. (1.8.5) is linear and homogeneous. The general solution is any linear combination of $\sin\xi/\xi$ and $\cos\xi/\xi$. The condition $\Theta(0) = 1$ fixes the solution to be simply $\Theta(\xi) = \sin\xi/\xi$. This gives $\xi_1 = \pi$ and $\xi_1^2 \Theta'(\xi_1) = -\pi$.
- For $\Gamma = 6/5$, the solution of Eq. (1.8.5) with $\Theta(0) = 1$ is

$$\Theta(\xi) = (1 + \xi^2/3)^{-1/2}.$$

  This reaches zero only at infinity, so $\xi_1 = \infty$, but $\xi^2 \Theta'(\xi)$ approaches the finite value $-\sqrt{3}$ for $\xi \to \infty$, so, though the radius is infinite, the mass is finite.

For other values of $\Gamma > 1$ a numerical computation is needed.[42] Here are some values of ξ_1 and $\xi_1^2 |\Theta'(\xi_1)|$ for several values of Γ:

Γ	ξ_1	$\xi_1^2 \lvert\Theta'(\xi_1)\rvert$
6/5	∞	$\sqrt{3}$
4/3	6.89685	2.01824
3/2	4.35287	2.41105
5/3	3.65375	2.71406
2	π	π
∞	$\sqrt{6}$	$2\sqrt{6}$

The isothermal case $\Gamma = 1$ is discussed in connection with galaxies in Section 4.2.

42 Chandrasekhar, *op. cit.*

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1.9 Instability

We noted in Section 1.1 that stars that are close to a polytrope with $\Gamma = 4/3$ are at the brink of a catastrophic instability. In this section we will prove a theorem that allows us to identify more precisely the threshold parameters at which such stars become unstable.

Suppose that a time-independent equilibrium stellar configuration is subject to an infinitesimal perturbation. As usual for perturbations of time-independent equilibrium, the perturbations $\delta\rho(\mathbf{x}, t)$, $\delta T(\mathbf{x}, t)$, etc. of various quantities can be expressed as a sum over normal modes, the contribution of each normal mode having a time-dependence given by a factor $e^{-i\omega t}$, with various values of ω (not necessarily real) for the various normal modes.[43] Each frequency ω is a function of the various parameters characterizing the equilibrium configuration, such as mass and/or central density.

In the absence of dissipative effects like heat conduction, the equations governing the time-dependence of the perturbations have the symmetry of time-reversal invariance, so that if $\delta\rho(\mathbf{x}, t)$, $\delta T(\mathbf{x}, t)$, etc. is a solution of these equations, then so is $\delta\rho(\mathbf{x}, -t)$, $\delta T(\mathbf{x}, -t)$, etc. This tells us that if ω is the frequency for some normal mode, then there is another normal mode with frequency $-\omega$. If ω is complex then $\exp(-i\omega t)$ grows exponentially unless the imaginary part of ω is negative, in which case $\exp(i\omega t)$ grows exponentially. Hence the equilibrium configuration is unstable unless all the frequencies ω characterizing the various normal modes are real.

Now, consider an equilibrium configuration with parameters for which all ω are real. Small perturbations will oscillate, but not grow. If we vary the star's parameters some ω may become complex, marking a transition to instability, but this faces an obstacle. Everything in these equations is real, so if $\delta\rho(\mathbf{x}, t)$, $\delta T(\mathbf{x}, t)$, etc. is a solution of these equations, then so is its complex conjugate $\delta\rho(\mathbf{x}, t)^*$, $\delta T(\mathbf{x}, t)^*$, etc., which tells us that if ω is the frequency for some normal mode, then there is another normal mode with frequency $-\omega^*$, as well as time-reversed modes with frequencies $-\omega$ and ω^*. Thus, if a generic real frequency ω became complex for some value of a stellar parameter, then the two modes with real frequencies ω and $-\omega$ would become four modes with frequencies ω, $-\omega$, $-\omega^*$, and ω^*. This is impossible; the number of modes is set by the dimensionality of the problem, and cannot suddenly increase or decrease.

43 This is a consequence of the time-translation symmetry of the problem. If $\delta\rho(\mathbf{x}, t)$, $\delta T(\mathbf{x}, t)$, etc. is a solution of the differential equations for small perturbations, then so is $\delta\rho(\mathbf{x}, t + \delta t)$, $\delta T(\mathbf{x}, t + \delta t)$, etc. Since the equations governing these very small perturbations are linear, the solution at $t + \delta t$ must be a linear combination of the various solutions at t. By diagonalizing the matrix in this linear combination, we obtain an equal number of solutions in which each $\delta\rho(\mathbf{x}, t + \delta t)$, $\delta T(\mathbf{x}, t + \delta t)$, etc. is simply proportional to the corresponding $\delta\rho(\mathbf{x}, t)$, $\delta T(\mathbf{x}, t)$, etc., with a coefficient of proportionality that differs from unity by a term of first order in δt. That is, $\delta\rho(\mathbf{x}, t + \delta t) = [1 - i\omega\delta t]\delta\rho(\mathbf{x}, t)$, $\delta T(\mathbf{x}, t + \delta t) = [1 - i\omega\delta t]$ $\delta T(\mathbf{x}, t)$, etc., with ω some constant, This implies the desired time-dependence, proportional to $\exp(-i\omega t)$.

There is, however, a way in which a real frequency ω can become complex, and the star thereby become unstable. If for some set of parameter values the two real frequencies ω and $-\omega$ come together, so that ω vanishes, then for slightly different parameters the frequency can become pure imaginary, so that $\omega = -\omega^*$, and there are still just two normal modes, with frequencies ω and $-\omega = \omega^*$. We conclude that the transition from stability, with all ω real, to instability, with some ω complex (actually imaginary), takes place for parameter values at which some ω vanishes.[44]

For the parameter values at which the ω for some normal mode vanishes, this normal mode becomes a time-independent perturbation of the stellar configuration, satisfying the equations of stellar structure. Since this perturbation becomes time-dependent for infinitesimal ω, it must preserve the values of conserved quantities, such as the total energy and baryon number. Thus (with the possible exceptions described in footnote 2) *a time-independent stellar configuration can become unstable only at values of stellar parameters at which there exists a time-independent perturbation that preserves the values of all conserved parameters.*

In cases where the effects of general relativity can be neglected at the transition to instability, we can take the two quantities that have to be conserved as the energy E, not counting rest masses, and the total rest mass M, defined equal to the baryon number B times the rest mass m_B per baryon. As we saw at the end of Section 1.7, at least for stars with a uniform entropy per rest mass, if one of these is stationary the other is too, so we can concentrate on perturbations that leave just E conserved. We will see an example of this in Section 1.10 for iron white dwarfs.

There are other cases, where the instability arises because of effects of general relativity. Here again there are two conserved quantities, the mass M in the Schwarzschild metric (see Eq. (1.9.A1) below) and the total baryon number B. As in the non-relativistic case, it is especially convenient to look for values of stellar parameters at which the total internal energy $E \equiv M - m_B c^2 B$ (which includes gravitational energy and everything else except the energy in rest masses) is stationary. Obviously if M and B are stationary then so is E, and the theorem mentioned at the end of Section 1.7 tells us that at least for stars with a uniform entropy per baryon, in general relativity the condition that E is stationary is sufficient as well as necessary for both M and B to be stationary.

44 Strictly speaking, there are other possible ways in which, at the transition to instability, several modes may come together to have the same frequency. As an example, suppose that for some set of parameters we have four normal modes with distinct real frequencies ω_1, $-\omega_1$, ω_2, and $-\omega_2$. If we vary the parameters in such a way that ω_1 and ω_2 become equal to the same real value ω_0, then for a further variation of parameters we could again have four distinct frequencies, $\omega_0 + i\epsilon$, $-\omega_0 - i\epsilon$, $-\omega_0 + i\epsilon$, and $\omega_0 - i\epsilon$ with $\epsilon \neq 0$ real. For instance, this happens if the frequencies of the four normal modes are the roots of the equation $(\omega^2 - a^2)^2 - b = 0$, with a and b real. The roots are real for $b > 0$, but become complex (though not pure imaginary) as b moves to negative values. I am not aware of these possibilities actually occurring in stars, and they will not be considered in what follows.

To find where E is stationary in a relativistic context, we will use an expansion for E in powers of the dimensionless quantities $p/\rho c^2$ and $G\mathcal{M}/rc^2$. These two quantities according to Eq. (1.1.4) are roughly of the same order of magnitude, which will be denoted v^2/c^2, and are assumed to be very small.[45] The expansion reads

$$E = \int_0^R \mathcal{E}(r)\, 4\pi r^2\, dr - 3\int_0^R p(r)\, 4\pi r^2\, dr + \int_0^R 6\pi r^4 dr\; p'^2(r)/c^2\rho(r) - \int_0^R 8\pi r^3 dr\; p(r)p'(r)/c^2\rho(r) + \cdots, \tag{1.9.1}$$

where $\mathcal{E}$ is the thermal energy density, excluding only gravitational energy and the energy in rest masses. The individual terms on the first line are of order Mv^2, while the terms on the second and third lines are of order Mv^4/c^2, and the dots denote terms no larger than of order Mv^6/c^4.

The derivation of the expansion (1.9.1) is given at the end of this section. This derivation does not rely on any assumption about the star being a polytrope, but the expansion finds its most important application when the terms of order Mv^2 on the first line nearly cancel, so that the relativistic corrections on the second and third lines become important. This occurs when the pressure is close to $\mathcal{E}/3$, i.e., when the star is close to a polytrope with $\Gamma = 4/3$. As already remarked in Section 1.1, when a star is very close to having $\mathcal{E} = 3p$, as for a polytrope with $\Gamma \simeq 4/3$, very small corrections to a stellar model can make the difference between stability and instability. Because the relativistic corrections on the second and third lines of Eq. (1.9.1) are already much smaller than the individual terms on the first line, in the case at hand these terms can be calculated using the non-relativistic equations of stellar structure for a polytrope with $\Gamma = 4/3$, that is with $p = K\rho^{4/3}$ for some constant K, the inaccuracies in this calculation being even smaller. Using Eqs. (1.8.3), (1.8.4), and (1.8.1), we easily find that Eq. (1.9.1) becomes

$$E = \int_0^R 4\pi r^2\, dr\big[\mathcal{E}(r) - 3p(r)\big] + \frac{16\pi}{(\pi G)^{3/2}c^2}\rho^{2/3}(0)K^{7/2}\eta, \tag{1.9.2}$$

[45] For ordinary stars supported by non-relativistic gas pressure, $p/\rho c^2 \approx v_{\rm th}^2/c^2$, where $v_{\rm th}$ is a typical thermal velocity, generally much less than c. Even when the pressure is dominated by relativistic particles, such as electrons in the most massive white dwarfs or photons in very massive stars, the ratio $p/\rho c^2$ is of the order of the ratio of the energy in these relativistic particles to the energy in baryon rest masses, and is still generally very small.

where η is the positive numerical constant[46]

$$\eta = -2\int_0^{\xi_1} \Theta^4(\xi)\Theta'(\xi)\xi^3\,d\xi + 6\int_0^{\xi_1} \Theta^3(\xi)\Theta'^2(\xi)\xi^4\,d\xi = 3.49815, \quad (1.9.3)$$

with $\xi_1 = 6.89685$ corresponding to the radius at the star's surface. We still need to calculate the first term in Eq. (1.9.2) separately for individual cases, such as massive white dwarfs and supermassive stars, as will be done in Sections 1.10 and 1.11, but the second term in Eq. (1.9.2) represents a universal relativistic correction for stars that are close to polytropes with $\Gamma = 4/3$.

Appendix: Derivation of Relativistic Correction to Energy

As in the non-relativistic case, in general relativity there are two quantities that must be conserved, at least in any spherically symmetric perturbations of the star, and that therefore must both be stationary at values of parameters such as central density at which there is a transition from stability to instability. One of them is the mass M appearing in the Schwarzschild solution for the metric outside the star:

$$-g_{tt} = g_{rr}^{-1} = 1 - 2MG/rc^2, \quad g_{\theta\theta} = r^2, \quad g_{\phi\phi} = r^2\sin^2\theta. \quad (1.9.\text{A1})$$

(Here and below we are using "standard" coordinates, for which inside or outside the star g_{tt} and g_{rr} are functions of r and t, while $g_{\phi\phi}$ and $g_{\theta\theta}$ are the same as for a flat space.) The Schwarzschild solution gives

$$M = \int_0^R \rho(r)\,4\pi r^2\,dr, \quad (1.9.\text{A2})$$

where now $\rho(r)c^2$ is the total energy density (that is, the time–time component of the energy-momentum tensor $T^{\mu\nu}$), including mass energy and everything else except gravitational energy. (Gravitational energy is included in M in the difference between $4\pi r^2\,dr$ and the spatial volume element $4\pi r^2\sqrt{g_{rr}}\,dr$.)

The other conserved quantity is B, the total baryon number of the star:

$$B = \int_0^R B^t(r)\,4\pi r^2\sqrt{-g_{rr}(r)g_{tt}(r)}\,dr, \quad (1.9.\text{A3})$$

where B^μ is the conserved current of baryon number. We can write B^t in terms of the scalar baryon density $n \equiv U_\mu B^\mu$, where U^μ is the velocity four-vector, normalized so that $U^\mu U^\nu g_{\mu\nu} = -1$. For a fluid at rest $U^r = U^\theta = U^\phi = 0$, so $U^t = 1/\sqrt{-g_{tt}}$, $U_t = \sqrt{-g_{tt}}$, $n = \sqrt{-g_{tt}}\,B^t$, and therefore

[46] The numerical value given here is inferred from Eqs. (6.9.29)–(6.9.31) of S. Shapiro and S. Teukolsky, *Black Holes, White Dwarfs, and Neutron Stars* (Wiley, New York, 1983).

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$$B = \int_0^R n(r)\, 4\pi r^2 \sqrt{g_{rr}(r)}\, dr. \tag{1.9.A4}$$

The Schwarzschild solution inside the star gives

$$g_{rr}(r) = \left(1 - \frac{2G\mathcal{M}(r)}{rc^2}\right)^{-1}, \tag{1.9.A5}$$

where, as before,

$$\mathcal{M}(r) = \int_0^r 4\pi r'^2 \rho(r')\, dr'. \tag{1.9.A6}$$

Thus

$$B = \int_0^R n(r)\, 4\pi r^2 \left(1 - \frac{2G\mathcal{M}(r)}{rc^2}\right)^{-1/2} dr. \tag{1.9.A7}$$

Of course, in the non-relativistic limit ρ is the rest mass density, and $M = m_{\rm B} B$, where $m_{\rm B} \simeq 938$ MeV$/c^2$ is the rest mass per baryon. More generally, we can define an internal energy density $\mathcal{E}$ excluding rest masses, by

$$\rho(r) \equiv n(r) m_{\rm B} + \mathcal{E}(r)/c^2. \tag{1.9.A8}$$

We will eventually be assuming that the star, though not necessarily a polytrope, is close to a non-relativistic polytrope with $\Gamma = 4/3$, and therefore has a pressure $p(r)$ close to $3\mathcal{E}(r)$. Without yet making this approximation, we can anticipate that it will be convenient to express ρ as

$$\rho(r) c^2 = m_{\rm B} n(r) c^2 + 3p(r) + \Delta\mathcal{E}(r), \tag{1.9.A9}$$

where

$$\Delta\mathcal{E}(r) \equiv \mathcal{E}(r) - 3p(r), \tag{1.9.A10}$$

which will eventually be treated as a small perturbation, arising from the finite electron mass for white dwarfs and the finite baryon kinetic energy for supermassive stars.

It is important to be clear about the order of magnitude of the terms in M and $m_{\rm B} B$. The leading term in both, $\int_0^R 4\pi r^2 m_{\rm B} n(r)\, dr$, is the non-relativistic approximation to M, and hence is of order M. The next-to-leading terms, $\int_0^R 12\pi r^2 p(r)\, dr/c^2$ in M and $\int_0^R 4\pi r^2 m_{\rm B} n(r) \left(G\mathcal{M}(r)/c^2 r\right) dr$ in $m_{\rm B} B$, are of order Mv^2/c^2, where v is a characteristic gas particle velocity, with $v^2 \approx G\mathcal{M}/r \approx p/m_{\rm B} n$, assumed much less than c. The general relativistic correction $\int_0^R 4\pi r m_{\rm B} n(r) \left(G\mathcal{M}(r)/rc^2\right)^2 dr$ in $m_{\rm B} B$ is of order Mv^4/c^4.

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To eliminate the terms of order M, we consider the difference, which gives the total internal energy E:

$$E/c^2 \equiv M - m_{\rm B}B = \int_0^R 4\pi r^2\,dr\left[3p(r)/c^2 + \Delta\mathcal{E}(r)/c^2 + m_{\rm B}n(r)\big(1-(1-2G\mathcal{M}(r)/rc^2)^{-1/2}\big)\right]. \quad (1.9.\text{A}11)$$

The terms of order Mv^2/c^2 also cancel in Eq. (1.9.A11), as can be seen by integrating the pressure term by parts,

$$\int_0^R 12\pi r^2 p(r)\,dr = \int_0^R p(r)\,d(4\pi r^3) = -\int_0^R p'(r)\,4\pi r^3\,dr,$$

and then using the relativistic equilibrium condition[47]

$$-r^2p'(r) = G\big(\rho(r)+p(r)/c^2\big)\big(\mathcal{M}(r)+4\pi r^3p(r)/c^2\big)\big(1-2\mathcal{M}(r)G/rc^2\big)^{-1}, \quad (1.9.\text{A}12)$$

which together with formula (1.9.A9) for the total energy density gives

$$\int_0^R 12\pi r^2 p(r)\,dr = \int_0^R 4\pi r G\big(m_{\rm B}n(r) + 4p(r)/c^2 + \Delta\mathcal{E}(r)/c^2\big) \times \big(\mathcal{M}(r)+4\pi r^3p(r)/c^2\big)\big(1-2\mathcal{M}(r)G/rc^2\big)^{-1}\,dr. \quad (1.9.\text{A}13)$$

Using this for the first term in Eq. (1.9.A11) gives the internal energy

$$E = \int_0^R 4\pi r^2\left[\Delta\mathcal{E}(r) + m_{\rm B}n(r)c^2\big(1-(1-2G\mathcal{M}(r)/rc^2)^{-1/2}\big)\right]dr + \int_0^R 4\pi r G\big(m_{\rm B}n(r)c^2 + 4p(r) + \Delta\mathcal{E}(r)\big) \times \big(\mathcal{M}(r)+4\pi r^3p(r)/c^2\big)\big(1-2\mathcal{M}(r)G/rc^2\big)^{-1}dr. \quad (1.9.\text{A}14)$$

So far, although we have been guided by order-of-magnitude estimates, the result (1.9.A14) is exact. Now, we note the term of order Mv^2 in the first line is $-\int_0^R 4\pi r^2 m_{\rm B}n(G\mathcal{M}/rc^2)\,dr$ and cancels the term of order Mv^2/c^2 in the second line, which is $+\int_0^R 4\pi rGm_{\rm B}n\mathcal{M}\,dr/c^2$. The leading terms are then of order Mv^4/c^2:

[47] For a textbook derivation, see Section 11.1 of S. Weinberg, *Gravitation and Cosmology* (Wiley, New York, 1972).

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$$E \simeq \int_0^R 4\pi r^2 \left[\Delta\mathcal{E}(r) - \frac{3}{2} m_B n(r) c^2 \big(G\mathcal{M}(r)/rc^2\big)^2 \right] dr$$
$$+ G \int_0^R 4\pi r\, dr \Big[m_B n(r) \mathcal{M}(r) \big(2\mathcal{M}(r) G/rc^2\big)$$
$$+ m_B n(r) 4\pi r^3 p(r) + 4p(r)\mathcal{M}(r) \Big]. \qquad (1.9.A15)$$

Because each relativistic correction term in Eq. (1.9.A15) is individually small, of order Mv^4/c^4, they can each be evaluated by using the non-relativistic approximation

$$\rho(r) = m_B\, n(r) \qquad (1.9.A16)$$

and the non-relativistic equation of equilibrium, Eq. (1.1.4), which gives

$$\mathcal{M}(r) = -r^2 p'(r)/G\rho(r). \qquad (1.9.A17)$$

Then, also combining the last term on the first line of Eq. (1.9.A15) with the first term on the second line,

$$E = \int_0^R 4\pi r^2\, dr\, \Delta\mathcal{E}(r) + \int_0^R 2\pi r^4\, dr\, p'^2(r)/c^2\rho(r)$$
$$+ \int_0^R 16\pi^2 G r^4\, dr\, \rho(r) p(r)/c^2 - \int_0^R 16\pi r^3\, dr\, p(r) p'(r)/c^2 \rho(r).$$
$$(1.9.A18)$$

This can be further simplified, by noting that to order Mv^4/c^2 the third term is a linear combination of the second and fourth terms. Using the non-relativistic equation (1.1.4) of hydrostatic equilibrium (which is justified since this term is already small), we have

$$4\pi G r^2 \rho = G\frac{d\mathcal{M}}{dr} = -\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dp}{dr}\right),$$

so integrating by parts gives

$$\int_0^R 16\pi^2 G r^4\, dr\, \rho(r) p(r) = -\int_0^R 4\pi r^2 p(r) \frac{d}{dr}\left(\frac{r^2}{\rho(r)} p'(r)\right)$$
$$= \int_0^R 4\pi r^4 p'(r)^2/\rho(r)$$
$$+ \int_0^R 8\pi r^3 p(r) p'(r)/\rho(r), \qquad (1.9.A19)$$

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so that (1.9.A18) becomes

$$E = \int_0^R 4\pi r^2\, dr\, \Delta\mathcal{E}(r) + \int_0^R 6\pi r^4\, dr\, p'^2(r)/c^2\rho(r) - \int_0^R 8\pi r^3\, dr\, p(r)p'(r)/c^2\rho(r), \tag{1.9.A20}$$

as was to be shown.

1.10 White Dwarfs and Neutron Stars

In a white dwarf star nuclear reactions have come to an end, the star has cooled to the point that temperature may be neglected in studying the interior, and pressure and kinetic energy are provided by cold degenerate electrons. To a good approximation, the mass density ρ is $m_1\mu$ times the electron number density, where $m_1 = 931.49$ GeV/c^2 is the nuclear mass for unit atomic weight, and here $\mu \equiv A/Z$ is the atomic weight per electron, equal to 55.847/26 for iron. According to the rules of Fermi statistics, this gives the mass density as[48]

$$\rho(r) = \frac{8\pi m_1\mu}{h^3}\int_0^{k_F(r)} k^2\, dk = \frac{8\pi m_1\mu k_F^3(r)}{3h^3}, \tag{1.10.1}$$

where k_F is the maximum momentum of the filled electron levels, known as the Fermi momentum, and $h = 2\pi\hbar$ is the original Planck constant. (The extra factor 2 in 8π takes account of the electron's two spin states.) The internal energy density (excluding rest masses) and pressure of the electrons are then

$$\mathcal{E}(r) = \frac{8\pi}{h^3}\int_0^{k_F(r)} \left[\sqrt{k^2c^2 + m_e^2c^4} - m_ec^2\right] k^2\, dk \tag{1.10.2}$$

and

$$p(r) = \frac{8\pi c^2}{3h^3}\int_0^{k_F(r)} \frac{k^4}{\sqrt{k^2c^2 + m_e^2c^4}}\, dk. \tag{1.10.3}$$

Using Eq. (1.10.1) to express k_F in terms of ρ/μ, Eqs. (1.10.2) and (1.10.3) become formulas for $\mathcal{E}$ and p in terms of ρ/μ (or each other). With μ assumed uniform in the star, for any given μ and central density $\rho(0)$, we can use the

48 Fermi statistics requires that no two electrons can have the same momentum and spin. The possible states of free particles are represented by wave functions of the form $\exp(i\mathbf{k}\cdot\mathbf{x}/\hbar)$, with $\mathbf{k}$ the momentum. To confine these particles in a finite volume L^3 without violating translation invariance, we require the wave function to be the same on opposite faces of a box with edge L, so that $\mathbf{k} = 2\pi\hbar\mathbf{n}/L$, where $\mathbf{n}$ is a vector with integer components. The number of such vectors $\mathbf{n}$ with magnitude between n and $n+dn$ is $4\pi n^2\, dn$, so the number of possible momenta with magnitude between k and $k+dk$ is $(2\pi\hbar/L)^{-3} \times 4\pi k^2\, dk$, and with two particles per momentum state, the number of particles per volume with momenta between k and $k+dk$ is $8\pi k^2\, dk/(2\pi\hbar)^3$.

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equations (1.1.4) and (1.1.5) of hydrostatic equilibrium to find $\rho(r)$ and $p(r)$ throughout the star, and in particular to find the stellar mass M and radius R.

In general a white dwarf star is not a polytrope, with $\mathcal{E}$ proportional to p and p proportional to a power of ρ, except in the limit of very small or very large density and Fermi momentum. According to Eq. (1.10.1), the critical density at which the Fermi momentum becomes equal to $m_e c$ is

$$\rho_c = \frac{8\pi m_1 \mu m_e^3 c^3}{3h^3} = 0.97 \times 10^6 \, \mu \text{ g/cm}^3. \tag{1.10.4}$$

For $\rho \ll \rho_c$ we have $k_F \ll m_e c$, so Eqs. (1.10.2) and (1.10.3) give

$$p = \frac{8\pi}{3m_e h^3} \int_0^{k_F} k^4 \, dk = \frac{8\pi k_F^5}{15 m_e h^3} = \frac{8\pi}{15 m_e h^3} \left(\frac{3h^3 \rho}{8\pi m_1 \mu} \right)^{5/3} \tag{1.10.5}$$

and $\mathcal{E} = 3p/2$. This is a polytrope, with $\Gamma = 5/3$, and

$$K = \frac{8\pi}{15 m_e h^3} \left(\frac{3h^3}{8\pi m_1 \mu} \right)^{5/3}. \tag{1.10.6}$$

Equations (1.8.7) and (1.8.8) then give the radius and mass of the star as

$$R = 3.65375 \times \left(\frac{8\pi G}{5K} \right)^{-1/2} \rho(0)^{-1/6} = 2.0 \times 10^4 \, \mu^{-1} \left(\frac{\rho(0)}{\rho_c} \right)^{-1/6} \text{ km} \tag{1.10.7}$$

and

$$M = 2.71406 \times 4\pi \rho(0)^{1/2} \left(\frac{5K}{8\pi G} \right)^{3/2} = 2.79 \mu^{-2} \left(\frac{\rho(0)}{\rho_c} \right)^{1/2} M_\odot. \tag{1.10.8}$$

Thus low-mass white dwarfs, with $\rho(0) \ll \rho_c$, have radii somewhat greater than the Earth's, and masses somewhat less than the Sun's. Also, their thermal plus gravitational energy (1.1.13) is

$$\begin{aligned} E &= \int_0^R \big(\mathcal{E}(r) - 3p(r)\big) 4\pi r^2 \, dr = -6\pi \int_0^R p(r) r^2 \, dr \\ &= -6\pi \left(\frac{5}{8\pi G} \right)^{3/2} K^{-1/2} \rho(0)^{7/6} \int_0^{\xi_1} \Theta^{5/2}(\xi) \xi^2 \, d\xi, \end{aligned} \tag{1.10.9}$$

where $\xi_1 = 3.65375$.

For $\rho \gg \rho_c$ we have $k_F \gg m_e c$, so Eqs. (1.10.2) and (1.10.3) give

$$p = \frac{8\pi c}{3h^3} \int_0^{k_F} k^3 \, dk = \frac{2\pi c k_F^4}{3h^3} = \frac{2\pi c}{3h^3} \left(\frac{3h^3 \rho}{8\pi m_1 \mu} \right)^{4/3} \tag{1.10.10}$$

and $\mathcal{E} = 3p$. This is a polytrope, with $\Gamma = 4/3$, and

$$K = \frac{2\pi c}{3h^3}\left(\frac{3h^3}{8\pi m_1 \mu}\right)^{4/3}. \tag{1.10.11}$$

Equations (1.8.7) and (1.8.8) then give the radius and mass of the star as

$$R = 6.89685 \times \left(\frac{\pi G}{K}\right)^{-1/2} \rho(0)^{-1/3} = 5.3 \times 10^4 \mu^{-1} \left(\frac{\rho(0)}{\rho_c}\right)^{-1/3} \text{ km} \tag{1.10.12}$$

and

$$M = 2.01824 \times 4\pi \left(\frac{K}{\pi G}\right)^{3/2} = 5.87\mu^{-2} M_\odot. \tag{1.10.13}$$

It is striking that although R decreases and M increases with increasing central density, the mass approaches the limiting value (1.10.13), known as the *Chandrasekhar bound*. Of course, with $\Gamma = 4/3$, the energy (1.1.13) is $E = 0$.

White dwarfs with $\rho(0) \ll \rho_c$ have Γ considerably above 4/3, so according to the arguments in Section 1.1 they are stable at least against complete dispersal. Also, Eqs. (1.10.8) and (1.10.9) show that in this region M and E both vary monotonically with $\rho(0)$, while according to the theorem cited in the previous section, in order for a white dwarf to become unstable with increasing central density it is necessary for $\rho(0)$ to reach a value at which the conserved quantities M and E have vanishing derivatives with respect to central density.

If our results so far were the whole story, white dwarfs would also be stable for $\rho(0) \gg \rho_c$. For $\Gamma = 4/3$ both E and M are constants, but as we shall see, by itself the small departure from the polytropic equation of state due to the finite electron mass would give both $-E$ and M a continued monotonic increase with $\rho(0)$. But there are two complications that make instability possible.

One complication is provided by neutronization: the baryon number per electron μ is not really constant. For sufficiently large central density, the Fermi momentum is large enough for it to be energetically favorable for electrons to be absorbed by protons, in the reaction $e^- + p \to \nu_e + n$. For an iron white dwarf, this occurs when $\rho(0)$ exceeds 1.14×10^9 g/cm^3, and has the effect that ^{56}Fe nuclei with $\mu = 2.15$ are converted to ^{56}Mn, with $\mu = 2.24$. The increase in μ eventually causes the mass (1.10.13) to stop rising toward a limit, and instead to reach a maximum close to the value (1.10.13), and then begin to decrease. This maximum marks the transition to instability, and thus represents the true upper bound on the masses of iron white dwarfs.

Another complication arises from general relativity. According to the results of the previous section, even if μ is constant there is a transition to instability at a value of central density for which a maximum is reached by the energy

$$E = \int 4\pi r^2\, dr \big[\mathcal{E}(r) - 3p(r)\big] + \frac{16\pi}{(\pi G)^{3/2} c^2} \rho^{2/3}(0) K^{7/2} \eta, \tag{1.10.14}$$

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where $\eta = 3.49815$. The second term is a universal general relativistic correction for stars with $\mathcal{E}$ near $3p$, except that we must use the appropriate value for K, which for white dwarfs is given by Eq. (1.10.11). To calculate the first term for white dwarfs with $\rho(0) \gg \rho_c$, we use Eqs. (1.10.2) and (1.10.3) for $m_e^2c^2 \ll k_F^2$, together with the familiar expansions

$$\sqrt{k^2c^2 + m_e^2c^4} = kc + m_e^2c^3/2k + \cdots,$$
$$1/\sqrt{k^2c^2 + m_e^2c^4} = 1/kc - m_e^2c/2k^3 + \cdots.$$

Taking k_F from Eq. (1.10.1), we obtain expansions in powers of electron mass[49]

$$\mathcal{E} = \frac{3hc}{4}\left(\frac{3}{8\pi}\right)^{1/3}\left(\frac{\rho}{\mu m_1}\right)^{4/3} - \left(\frac{m_e}{\mu m_1}\right)\rho c^2 + \frac{3m_e^2c^3}{4h}\left(\frac{3}{8\pi}\right)^{-1/3}\left(\frac{\rho}{\mu m_1}\right)^{2/3} + \cdots, \tag{1.10.15}$$

$$3p = \frac{3hc}{4}\left(\frac{3}{8\pi}\right)^{1/3}\left(\frac{\rho}{\mu m_1}\right)^{4/3} - \frac{3m_e^2c^3}{4h}\left(\frac{3}{8\pi}\right)^{-1/3}\left(\frac{\rho}{\mu m_1}\right)^{2/3} + \cdots. \tag{1.10.16}$$

The leading terms give the equation of state (1.10.10) of a $\Gamma = 4/3$ polytrope. The terms in Eqs. (1.10.15) and (1.10.16) of first and second order in m_e give

$$\Delta\mathcal{E} \equiv \mathcal{E} - 3p = -\left(\frac{m_e}{\mu m_1}\right)\rho c^2 + \frac{3m_e^2c^3}{2h}\left(\frac{3}{8\pi}\right)^{-1/3}\left(\frac{\rho}{m_1\mu}\right)^{2/3} + \cdots$$
$$= -\left(\frac{m_e}{\mu m_1}\right)\rho c^2 + \frac{3m_e^2c^4}{8m_1^2\mu^2K}\rho^{2/3} + \cdots. \tag{1.10.17}$$

Since the factor m_e^2/m_1^2 makes the term in $\int 4\pi r^2\,dr\,\Delta\mathcal{E}$ of second order in m_e very small, it can be evaluated by using the solution given in Section 1.8 for a non-relativistic polytrope with $\Gamma = 4/3$. Equation (1.10.17) then gives the expansion

$$\int 4\pi r^2\,dr\,\Delta\mathcal{E} = -\left(\frac{m_e}{m_1\mu}\right)Mc^2 + \frac{3\pi m_e^2c^4K^{1/2}\zeta}{2m_1^2\mu^2(\pi G)^{3/2}}\rho(0)^{-1/3} + \cdots, \tag{1.10.18}$$

where[50]

$$\zeta = \int_0^{\xi_1}\Theta^2(\xi)\xi^2\,d\xi = 4.3267. \tag{1.10.19}$$

[49] The term in Eq. (1.8.15) of first order in m_e is present because we have chosen to define E excluding all rest masses, including the electron rest mass. As we shall see in Eq. (1.10.18), it leads to a term in E that is independent of central density, and therefore has no effect on the threshold for instability.

[50] The numerical value here is inferred from Eqs. (6.10.19) and (6.10.20) of S. Shapiro and S. Teukolsky, *Black Holes, White Dwarfs, and Neuton Stars* (Wiley, New York, 1983).

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The critical central density ρ_{inst} for a transition from stability to instability is then the stationary point of Eq. (1.10.14):

$$0 = \left.\frac{\partial E}{\partial \rho(0)}\right|_{\rho(0)=\rho_{\text{inst}}} = -\frac{\pi m_e^2 c^4 K^{1/2}\zeta}{2m_1^2\mu^2(\pi G)^{3/2}}\rho_{\text{inst}}^{-4/3} + \frac{32\pi}{3(\pi G)^{3/2}c^2}\rho_{\text{inst}}^{-1/3}K^{7/2}\eta. \tag{1.10.20}$$

This gives[51]

$$\rho_{\text{inst}} = \frac{3m_e^2c^6\zeta}{64m_1^2\mu^2K^3\eta} = \frac{8\pi m_e^2 m_1^2 c^3\mu^2\zeta}{h^3\eta} = 6.6\times 10^9\mu^2\ \text{g/cm}^3. \tag{1.10.21}$$

The critical densities for general relativistic instability along with neutronization thresholds[52] are given (both in g/cm^3) for three commonly considered chemical compositions in the table below. Since white dwarfs of low central density are stable, the transition to instability for increasing central density occurs at the *lower* of the neutronization threshold and ρ_{inst}. This transition is evidently produced by neutronization for ^{56}Fe, and by general relativity for ^{12}C and ^{4}He.

Critical densities and neutronization thresholds

Composition	*Neutronization threshold*	ρ_{inst}
^{56}Fe	1.14×10^9	3.06×10^{10}
^{12}C	3.9×10^{10}	2.63×10^{10}
^{4}He	1.37×10^{14}	2.63×10^{10}

Neutronization in a white dwarf star can only go so far before the star becomes unstable. But when a star that is too massive to form a stable white dwarf exhausts its nuclear fuel it collapses, becoming a supernova. The density increases, and the rapid rise in the electron Fermi momentum forces a nearly complete neutronization. Almost all of the star's protons and electrons are converted to neutrons, with just enough electrons left for the neutron decay $\text{n}\to\text{p}+\text{e}^-+\nu$ to be blocked by the Pauli exclusion principle, and with an equal number of protons left over to balance the electron charges. After blowing off enough matter, what remains is a stable neutron star.[53]

In a neutron star, it is neutrons rather than electrons that fill all quantum levels up to a Fermi momentum $k_F(r)$. Here the mass density is given again by Eq. (1.10.1), but with the neutron mass m_n in place of $m_1\mu$:

51 When cancellations and different notation are taken into account, this formula turns out to be identical to the second line of Eq. (6.10.28) of Shapiro and Teukolsky, *op. cit.*, derived in a rather different manner. They give a numerical result $6.615\times 10^9\mu^2\ \text{g/cm}^3$ for ρ_{inst}.

52 Thresholds are taken from Shapiro and Teukolsky, *op. cit.*, Table 3.1.

53 W. Baade and F. Zwicky, *Phys. Rev.* **46**, 76 (1934).

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$$\rho(r) = \frac{8\pi m_n}{h^3} \int_0^{k_F(r)} k^2\, dk = \frac{8\pi m_n k_F^3(r)}{3h^3}. \tag{1.10.22}$$

There is again a critical density here, but now one for which the Fermi momentum becomes $m_n c$ rather than $m_e c$:

$$\rho_c = \frac{8\pi m_n^4 c^3}{3h^3} = 6.11 \times 10^{15}\ \text{g/cm}^3. \tag{1.10.23}$$

The mean separation between neutrons is

$$\left(\frac{\rho}{m_n}\right)^{-1/3} = \left(\frac{\rho_c}{\rho}\right)^{1/3} \times 0.52 \times 10^{-13}\ \text{cm}, \tag{1.10.24}$$

which for $\rho \ll \rho_c$ is greater than the range of nuclear forces, justifying the treatment of neutrons as free particles, as implicitly assumed in Eq. (1.10.22). Also, for $\rho \ll \rho_c$ even neutrons at the top of the Fermi sea are moving non-relativistically, so the neutron pressure is given by the same formula (1.10.5) as for low-mass white dwarfs, but with m_n in place of both m_e and $m_1\mu$:

$$p = \frac{8\pi}{15 m_n h^3} \left(\frac{3h^3 \rho}{8\pi m_n}\right)^{5/3}. \tag{1.10.25}$$

This is again a polytrope with $\Gamma = 5/3$. Since the neutrons are moving non-relativistically, the structure of the neutron star for $\rho(0) \ll \rho_c$ is governed by the Newtonian equations of gravitation and dynamics, just like the structure of white dwarfs, and can therefore be treated by the methods of Section 1.8. In particular, we can use Eqs. (1.10.6)–(1.10.8) for the neutron star's radius R and mass M, again with m_n in place of both m_e and $m_1\mu$:

$$R = 3.65375 \times \left(3G m_n h^3\right)^{-1/2} \left(\frac{3h^3}{8\pi m_n}\right)^{5/6} \rho(0)^{-1/6} = 11.0\ \text{km} \times \left(\frac{\rho_c}{\rho(0)}\right)^{1/6} \tag{1.10.26}$$

and

$$M = 2.71406 \times 4\pi \rho(0)^{1/2} (3G m_n h^3)^{-3/2} \left(\frac{3h^3}{8\pi m_n}\right)^{5/2} = 2.7 M_\odot \times \left(\frac{\rho(0)}{\rho_c}\right)^{1/2}. \tag{1.10.27}$$

The mass is again a few solar masses, like a white dwarf, but now in a radius of a few kilometers instead of a few thousand kilometers.

It may be surprising that both white dwarfs and neutron stars typically have masses of order $M_\odot$, even though they are supported by the degeneracy pressure of particles of very different mass: electrons for white dwarfs, and neutrons for neutron stars. This is because the electron mass cancels in Eq. (1.10.8) for white dwarf masses (though not in Eq. (1.10.7) for white dwarf radii). Indeed, Eqs. (1.10.8) and (1.10.27) give the masses of both white dwarfs and neutron stars as equal to different factors of order unity times the same combination of fundamental constants:

$$(\hbar c/G)^{3/2} m_1^{-3/2} = 1.90 M_\odot .$$

In the following section we will see the same combination of constants appearing in the mass of stars supported by radiation pressure.

The theory of neutron stars is much more complicated for $\rho(0)$ comparable to or greater than ρ_c. Here the neutron velocities are comparable to c, and since the neutrons are the source of the star's gravitational field, general relativity is needed to work out the structure of the neutron star. Calculations by Landau[54] and by Oppenheimer and Volkoff[55] showed that there is a maximum mass, beyond which neutron stars become unstable. Oppenheimer and Volkoff found this maximum mass to be $0.7 M_\odot$. But these calculations treated the neutrons as an ideal gas. For $\rho(0)$ greater than ρ_c, the separation (1.10.24) of neutrons is no greater than the range of nuclear forces, and a treatment of neutrons as free particles is no longer reliable. There have been various estimates of the maximum mass of stable neutron stars, all of the order of a few solar masses at most.

Because of their small size, neutron stars would naturally be expected to spin very rapidly. The Sun, with a radius $R_\odot$ of about 7×10^5 km, rotates with a frequency $\omega_\odot / 2\pi = 5 \times 10^{-7}$ revolutions per second, not an exceptionally rapid rotation. If a progenitor star core had a similar radius and rotation rate, and if angular momentum per mass $\propto \omega R^2$ were conserved in its collapse to a neutron star, then the decrease in its radius to a few kilometers would increase its rate of rotation by a factor of order 10^{10}, giving it a rotation rate of a few thousand revolutions per second. This is roughly the maximum possible rotation rate. A body of mass M and radius R that is held together only by gravitation cannot rotate at an angular frequency ω greater than $\omega_{\rm max} \approx \sqrt{GM/R^3}$, at which rate the centripetal acceleration $\omega^2 R$ equals the gravitational acceleration GM/R^2. For a neutron star at the Oppenheimer–Volkoff limit, $M \simeq 0.7 M_\odot$ and $R \simeq 10$ km, so its maximum rotation rate $\omega_{\rm max}/2\pi$ is about 1,600 revolutions per second.

The theoretical anticipation[56] of rapidly rotating neutron stars was borne out by the discovery of pulsars. First came the observation[57] in 1967 of a source of radio pulses with period 1.33 seconds. It was proposed[58] that this was a rapidly rotating neutron star, emitting radiation along the direction of a strong magnetic field, at an angle to the axis of rotation, which happens to point in the direction of Earth once in each rotation. This suggestion became widely

54 L. D. Landau, *Phys. Z. Sowjetunion* **1**, 285 (1933).
55 J. R. Oppenheimer and G. M. Volkoff, *Phys. Rev.* **55**, 374 (1938).
56 F. Pacini, *Nature* **216**, 567 (1967).
57 A. Hewish, S. J. Bell, J. D. H. Pilkington, P. F. Scott, and R. A. Collins, *Nature* **217**, 709 (1968).
58 T. Gold, *Nature* **218**, 731 (1968).

accepted with the discovery of a source of much more rapid pulses, with period 33 milliseconds, too rapid for anything but a neutron star, in a known supernova remnant, the Crab nebula. Since then pulsars have been found emitting radiation at various wavelengths, with pulse periods ranging from 8.5 seconds down to 1.4 milliseconds. There is uncertainty about the mechanism for producing this radiation, but there seems no doubt that the sources are rapidly rotating neutron stars.

Since the discovery of pulsars neutron stars have become even more interesting. As discussed in Section 2.3, pulsars were found in binary systems, whose decay gave the first observational evidence for the emission of gravitational radiation, and the coalescence of binary neutron stars was proposed to account for observed bursts of electromagnetic radiation (so-called *kilonovae*) with intrinsic luminosity between ordinary novae and supernovae. Most dramatically, as we shall see in Section 2.4, in 2017 gravitational waves as well as electromagnetic radiation were observed coming from the coalescence of a binary of neutron stars.

1.11 Supermassive Stars

There is an interesting class of stars in which the pressure of material particles is much less than radiation pressure, though not entirely negligible. As we shall see, these stars are necessarily supermassive, typically heavier than $100M_\odot$. Stars this massive are very rare in the present universe,[59] but they are plausible precursors of supernovae that have led to neutron stars or black holes.

The energy density and pressure of radiation are given by the well-known formulas

$$\mathcal{E}_{\rm rad} = aT^4, \quad p_{\rm rad} = \frac{1}{3}aT^4, \tag{1.11.1}$$

where a is the radiation energy constant

$$a = \frac{\pi^2 k_{\rm B}^4}{5\hbar^3 c^3} = 7.567 \times 10^{-15}\,\frac{\rm erg}{\rm cm^3\,K^4}. \tag{1.11.2}$$

The matter of the star is assumed to form a non-relativistic ideal gas of particles of average mass $\overline{m}$, in thermal equilibrium with the radiation, and with energy density and pressure

$$\mathcal{E}_{\rm gas} = \frac{\rho k_{\rm B} T}{\overline{m}(\gamma - 1)}, \quad p_{\rm gas} = \frac{\rho k_{\rm B} T}{\overline{m}}, \tag{1.11.3}$$

[59] One famous example is η Carinae A, the heavier star in a binary at a distance of 2,300 pc, with a mass estimated as $(100\text{–}200)M_\odot$. This is not a stable star; in 1837 it became the second brightest star in the sky, then faded to below naked-eye visibility, and in 1940 again became easily visible. It is estimated to have lost a mass of about $10M_\odot$ in a decade.

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where k_B is the Boltzmann constant, ρ is the mass density, γ is the polytrope index of the gas alone, not counting the radiation, and $\overline{m}$ is the mean mass of the gas particles. (For ionized hydrogen $\overline{m} \simeq m_p$, the proton mass, and $\gamma = 5/3$.) The ratio of gas pressure to radiation pressure is then

$$\beta \equiv p_{gas}/p_{rad} = \frac{3k_B}{a\overline{m}} \frac{\rho}{T^3}. \tag{1.11.4}$$

For $\beta \ll 1$ Eq. (1.11.1) shows that the star is close to a polytrope with index $\Gamma = 4/3$, so that $p \simeq K\rho^{4/3}$ with a constant K that can be expressed it terms of β:

$$K \simeq p_{rad}/\rho^{4/3} = \frac{aT^4}{3\rho^{4/3}} = \left(\frac{3}{a}\right)^{1/3} \left(\frac{k_B}{\overline{m}\beta}\right)^{4/3}. \tag{1.11.5}$$

Assuming the whole star to be in a state of effective convection, the constant K and hence also β must be constant through the star. This can be seen more quantitatively from considerations of entropy. As we saw in Section 1.7, the entropy per gram of an ideal gas with $\mathcal{E}_{gas} = p_{gas}/(\gamma - 1)$ is $R/(\gamma - 1)\ln\left(T\rho^{1-\gamma}\right)$, where $R = k_B/\overline{m}$ is the constant appearing in the ideal gas law $p_{gas} = R\rho T$. The entropy of the radiation per mass of gas is calculated from

$$T\,ds_{rad} = d\left(\frac{aT^4}{\rho}\right) + \frac{aT^4}{3} d\left(\frac{1}{\rho}\right),$$

from which it follows that $s_{rad} = 4aT^3/3\rho$. The total specific entropy is then

$$s = 4aT^3/3\rho + R/(\gamma - 1)\ln\left(T\rho^{1-\gamma}\right) = R\left[\frac{4}{\beta} + \frac{1}{\gamma - 1}\ln\left(p_{gas}/\rho^{\gamma}\right)\right].$$

We expect the logarithm to vary only by amounts of order unity (at least away from the star's surface), so in order for s to be constant $1/\beta$ can only change by amounts of order unity, and therefore for $\beta \ll 1$ by only a small fractional amount, at most of order β.

Using the general formula (1.8.8) for the mass of any non-relativistic polytrope, and setting $\Gamma = 4/3$, we have here

$$M = 4\pi \times 2.01824 \times \left(\frac{K}{\pi G}\right)^{3/2} = 18M_\odot \left(\frac{m_p}{\overline{m}}\right)^2 \frac{1}{\beta^2}. \tag{1.11.6}$$

With $\overline{m} \simeq m_p/2$ and β less, say, than 0.3, the mass must be above $800M_\odot$, and hence such stars are truly supermassive.

It should be noted that, even though we are interested here in the case $\beta \ll 1$, for which gas pressure is much less than radiation pressure, we can (and will) nevertheless confine our attention to the case $\rho c^2 \gg aT^4$, for which gas rest energy density is much greater than radiation energy density. The ratio is

$$\frac{\rho c^2}{aT^4} = \beta \frac{\overline{m}c^2}{3k_B T}.$$

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Hence, as long as the gas itself is non-relativistic, with $3k_B T \ll \overline{m}c^2$, we can assume that $\rho c^2 \gg aT^4$, provided only that β is greater than a lower bound $3k_B T/\overline{m}c^2$. For this reason, general relativity has so far played no role in our remarks about supermassive stars.

But, as shown in Section 1.9, in order to identify the critical density at which the star becomes unstable, we must find the stationary point of the internal energy E, which general relativity gives as the expression (1.9.2). To calculate the first term in Eq. (1.9.2), we need

$$\Delta\mathcal{E} \equiv \mathcal{E}_{\rm rad} + \mathcal{E}_{\rm gas} - 3p_{\rm rad} - 3p_{\rm gas} = -\frac{3\gamma-4}{\gamma-1}p_{\rm gas} \simeq -\frac{3\gamma-4}{\gamma-1}\beta p. \quad (1.11.7)$$

The factor β makes this small, so we can calculate its integral over the star by taking the star to be a polytrope with $\Gamma = 4/3$, the corrections to this being doubly small. Making the appropriate substitutions (1.8.3) and (1.8.4) for a $\Gamma = 4/3$ polytrope, we have

$$\int 4\pi r^2 p\, dr = 4\pi K\rho^{4/3}(0)\left[\rho^{1/3}(0)(\pi G/K)^{1/2}\right]^{-3}\alpha, \quad (1.11.8)$$

where α is another numerical constant,

$$\alpha \equiv \int_0^{\xi_1} \Theta^4(\xi)\xi^2\, d\xi = 1.18119. \quad (1.11.9)$$

Combining Eqs. (1.9.2), (1.11.7), and (1.11.8), we have then

$$E = -\frac{4\pi\beta K^{5/2}\alpha}{(\pi G)^{3/2}}\left(\frac{3\gamma-4}{\gamma-1}\right)\rho^{1/3}(0) + \frac{16\pi K^{7/2}\eta}{(\pi G)^{3/2}c^2}\rho^{2/3}(0). \quad (1.11.10)$$

The star is stable for sufficiently small central densities, where the second term, due to general relativistic corrections, can be neglected. With increasing central density, the star becomes unstable at a critical value $\rho_{\rm inst}$ of the central density at which $\partial E/\partial\rho(0) = 0$:

$$\rho_{\rm inst} = \left[\frac{\beta\alpha c^2}{8K\eta}\frac{3\gamma-4}{\gamma-1}\right]^3. \quad (1.11.11)$$

To bring out the physical significance of this result, it is useful to consider the radius of the star, which according to Eq. (1.8.7) is given by $R = \xi_1(K/\pi G)^{1/2}\rho^{-1/3}(0)$, where $\xi_1 = 6.89685$ is the value of ξ where $\Theta(\xi)$ drops to zero. At the critical central density, this is

$$R_{\rm inst} = \frac{8\eta\xi_1}{\beta\alpha c^2}(\pi G)^{-1/2}K^{3/2}\frac{\gamma-1}{3\gamma-4}. \quad (1.11.12)$$

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It is instructive to compare this with the Schwarzschild radius $2MG/c^2$. The mass of a $\Gamma = 4/3$ polytrope is given by Eq. (1.11.6), so

$$\frac{c^2 R_{\text{inst}}}{MG} = \frac{\gamma - 1}{\beta(3\gamma - 4)} \frac{2 \times 6.89685 \times 3.49815}{2.01824 \times 1.18119} = 20.24 \frac{\gamma - 1}{\beta(3\gamma - 4)}. \quad (1.11.13)$$

For a stable supermassive star we have $R > R_{\text{inst}}$, so for small β the star's radius is much larger than the Schwarzschild radius, and the redshift MG/Rc^2 from the surface of the star is quite small.

Finally, let us consider the evolution of a supermassive star. The mass M of the star is dominated by the rest mass of the nucleons it contains, and hence cannot appreciably change with time. According to Eq. (1.11.6), it follows that K does not change much, so the same is true of β. But the central density can and does evolve. The internal energy (1.11.10) may be written

$$E\big(\rho(0)\big) = E_0 \left[-2 \left(\frac{\rho(0)}{\rho_{\text{inst}}} \right)^{1/3} + \left(\frac{\rho(0)}{\rho_{\text{inst}}} \right)^{2/3} \right], \quad (1.11.14)$$

where E_0 is a positive constant. If initially a supermassive star has $\rho(0) < \rho_{\text{inst}}$ it will have an internal energy that decreases monotonically with central density, and be stable. As the star radiates, it loses internal energy, which must then become increasingly negative. Since for $\rho(0) < \rho_{\text{inst}}$ the internal energy E is a monotonically decreasing function of central density, this requires the central density to rise, until $\rho(0)$ reaches its critical value, whereupon the star explodes. It is not clear what happens after that, but it is plausible that a stable remnant is left, a star that is no longer supermassive.

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