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MULTIDIMENSIONAL ITERATIVE INTERPOLATION

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ABSTRACT. We define an iterative interpolation process for data spread over a closed discrete subgroup of the Euclidean space. We describe the main algebraic properties of this process. This interpolation process, under very weak assumptions, is always convergent in the sense of Schwartz distributions. We find also a convenient necessary and sufficient condition for continuity of each interpolation function of a given iterative interpolation process.

1. **Definition of the Interpolation Process.** The original data is a complex valued function f whose domain of definition is a subgroup G of the Euclidean space R^d. We will always assume that G is a closed discrete subgroup of R^d and that the vector subspace generated by G is all the space R^d. Our aim is to describe a scheme in order to extend f to an increasing sequence of subgroups G_k starting with G and such that the union of all subgroups is dense in R^d.

The extension of f will be carried through a linear transformation T of R^d and a complex valued weight function w. The following assumptions are made about T and w.

HYPOTHESIS 1.1. G is inside the image of G by T: $T(G) \supset G$.

HYPOTHESIS 1.2. The domain of definition of w is $T(G)$.

HYPOTHESIS 1.3. $w(0) = 1$ and $w(x) = 0$ if x is any element of G other than 0.

HYPOTHESIS 1.4. The support of w is finite: there is just a finite number of points x for which $w(x) \neq 0$.

We set $G_0 = G$ and for any positive integral value k, $G_k = T^k(G)$. G_∞ is by definition the subgroup of R^d obtained as the union of the increasing sequence of sets G_k. Note that if x belongs to G_∞, then $T^k x$ belongs to G_∞ for any integer k of Z.

LEMMA 1.1. *There is one and only one function g defined on G_∞ such that for any x in G, $g(x) = f(x)$ and for any x in G and for any nonnegative integral number k,*

$$g(T^{k+1}x) = \sum_{y \in G} w(Tx - y)g(T^k y). \tag{1.1}$$

Key Words and Phrases: Iterative interpolation process, fundamental interpolating function, characteristic function, continuous interpolation, temperate distribution, Fourier transform.

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PROOF. A first remark should be done. From the fact that the vector subspace generated by G is R^d and that $T(G)$ contains G, it follows that $T(R^d) = R^d$ and hence T is one to one.

By induction, a sequence of functions may be defined. f_0 is the original function f. If f_k is defined over G_k, then we set, for any x in G, $f_{k+1}(T^{k+1}x) = \sum_{y \in G} w(Tx - y) f_k(T^k y)$. Since T is one to one, f_{k+1} is well defined over G_{k+1}. From Hypothesis 1.4, it follows that the infinite series in the last formula is in fact a finite sum; so this formula is always meaningful.

The effect of the Hypothesis 1.3 (which in fact is a compatibility condition) is that for any integral value k, f_{k+1} is an extension of f_k: for any x of G_k, $f_{k+1}(x)$ is equal to $f_k(x)$. So, it is possible to define a function g defined on G_∞: if x belongs to G_∞, then there is a k for which x belongs to G_k; for such an index k, we set $g(x) = f_k(x)$. This function g is the one that is needed and is unique. ∎

DEFINITION. The extension g in (1.1) will be called the *iterative interpolation of f with respect to T and w.*

Now we describe a very simple way to find a linear mapping T that fulfills Hypothesis 1.1. Let us use d points $\{e_1, e_2, \ldots, e_d\}$ of G such that the subgroup generated by these points is G and let us use a nonsingular square matrix of order d with integral entries $(n_{j,k})$. There is a unique linear mapping H defined on R^d such that $H(e_j) = \sum_k n_{j,k} e_k$. H is invertible and $H(G)$ is contained in G. If $T = H^{-1}$, then T satisfies the Hypothesis 1.1.

We outline the content of this paper. In Section 2, we present six examples of iterative interpolation processes, the last two of them being new fractal surfaces. Simple algebraic properties of interpolation processes are studied in Sections 3 and 4. A fundamental function F, which is the kernel of the interpolation process, is introduced; this function F fulfills an interesting functional equation. In Section 6, we show that under very weak conditions a Schwartz distribution is associated to the fundamental function F and may be viewed as an extension to the space R^d of F. In Section 7, we prove that the subgroup G_∞ is dense in R^d if the spectral radius of T is smaller than one. In Section 8, a necessary and sufficient condition for the continuity of F is found. As a conclusion, applications of the previous theory are done in Section 9.

The first people to use iteration for unidimensional interpolation were Aitken and Neville. Their formulas for linear iterative interpolation can be found in [9]. This idea of using iteration for interpolation was pursued by various authors, see for example [1–7] and [12–13]. Iterative interpolation provides a very large class of interpolating functions, which may be more or less regular according to the choice of the linear transformation T and weight function w.

2. **Examples.**

Now we give some examples of iterative interpolation processes. The first three examples we give are one-dimensional. So in these cases, d is one and G is the subgroup of all integers in R.

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EXAMPLE 1 (SEE [2] AND [5]). The linear transformation T is $Tx = x/2$, the weight function is $w(0) = 1$, $w(\pm 1/2) = 9/16$, $w(\pm 3/2) = -1/16$ and $w(x) = 0$ for any other integral multiple x of $1/2$. In [2], this interpolation process has been called dyadic interpolation.

EXAMPLE 2 (SEE [4]). The linear transformation T is $Tx = x/b$ where b is an integral value larger than one. Weights are defined through some Lagrange polynomials.

If $\{-1, 0, 1, 2\}$ are used as nodes, we get four Lagrange polynomials:

$$L_{-1}(x) = -x(1-x)(2-x)/6, \qquad L_0(x) = (x+1)(1-x)(2-x)/2,$$
$$L_1(x) = (x+1)x(2-x)/2, \qquad L_2(x) = -(x+1)x(1-x)/6.$$

We define weights w as follows : $w(n + j/b) = L_{-n}(j/b)$ for $n = -2, -1, 0, 1$ and for $j = 0, 1, \ldots, b-1$; otherwise $w(x) = 0$. (In [4], this is called a Lagrange iterative interpolation process.)

EXAMPLE 3 (SEE [3] AND [11]). If complex-valued functions and weights are allowed, the same scheme can be used to produce fractal curves like von Koch curve. That curve is generated if $Tx = x/4$, $w(0) = 1$, $w(\pm 3/4) = 1/3$, $w(-1/2) = 1/2 + i\sqrt{3}/6$, $w(\pm 1/4) = 2/3$, $w(1/2) = 1/2 - i\sqrt{3}/6$ and $w(x) = 0$ for other integral multiple of $1/4$. The function f that can be used is any function f such that $f(0) = 0$ and $f(1) = 1$. If $g(x)$ is the iterative interpolation of f, then von Koch curve is produced by the continuous extension of g to the unit interval.

We reproduce the elementary computations. $g(0) = f(0) = 0$, $g(1) = f(1) = 1$. If the formula (1.1) is used with $k = 0$, then the values of g at $1/4, 1/2$, and $3/4$ are found.

$$g(1/4) = w(1/4)g(0) + w(-3/4)g(1) = 1/3$$
$$g(1/2) = w(1/2)g(0) + w(-1/2)g(1) = 1/2 + i\sqrt{3}/6$$
$$g(3/4) = w(3/4)g(0) + w(-1/4)g(1) = 2/3.$$

In the same way, if formula (1.1) is used with $k = 1$, then the function g will be known at any integral multiple of $1/16$.

$$g(1/16) = w(1/4)g(0) + w(-3/4)g(1/4) = 1/9$$
$$g(1/8) = w(1/2)g(0) + w(-1/2)g(1/4) = 1/6 + i\sqrt{3}/18$$
$$g(3/16) = w(3/4)g(0) + w(-1/4)g(1/4) = 2/9$$
$$g(1/4) = w(0)g(1/4) = 1/3$$
$$g(5/16) = w(1/4)g(1/4) + w(-3/4)g(1/2) = 7/18 + i\sqrt{3}/18, \ldots.$$

We get 17 points on the von Koch curve: $\big(n/16, g(n/16)\big)$ for $n = 0, 1, \ldots, 16$. If the formula (1.1) is applied again and again, for $k = 2, 3, 4, \ldots$, more and more points on the von Koch curve are drawn.

EXAMPLE 4. If two iterative interpolation processes are defined, one in R^{d_1}, the other in R^{d_2}, then it is possible to define the cartesian product of these processes on $R^{d_1} \times R^{d_2}$.

If the first process uses a subgroup G_1, a linear transformation T_1 and a weight function w_1 and that the second process uses G_2, T_2 and w_2 , then we set $G = G_1 \times G_2, T(x,y) = (T_1x, T_2y)$ for x in R^{d_1} and y in R^{d_2} and $w(x,y) = w_1(x)w_2(y)$.

EXAMPLE 5. We take $d = 2$, $G = Z \times Z$, $T(x_1,x_2) = \big((x_1-x_2)/2, (x_1+x_2)/2\big)$. In order to define the weight function w, we will use four positive numbers whose sum is one, p_1, p_2, p_3, p_4. $w(0,0) = 1$, $w(1/2, 1/2) = p_1$, $w(-1/2, 1/2) = p_2$, $w(-1/2, -1/2) = p_3$, $w(1/2, -1/2) = p_4$ and $w(x,y) = 0$ otherwise.

EXAMPLE 6. We take $d = 2$, G is a triangular mesh $= \{(m + n/2, n\sqrt{3}/2) : m, n \in Z\}$, $T(x,y) = \big((3x - \sqrt{3}y)/6, (\sqrt{3}x + 3y)/6\big)$. $w(0,0) = 1$; $w(x,y) = 4/9$ if (x,y) is one of the six closest neighbours of $(0,0)$ in $T(G)$, i.e. if $x^2 + y^2 = 1/3$; $w(x,y) = -1/9$ if $x^2 + y^2 = 4/3$; $w(x,y) = 0$ otherwise.

We will come back to these examples at the end of this paper.

3. **Elementary properties of the interpolation process.** The interpolation process we have defined satisfies some simple properties. The interpolation process is a *linear process*: if $f_1(x)$ and $f_2(x)$ are two functions defined on G and if $f(x) = f_1(x) + f_2(x)$, then the respective interpolations $g_1(x), g_2(x)$ and $g(x)$ defined on G_∞ satisfy the linear relation: $g(x) = g_1(x) + g_2(x)$. Likewise, for any real scalar c, the interpolation corresponding to $cf(x)$ is $cg(x)$. Another basic property is the *translation property* of the interpolation when translation is done through elements of G. If $f(x)$ is a given function defined on G with g as interpolation and if a is an element of G , the interpolation of the function $f(x+a)$ is the function $g(x+a)$.

LEMMA 3.1. *If g is the interpolation of the function f defined over G, if k is a positive integer, then the interpolation of the function h defined on G by $h(x) = g(T^kx)$ is the function $g(T^kx)$ already defined on G_∞.*

The proof is almost trivial, but the *homogeneity* of the interpolation process is important.

4. **The fundamental interpolating function.** As in Section 1, we consider a subgroup G of R^d, a linear transformation T of R^d and a complex-valued weight function w. The hypotheses 1.1, 1.2, 1.3 and 1.4 are fulfilled. Let us start with the following function: $f(0) = 1$ and $f(x) = 0$ when x is any other element of G. Using the interpolation process, this function can be extended to G_∞. Let us denote by $F(x)$ this extension; F will be called the *fundamental interpolating function*.

REMARK. There is a close connection between F and w. For any x in $T(G)$,

$$F(x) = w(x).$$

LEMMA 4.1. *If S is the support of w and if x is an element of $G_n = T^n(G)$ which is not in $\sum_{0\le k<n} T^k(S)$ (the Minkowski sum of the sets $T^k(S)$), then $F(x) = 0$.*

PROOF. This can be easily proven by induction. ∎

LEMMA 4.2. *If $f(x)$ is a function defined on G, the extension g to G_∞ according to the interpolation process is $g(x) = \sum_{y \in G} f(y)F(x-y)$.*

PROOF. If x is in G_n, then the series $\sum_{y\in G} f(y)F(x-y)$ is in fact a finite sum according to the previous lemma. Since the interpolating process is linear and invariant through translation, the last series is in fact the interpolation function for the function $f(x)$. By unicity, the series is equal to $g(x)$. ■

THEOREM 4.3. *The function F satisfies the following functional equation:*

$$F(T^k x) = \sum_{y \in G} F(T^k y)F(x-y) \text{ where } x \in G_\infty \text{ and } k \in N. \tag{4.1}$$

PROOF. This follows right away from Lemma 3.1 and Lemma 4.2. ■

Formula (4.1) can be used for numerical purposes in the computation of F. For example, if the values of $F(x)$ are known for x in G_k, in one step it is possible to compute F at any point of G_{2k}:

$$F(T^{2k}x) = \sum_{y \in G} F(T^k y)F(T^k x - y).$$

5. **Characteristic polynomial.** A sequence P_n of trigonometric polynomials will be associated to the fundamental function F: if y is a vector of R^d, then

$$P_n(y) = \sum_{x \in G} F(T^n x)e^{i\langle x,y\rangle} \tag{5.1}$$

where $\langle x, y\rangle$ is the usual scalar product between vectors of R^d.

We set $P(y) = P_1(y)$. $P(y)$ will be called the *characteristic function* of the interpolation process.

LEMMA 5.1. *If T^* is the adjoint of T, if $U = (T^*)^{-1}$, then*

$$P_n(y) = \prod_{0 \le k \le n-1} P(U^k y). \tag{5.2}$$

PROOF. If, in Formula 4.1, x is replaced by Tx, then $F(T^{k+1}x) = \sum_{z\in G} F(T^k z)F(Tx - z)$ where $x \in G_\infty$ and $k \in N$. Hence: $\sum_{x\in G} F(T^{k+1}x)e^{i\langle x,y\rangle} = \sum_{x\in G}\sum_{z\in G} F(T^k z)F(Tx-z)e^{i\langle x,y\rangle}$, and so

$$\begin{aligned}
P_{k+1}(y) &= \sum_{z\in G}\sum_{x\in G} F(T^k z)F(Tx-z)e^{i\langle x,y\rangle} \\
&= \sum_{z\in G}\sum_{x\in G} F(T^k z)F(Tx)\exp(i\langle (x+T^{-1}z), y\rangle) \\
&= \sum_{z\in G} F(T^k z)P(y)\exp(i\langle T^{-1}z, y\rangle) = \sum_{z\in G} F(T^k z)P(y)e^{i\langle z,Uy\rangle} \\
&= P_k(Uy)P(y).
\end{aligned}$$

The recurrence relation $P_{k+1}(y) = P_k(Uy)P(y)$ has been proved. Hence

$$P_n(y) = \textstyle\prod_{0\le k\le n-1} P(U^k y).$$ ■

This last identity will be useful in the next section.

6. **Distribution of Schwartz and iterative interpolation.** In this section, we will show that under weak conditions, a temperate Schwartz distribution is associated to the fundamental function F and may be viewed as an extension to the space R^d of F.

Let us introduce a sequence of distributions. If ϕ is a C^∞-function with compact support, then we set

$$D_n(\phi) = \sum_{x \in G} F(T^n x)\phi(T^n x)(|\det T|)^n. \tag{6. 1}$$

D_n is a distribution in the sense of Schwartz; in fact, it is an absolutely convergent sum of Dirac masses located on G_n. We recall a general result which will help in the study of the sequence of D_n.

THEOREM 6.1. *If D_n is a sequence of temperate distributions on R^d whose Fourier transforms $G_n(y)$ fulfill the two following conditions:*

a) $G_n(y)$ converges pointwise to $G(y)$,

b) There is a polynomial B such that for every n and for every y, $|G_n(y)| \leq B(y)$,

then the sequence D_n of distributions is weakly convergent to a temperate distribution D whose Fourier transform is $G(y)$.

PROOF. The proof of this result when $d = 1$ has been given in [3]. A similar proof holds here. ∎

THEOREM 6.2. *If every eigenvalue of T has a modulus smaller than one and if $\sum_{x \in T(G)} w(x) = 1/|\det T|$, then the sequence D_n of distributions as defined in (6.1) weakly converges to a distribution D. If $P(y)$ is the characteristic polynomial of the interpolation process, then the Fourier transform $G(y)$ of D is $\prod_{k \geq 1}[|\det T| P(-T^{*k}y)]$.*

PROOF. The proof will follow from the study of the sequence of Fourier transforms $G_n(y)$ of D_n. We set $\Delta = |\det T|$. We have that $G_n(y) = \sum_{x \in G} F(T^n x)\exp(-i\langle T^n x, y\rangle)\Delta^n$ and the two following properties will be shown: the sequence $G_n(y)$ converges and every function $|G_n(y)|$ is bounded by the same polynomial.

According to formula (5.1) and Lemma 5.1, $G_n(y) = P_n(-T^{*n}y)\Delta^n = \prod_{1 \leq k \leq n}[P(-T^{*k}y)\Delta]$. The convergence of the sequence $G_n(y)$ will come from the finiteness of the series $\sum_{k \geq 1} |P(T^{*k}y) - 1/\Delta|$. By Hypothesis $P(0) = 1/\Delta$; P is differentiable and $\sum_{k \geq 1} \|T^{*k}y\| < \infty$ (this is true because the spectral radius of T is less than one). According to Weierstrass criterion, $G_n(y)$ converge uniformly on every compact subset of R^d.

For any y, $G_n(y)$ converge to $G(y) = \prod_{k \geq 1}[P(-T^{*k}y)\Delta]$.

Let us now show that there is a polynomial $B(y)$ such that for any y and any n, $|G_n(y)| \leq B(y)$. There is a number M such that for any y in the unit ball of R^d and for any n, $|G_n(y)| \leq M$. If C is the maximal value of $|P(y)\Delta|$, this parameter C with M can be used in order to bound $G_n(y)$. Let us use again the fact that the spectral radius ρ of T is smaller than one. If r is a real number from the interval $(\rho, 1)$, there is a number K such that $\|T^{*k}\| \leq r^k$ for every integral value k larger than K.

If y is a vector such that $1/r^K < \|y\| \leq 1/r^k$, then $T^{*k}y$ is in the unit ball and in that case, for every n, $|G_n(y)| = |G_{n-k}(T^{*k}y)\prod_{1\leq j\leq k}(P(T^{*j}y)\Delta)| \leq MC^k$. Thus it is possible to find a number E such that $|G_n(y)| \leq M(1+\|y\|)^E$. This last function is bounded by a polynomial.

Therefore we can apply Theorem 6.1 and it follows that the sequence of distributions D_n converges to a distribution D whose Fourier transform is $G(y)$. ∎

REMARK 1. The representation of G as an infinite product gives the functional relation: $G(y) = G(T^*y)P(-T^*y)(|\det T|)$ and shows that $G(0) = 1$.

REMARK 2. If we assume that F has a continuous extension to R^d, if, for x in R^d, $F(x)$ is this extension and if the support of F is bounded, then the sequence of distributions D_n converges to the following distribution: $\lim_{n\to\infty} D_n(\phi) = \int F(x)\phi(x)\,dx = D(\phi)$. That fact can be shown.

For that reason, the distribution D can be viewed as an extension of F to the space, even in the case when the original fundamental function cannot be extended continuously to R^d.

7. **Denseness of G_∞.** In this section, we look for conditions which will ensure that the domain of definition of the extension according to the iterative process is dense in R^d.

THEOREM 7.1. *If G is a subgroup of R^d, if the vector subspace generated by G is all R^d, if T is a linear transformation such that*

a) G is contained in $T(G)$,

b) every eigenvalue of T is smaller than one in modulus, then the subgroup $\cup_{k\geq 0}T^k(G)$ is dense in R^d.

PROOF. We begin with two remarks.

1) Since the vector subspace generated by G is the Euclidean space R^d, there is a number r such that for any point y of R^d, there is a point x of G whose distance to y is smaller than r.

2) Since the spectral radius of T is smaller than one, for any positive real number ε, there is a positive integral number n for which $\|T^n\| < \varepsilon$.

Let ε be a given positive real number. Let us choose an integral value of n for which $\|T^n\| < \varepsilon$. If y is a vector of R^d, there is a vector z of R^d such that $T^n z = y$. According to Remark 1), there is a point x in G such $\|x - z\| < r$. The distance of $T^n x$ to y will not exceed $r\varepsilon$. So any point of $R^d$is arbitrarily close to $G_\infty = \cup_{k\geq 0}T^k(G)$. ∎

8. **Continuous Interpolation.** An interpolation process will be called *continuous* if the fundamental interpolating function is uniformly continuous. In this section we will give a necessary and sufficient condition for continuous interpolation when the subgroup G_∞ is dense in R^d. We will say that a function g defined on G_∞ is an *interpolationfunction* (for an iterative interpolation process arising with a linear transformation T and a weight

function w) if g satisfies the relation (1.1). In a continuous interpolation process, any interpolation function is continuous as it can be seen from Lemma 4.2.

The continuity of the interpolation process will be related to two kinds of modulus of continuity. If h is a positive real number, if F is the fundamental function, if $f(x)$ is a function defined on G_∞, then we set

$$(8.1) \qquad C(h) = \max\{ \sum_{z \in G} |F(Tx - z) - F(Ty - z)| : x \in G, y \in G, |x - y| \leq h\}.$$

$$(8.2) \quad \omega_n(f, h) = \sup\{ |f(T^n x) - f(T^n y)| : x \text{ and } y \text{ belong to } G, |x - y| \leq h\}.$$

We use a given norm in R^d, $x \longrightarrow |x|$.

As usual, we will set $\|T\| = \sup\{ |Tx| : x \in R^d, |x| = 1\}$.

We introduce the set $S = \{x \in G_1 : F(x) \neq 0\}$. R is the radius of the smallest ball centered at the origin covering S.

LEMMA 8.1. *We assume that* $\|T\| < 1$, *h is a real number at least as large as* $R/(1 - \|T\|)$, *a is a point of* $R^d, x \in G$ *and* $|x - a| \leq h$ *and that f is an interpolation function, then* $f(Tx) = \sum_{y \in G, |y - Ta| \leq h} F(Tx - y) f(y)$.

PROOF. We know that $f(Tx) = \sum_{y \in G} F(Tx - y) f(y)$. If x belongs to G and y is a point of G for which $F(Tx - y) \neq 0$, then $y - Ta = T(x - a) - z$ where $z = Tx - y$; z belongs to S. So $|y - Ta| \leq \|T\| h + R \leq h$. ∎

LEMMA 8.2. *If $f(x)$ is an interpolation function, if for any x in G_1,* $\sum_{y \in G} F(x - y) = 1$ *and if* $\|T\| < 1$, $h \geq 2R/(1 - \|T\|)$, *then* $\omega_n(f, h) \leq [C(h)/2]^n \omega_0(f, h)$.

PROOF. We consider the case where $n = 1$. We can find two points x_1 and x_2 of G such that $\omega_1(f, h) = |f(Tx_1) - f(Tx_2)|$ and $|x_1 - x_2| \leq h$. If we set $a = (x_1 + x_2)/2$, then $|x_1 - a| = |x_2 - a| = |x_1 - x_2|/2 \leq h/2$. As told in Lemma 8.1,

$$f(Tx_i) = \sum_{y \in G, |y - Ta| \leq h/2} F(Tx_i - y) f(y), \; i = 1 \text{ and } 2.$$

Let $M = \max\{f(y) : y \in G, |y - Ta| \leq h/2\}$, $m = \min\{f(y) : y \in G, |y - Ta| \leq h/2\}$ and $c = (M + m)/2$, then

$$f(Tx_i) - c = \sum_{y \in G, |y - Ta| \leq h/2} F(Tx_i - y)(f(y) - c), \; i = 1 \text{ and } 2.$$

(Here we used the hypothesis that for any x in G_1, $\sum_{y \in G} F(x - y) = 1$.) It follows that

$$|f(Tx_1) - f(Tx_2)| \leq \sum_{y \in G, |y - Ta| \leq h/2} |F(Tx_1 - y) - F(Tx_2 - y)| \, |f(y) - c|,$$

$$|f(Tx_1) - f(Tx_2)| \leq \sum_{y \in G, |y - Ta| \leq h/2} |F(Tx_1 - y) - F(Tx_2 - y)| (M - m)/2.$$

And so : $\omega_1(f, h) \leq C(h)\omega_0(f, h)/2$. The inequality of the theorem for $n = 1$ has been proved. The result then follows by induction. ∎

LEMMA 8.3. *If the subgroup generated by* $\{x \in G : |x| \leq h\}$ *is exactly* G *and if* h^* *is a positive number, then there is a positive integral value* L *such that for positive integers* n, $\omega_n(f, h^*) \leq L\omega_n(f, h)$ *for any function* f *defined on* G_n. *(*L *is a function of* h *and* h^*, *but may be chosen independently of* f *and* n.*)*

PROOF. If $B = \{x \in G : |x| \leq h\}$ and $B^* = \{x \in G : |x| \leq h^*\}$, any point of B^* is a sum of points of B. If N is sufficiently large, any point of B^* will be the sum of N points of B. If x and y belong to G, $|x - y| \leq h^*$, it is possible to find $N + 1$ points of G, $\{x_k\}_{0 \leq k \leq N}$, such that $x_0 = x, x_N = y$, $|x_k - x_{k-1}| \leq h$ for $k = 1, 2, \ldots, N$.

Since $f(T^n x) - f(T^n y) = \sum_{1 \leq k \leq N} f(T^n x_{k-1}) - f(T^n x_k)$, then $|f(T^n x) - f(T^n y)| \leq N\omega_n(f, h)$. The inequality $\omega_n(f, h^*) \leq N\omega_n(f, h)$ is true. ∎

THEOREM 8.4. *Let us assume that for any* x *in* G_1, $\sum_{y \in G} F(x - y) = 1$, *and that* $\|T\| < 1$. *If there exists a number* h *such that* $h \geq 2R/(1 - \|T\|)$, $C(h) < 2$ *and if the subgroup generated by* $\{x \in G : |x| \leq h\}$ *is exactly* G, *then the interpolation process is continuous.*

PROOF. If ε is a given positive number, we choose an integer n so large that $2\sum_{k>n} \omega_k(F, \|T^{-1}\|R) + \omega_n(F, 4h) < \varepsilon$. This is a consequence of Lemma 8.2 and Lemma 8.3. We choose δ as $2h/\|T^{-n}\|$. Let us prove that if x and y are two points of G_∞ such that $|x - y| < \delta$, then $|F(x) - F(y)| < \varepsilon$.

Let x and y be two points of G_∞ such that $|x - y| < \delta$. We may assume that the points x and y are in G_m for an integer $m \geq n$. We will construct a sequence of points $\{x_k\}_{n \leq k \leq m}$ such that $x_m = x$ and as k decreases from m to $n + 1$, x_{k-1} will belong to G_{k-1} and will be found from x_k. Since x_k belongs to G_k, there is a point z_k of G such that $x_k = T^k z_k$. Since $\sum_{w \in G} F(Tz_k - w) = 1$, there is an w in G such that $Tz_k - w \in S$. We set $z_{k-1} = w$ and $x_{k-1} = T^{k-1} z_{k-1}$; x_{k-1} belongs to G_{k-1}.

By the definition of R, $|Tz_k - z_{k-1}| \leq R$. The definition of $\omega_k(F, h)$ shows that $|F(x_k) - F(x_{k-1})| = |F(T^k z_k) - F(T^k T^{-1} z_{k-1})| \leq \omega_k(F, \|T^{-1}\|R)$.

So $|F(x) - F(x_n)| \leq \sum_{k>n} \omega_k(F, \|T^{-1}\|R)$.

In the same way, we may construct a sequence of points $\{y_k\}_{n \leq k \leq m}$ such that $y_m = y$ and as k decreases from m to $n + 1$, y_{k-1} will belong to G_{k-1} and will be found from y_k. Since y_k belongs to G_k, there is a point w_k of G such that $y_k = T^k w_k$. Since $\sum_{w \in G} F(Tw_k - w) = 1$, there is a w in G such that $Tw_k - w \in S$. We set $w_{k-1} = w$ and $y_{k-1} = T^{k-1} w_{k-1}$; y_{k-1} belongs to G_{k-1}.

So $|F(y) - F(y_n)| \leq \sum_{k>n} \omega_k(F, \|T^{-1}\|R)$.

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We also notice that the distance between z_n and w_n cannot be so large:

$$|z_n - w_n| \leq \sum_{0 \leq k < m-n} |T^k z_{n+k} - T^{k+1} z_{n+k+1}| + |T^{m-n} z_m - T^{m-n} w_m| + \sum_{0 \leq k < m-n} |T^k w_{n+k} - T^{k+1} w_{n+k+1}|$$

$$|z_n - w_n| \leq \sum_{0 \leq k < m-n} \|T^k\| \, |z_{n+k} - T z_{n+k+1}| + |T^{-n} x - T^{-n} y| + \sum_{0 \leq k < m-n} \|T^k\| \, |w_{n+k} - T w_{n+k+1}|$$

$$|z_n - w_n| \leq 2 \sum_{0 \leq k < m-n} \|T^k\| R + |T^{-n} x - T^{-n} y| \leq 2h + \|T^{-n}\| |x - y| \leq 4h.$$

So $|F(x_n) - F(y_n)| \leq \omega_n(F, 4h)$. It follows that

$$|F(x) - F(y)| \leq 2 \sum_{k > n} \omega_k(F, \|T^{-1}\| R) + \omega_n(F, 4h) < \varepsilon.$$

The function F is uniformly continuous. ∎

The previous theorem can be improved by using the following notation. We introduce the sets $S_n = \{x \in G_n : F(x) \neq 0\}$. R_n is the radius of the smallest ball centered at the origin covering S_n. If h is a positive real number, we set

$$C_n(h) = \max\{\sum_{z \in G} |F(T^n x - z) - F(T^n y - z)| : x \in G, y \in G, |x - y| \leq h\}.$$

COROLLARY 8.5. *If for any x in G_1, $\sum_{y \in G} F(x - y) = 1$, if there exists an integral number $n \geq 1$ and a positive real number h such that $\|T^n\| < 1$, $h \geq 2R_n/(1 - \|T^n\|)$, $C_n(h) < 2$ and if the subgroup generated by $\{x \in G : |x| \leq h\}$ is exactly G, then the interpolation process is continuous.*

LEMMA 8.6. *If the interpolation process is continuous and if the spectral radius of T is smaller than one, then for any x in G_1, $\sum_{y \in G} F(x - y) = 1$.*

PROOF. In Formula (4.1), we take x in G_1 and as k increases to ∞:
$1 = \lim_{k \to \infty} F(T^k x) = \lim_{k \to \infty} \sum_{y \in G} F(T^k y) F(x - y) = \sum_{y \in G} F(x - y)$. This is true because $T^k x$ converge to 0, $F(0) = 1$ and the sum over G is in fact a finite sum whose number of terms is independent of k since the support of F is bounded. ∎

THEOREM 8.7. *If the interpolation process is continuous and if the spectral radius of T is smaller than one, then there are an integer n and a positive number h such that $h \geq 2R_n/(1 - \|T^n\|)$, $C_n(h) < 2$ and the subgroup generated by $\{x \in G : |x| \leq h\}$ is exactly G.*

PROOF. Since the spectral radius of T is smaller than one, the support of F is bounded and the sequence R_n is bounded. It is possible to find a number h such that the subgroup generated by $\{x \in G : |x| \leq h\}$ is exactly G and for any $n \geq 1$, $h \geq 2R_n/(1 - \|T^n\|)$.

We set N as the largest cardinality amongst these sets: $\{z \in G : F(a - z) \neq 0\}$ where a is a vector of R^d. Since F is uniformly continuous, there is a number $\delta > 0$ such that when x and y are in G_∞ and when $|x-y| < \delta$, then $|F(x)-F(y)| < 1/N$. If n is an integer for which $\|T^n\|h < \delta$, then $C_n(h) < 2$; since such an integer n exists, the proof is complete. ∎

COMMENT. A necessary and sufficient condition for the continuity of the interpolation process is given by Corollary 8.5 and Theorem 8.7.

9. **Examples of Continuous Interpolation.** We will show in this section that some specific one or two-dimensional interpolation processes are continuous. These examples were already described in Section 2.

9.1. Study of Example 1 from Section 2. The group G is the set of integers Z. The linear transformation T is $Tx = x/2$, the weight function is $w(0) = 1$, $w(\pm 1/2) = 9/16$, $w(\pm 3/2) = -1/16$ and $w(x) = 0$ for any other integral multiple x of $1/2$. We apply Corollary 8.5 with its notations. We take n as 3, the number $R_3 = 21/8$, $\|T^3\| = 1/8$, we choose h as $2R_3/(1-\|T^3\|) = 6$; after computations, it is seen that $C_3(6) = 7/4 < 2$. Those computations can be done by hand. So the fundamental function F is uniformly continuous. This has been proved with other arguments in [5] and in [2]. The functional equation of F is $F(x/2) = [-F(x-3)+9F(x-1)+16F(x)+9F(x+1)-F(x+3)]/16$.

9.2. Study of Example 3 from Section 2. Von Koch's curve may be generated by iterative interpolation if complex-valued functions and weights are allowed.

The group G is again the set of integers Z. The linear transformation T is $Tx = x/4$. Weights are $w(\pm 3/4) = 1/3$, $w(-1/2) = 1/2 + i\sqrt{3}/6$, $w(1/2) = 1/2 - i\sqrt{3}/6$, $w(\pm 1/4) = 2/3$, $w(0) = 1$ and $w(x) = 0$ for other integral multiple of $1/4$. We apply Corollary 8.5 with its notations. The number $R = 3/4$, $\|T\| = 1/4$, we choose h as $2R/(1-\|T\|) = 2$; it is easily seen that $C(2) = 2/\sqrt{3} < 2$. The corresponding iterative interpolation process is continuous. The same argument can be repeated for other self-similar curves.

9.3. Study of Example 5 from Section 2. We consider the following situation: $G = Z \times Z$, $T(x_1,x_2) = ((x_1 - x_2)/2, (x_1 + x_2)/2)$, $w(0,0) = 1$, $w(1/2,1/2) = p_1$, $w(-1/2,1/2) = p_2$, $w(-1/2,-1/2) = p_3$, $w(1/2,-1/2) = p_4$ and $w(x_1,x_2) = 0$ otherwise.

THEOREM 9.1. *If the four parameters p_k are positive and $\sum_k p_k = 1$, then the fundamental interpolating function F is uniformly continuous.*

PROOF. We use the L^∞-norm in R^2 : $|(x_1,x_2)| = \max(|x_1|,|x_2|)$. In order to prove that F is uniformly continuous, the Corollary 8.5 will be applied with $n = 6$ and $h = 4$. Because $\sum_k p_k = 1$, then for every x in G_1, $\sum_{z\in G} F(x-z) = 1$. The subgroup generated by $G \cap \{|x| \le h\}$ is G. We remark that $T^6(x_1,x_2) = (x_2/8, -x_1/8)$. It can be shown that $R_6/(1-\|T^6\|) = R_2/(1-\|T^2\|) = 2$; so $h \ge 2R_6/(1-\|T^6\|)$.

Let us check that $C_6(4) < 2$. $F(x)$ is always between 0 and 1. From the property that for every x in G_1, $\sum_{z\in G} F(x-z) = 1$ and under the Hypotheses 1.1–1.4, it follows that for

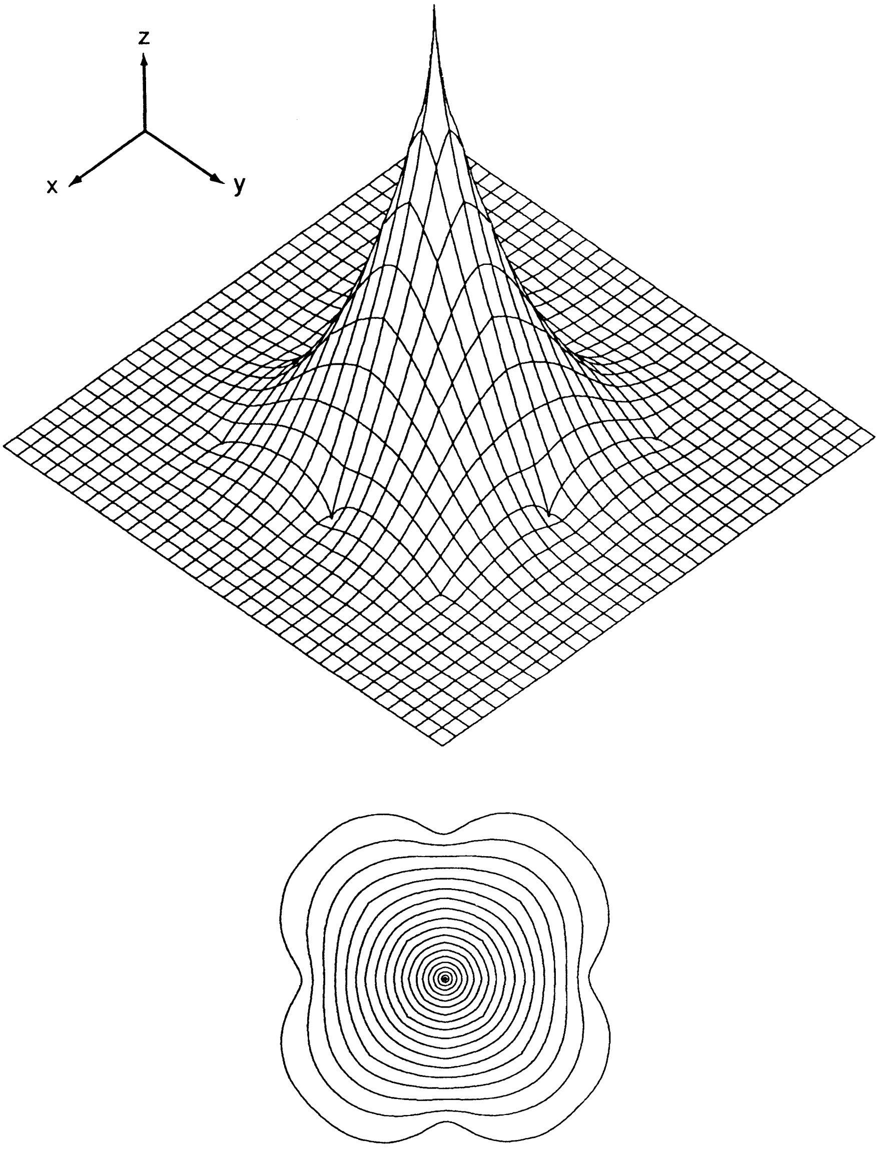


FIGURE 1. Surface and level curves of Example 5 with four equal weights

every x in G_∞, $\sum_{z\in G} F(x-z) = 1$. So for any u and v in G_∞, $\sum_{z\in G} |F(u-z)-F(v-z)| \le 2$. But equality to 2 is impossible when $u = T^6x$, $v = T^6y$, $|x-y| \le 4$. In that case $|u-v| \le 1/2$ and there is a z in G such that $F(u-z) > 0$ and $F(v-z) > 0$ (when $|u-z| \le 1/2$ then $F(u-z) > 0$); for this choice of u, v and z, $|F(u-z)-F(v-z)| < F(u-z)+F(v-z)$.

So $\sum_{z\in G} |F(u-z) - F(v-z)| < \sum_{z\in G} F(u-z) + F(v-z) = 2$.

The hypotheses of the Corollary 8.5 are fulfilled; F is uniformly continuous. ∎

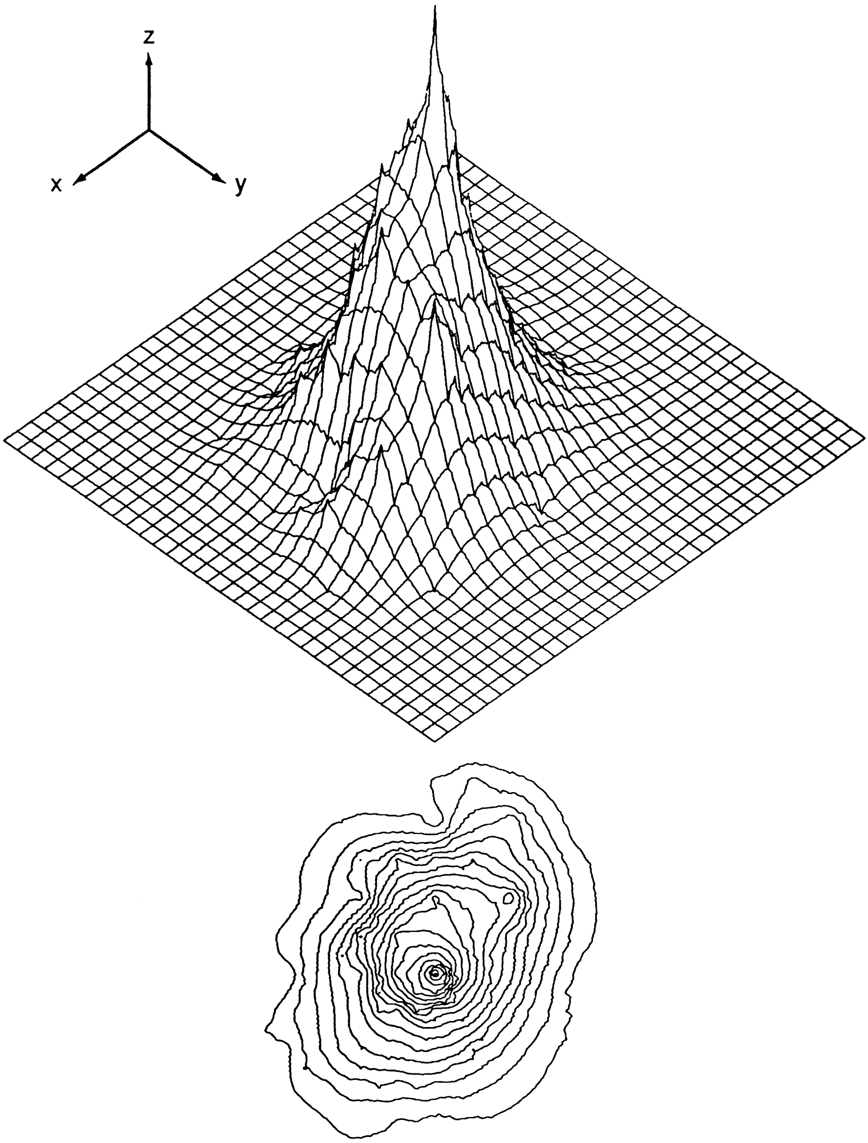
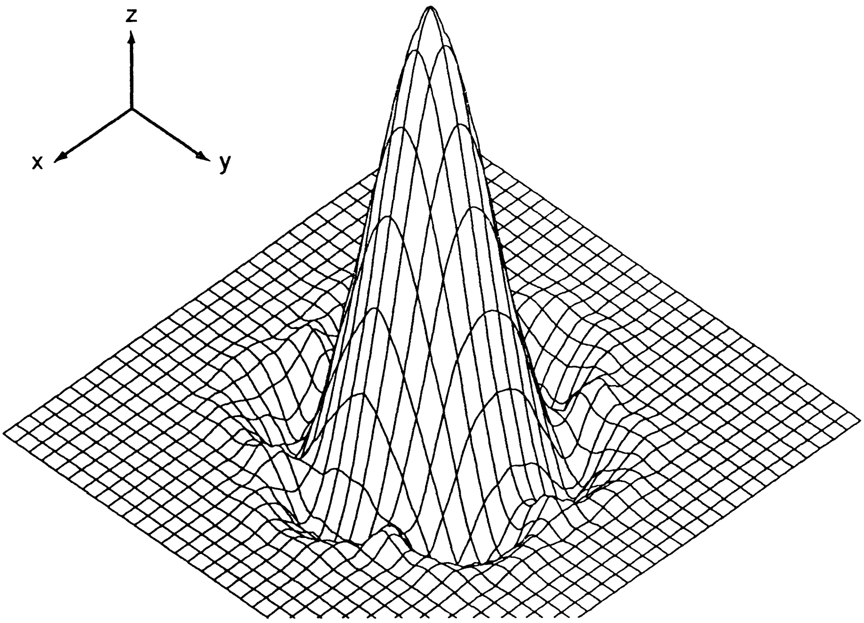


FIGURE 2. Surface and level curves of Example 5 with four unequal weights

9.4. Study of Example 6 from Section 2. G is a triangular mesh: $G = \{m + n/2, n\sqrt{3}/2)\}$; $T(x,y) = ((3x - \sqrt{3}y)/6, (\sqrt{3}x + 3y)/6))$. $G_1 = T(G) = \{(m/2, (m+2n)\sqrt{3}/6) : m, n \in Z\}$. The values of w are $w(x,y) = 4/9$ if (x,y) is one of the six closest neighbours of $(0,0)$ in $T(G)$, i.e. $x^2+y^2 = 1/3$, $w(x,y) = -1/9$ if $x^2+y^2 = 4/3$ and $w(0,0) = 1$ and $w(x,y) = 0$ otherwise in $T(G)$.

We use the usual Euclidean norm in R^2: $|(x,y)| = (x^2+y^2)^{1/2}$. Corollary 8.5 will be



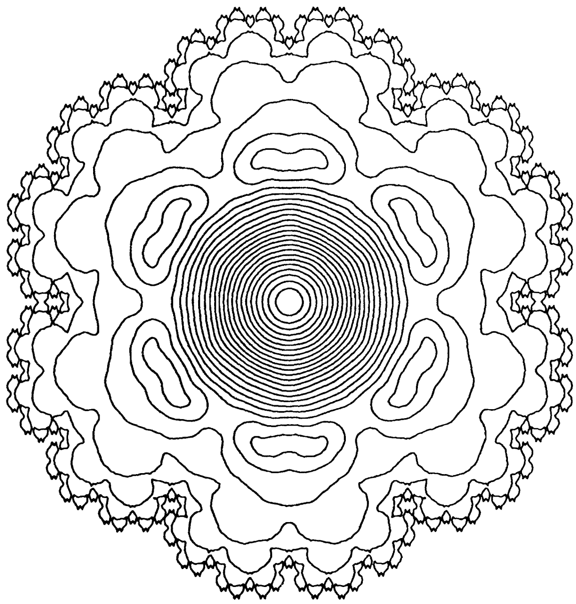


FIGURE 3. Surface and level curves of Example 6

applied with $n = 5$ and $h = 2R_5/(1-\|T^5\|)$. The norm of T^5, $\|T^5\|$, is $3^{-5/2} < 1$. It can be shown that the radius $R_5 = \max\{\,|(x,y)| : (x,y) \in G_5, F(x,y) \neq 0\}$ is $(4476)^{1/2}/27$. As it can be checked with a computer, after thousands and thousands of arithmetical operations, the value of $C_5(h)$ is found to be $1.443\ldots$. So according to Corollary 8.5, this interpolation process is continuous.

We point out one property of this last interpolation process: any quadratic polynomial in two variables is an interpolation function in that interpolation process. In fact, weights have been chosen in order to get that property.

Three surfaces are given by the accompanying figures. The first surface is the graph of the fundamental interpolating function $F(x)$ of Example 5 with $p_1 = p_2 = p_3 = p_4 = 1/4$. Also given are 19 level curves : $F(x) = j/20, 1 \leq j \leq 19$. (see Figure 1).

The second one is the graph of Example 5 with $p_1 = 1/2, p_2 = 1/8$, $p_3 = 1/4$ and $p_4 = 1/8$. The same 19 level curves are also drawn in this case. (see Figure 2).

Finally, Figure 3 gives the graph of the fundamental interpolating function of Example 6 and the 22 level curves $F(x) = j/20, -2 \leq j \leq 19$.

10. **Conclusion.** Iterative interpolation provides a very large class of interpolating functions, which may be more or less regular according to the choice of the linear transformation T and weight function w. We described a simple criterion for continuous interpolation. We know that continuous interpolation functions always belong to a Hölder class, but we do not know the critical Hölder's exponent. The computation of the critical Hölder's exponent or of the dimension (Hausdorff or box counting dimension) of the graph of any interpolating function is an open problem. Maybe Falconer's work like [8] and Kôno's paper [10] could be used for this task.

Smooth functions as well as fractal functions can be generated by the same scheme and new curves or surfaces can be found. Iterative interpolation could also compete with other fast surface interpolation techniques as studied by authors like Smith [14].

By the same time this paper has been accepted, we became aware that the subject of iterative interpolation was already considered by other authors in recent years. The continuity of the interpolating functions in the univariate subdivision schemes has been studied by at least the following authors: Daubechies-Lagarias [1], Dyn-Levin-Gregory [6] and [7], Micchelli-Prautzsch [12] and [13]. Multivariate controlled approximation has been considered by Micchelli with other collaborators, Dahmen, Höllig, Prautzsch, ... There is no doubt that the subject of iterative interpolation is in expansion.

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