ANZIAM J. 55(2014), 245–266
doi:10.1017/S1446181114000030

OPTIMAL L^2 ESTIMATES FOR THE SEMIDISCRETE GALERKIN METHOD APPLIED TO PARABOLIC INTEGRO-DIFFERENTIAL EQUATIONS WITH NONSMOOTH DATA

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(Received 23 April, 2012; revised 18 September, 2013; first published online 5 June 2014)

Abstract

We propose and analyse an alternate approach to a priori error estimates for the semidiscrete Galerkin approximation to a time-dependent parabolic integro-differential equation with nonsmooth initial data. The method is based on energy arguments combined with repeated use of time integration, but without using parabolic-type duality techniques. An optimal L^2-error estimate is derived for the semidiscrete approximation when the initial data is in L^2. A superconvergence result is obtained and then used to prove a maximum norm estimate for parabolic integro-differential equations defined on a two-dimensional bounded domain.

2010 *Mathematics subject classification*: primary 35K20; secondary 65M15, 65M60.

Keywords and phrases: parabolic integro-differential equation, finite element method, semidiscrete solution, energy argument, optimal error estimate, nonsmooth initial data, superconvergence, maximum norm estimate.

1. Introduction

In this paper, we discuss an alternate approach to a priori L^2-error estimates for a semidiscrete finite element Galerkin approximation to the following parabolic integro-differential equation (PIDE):

$$\begin{cases} u_t + A(t)u = \int_0^t B(t,s)u(s)\,ds & \text{in } \Omega \times J, \\ u = 0 & \text{on } \partial\Omega \times J, \\ u(\cdot,0) = u_0 & \text{in } \Omega, \end{cases} \tag{1.1}$$

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with $u_0 \in L^2(\Omega)$, where $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$ is a bounded convex polygon or polyhedron, $J = (0, T]$, $0 < T < \infty$. Here $u = u(x, t)$ is a real-valued function in $\Omega \times J$ and $u_t = \partial u/\partial t$. Further, $A(t)$ is a second-order self-adjoint, uniformly positive-definite elliptic operator of the form

$$A(t) = -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_j}\Big(a_{ij}(x, t)\frac{\partial}{\partial x_i}\Big) + a_0(x, t)I,$$

and $B(t, s)$ is a general second-order elliptic differential operator,

$$B(t, s) = -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_j}\Big(b_{ij}(x; t, s)\frac{\partial}{\partial x_i}\Big) + \sum_{j=1}^{d} b_j(x; t, s)\frac{\partial}{\partial x_j} + b_0(x; t, s)I.$$

Equations of the type described above arise naturally in nonlocal flows in porous media [4, 5] and heat conduction through materials with memory [19].

We use the usual notations for L^2, H_0^1 and H^2 spaces and their norms. Let $\mathcal{A}(t\,;\,\cdot,\cdot)$ and $\mathcal{B}(t, s\,;\,\cdot,\cdot)$ be bilinear forms on $H_0^1 \times H_0^1$ corresponding to the operators $A(t)$ and $B(t, s)$, respectively. That is,

$$\mathcal{A}(t; \phi, \psi) = \int_\Omega \Big(\sum_{i,j=1}^{d} a_{ij}(x, t)\frac{\partial\phi}{\partial x_i}\frac{\partial\psi}{\partial x_j} + a_0(x, t)\phi\psi\Big)\,dx$$

and

$$\mathcal{B}(t, s; \phi(s), \psi) = \int_\Omega \Big(\sum_{i,j=1}^{d} b_{ij}(x; t, s)\frac{\partial\phi(s)}{\partial x_i}\frac{\partial\psi}{\partial x_j} + \sum_{j=1}^{d} b_j(x; t, s)\frac{\partial\phi(s)}{\partial x_j}\psi + b_0(x; t, s)\phi(s)\psi\Big)\,dx.$$

The weak formulation for (1.1) may be stated as follows: find $u : J \to H_0^1$ such that

$$\begin{cases} (u_t, \phi) + \mathcal{A}(t; u, \phi) = \int_0^t \mathcal{B}(t, s; u(s), \phi)\,ds, & \phi \in H_0^1, t \in J \\ u(0) = u_0. \end{cases} \tag{1.2}$$

Now we define a semidiscrete Galerkin approximation of u. Let h with $0 < h < 1$ be the discretizing parameter of a regular triangulation of Ω. Let S_h be the corresponding finite-dimensional subspace of H_0^1 such that the following approximation properties hold for all $v \in H_0^1 \cap H^2$, $k \in \{1, 2\}$:

$$\inf_{\phi_h \in S_h} \|v - \phi_h\|_j \le \rho_0 h^{k-j}\|v\|_k, \quad j \in \{0, 1\}, \tag{1.3}$$

where ρ_0 is independent of h.

The semidiscrete Galerkin approximation to a solution u of (1.1) is to find $u_h(t) \in S_h$ for $t \in J$ satisfying

$$(u_{ht}, \phi_h) + \mathcal{A}(t; u_h, \phi_h) = \int_0^t \mathcal{B}(t, s; u_h(s), \phi_h)\,ds, \quad \phi_h \in S_h, t > 0, \tag{1.4}$$

with $u_h(0) = P_h u_0$, where $P_h u_0$ is an L^2-projection of u_0 onto S_h.

https://doi.org/10.1017/S1446181114000030 Published online by Cambridge University Press

Below we present our main result on L^2-error estimates of $e = u - u_h$, when the initial data $u_0 \in L^2$.

THEOREM 1.1. *Let u and u_h be the solutions of (1.2) and (1.4), respectively, with $u(0) = u_0$ and $u_h(0) = P_h u_0$. Then there exists a positive constant C independent of h such that the following estimate holds for $t > 0$:*

$$\|u(t) - u_h(t)\| \le Ch^2 t^{-1} \|u_0\|.$$

Yanik and Fairweather [23] have derived optimal error estimates for smooth solutions to a class of nonlinear problems with only the first-order partial differential operator B. Cannon and Lin [2, 3], Lin et al. [10], Lin and Zhang [11] and Pani et al. [18] have proved a priori error estimates for PIDEs for smooth initial data using Ritz–Volterra projection, in place of elliptic projection, which is normally used for the derivation of optimal error estimates for Galerkin approximations to parabolic-type equations. Thomée and Zhang [21] have obtained optimal L^2-error estimates for smooth and nonsmooth initial data using a semigroup theoretic approach combined with a use of the inverse of an associated elliptic operator, when A is independent of time. Subsequently, based on an energy argument and parabolic-type duality, Pani and Sinha [16] have proved an optimal L^2-estimate for the semidiscrete Galerkin approximation to a more general time-dependent PIDE with nonsmooth initial data. Pani and Peterson [13] and Pani and Sinha [17] have discussed the effect of quadrature for nonsmooth initial data using a combination of integration in time and a use of the inverse of an associated elliptic operator. For a completely discrete scheme based on the backward Euler method, optimal error estimates are derived by Pani and Sinha [15] and Thomée and Zhang [22].

In order to continue our investigation on an alternate approach, which started with optimal L^2-estimates for semidiscrete Galerkin approximations to parabolic problems with nonsmooth data [6], in this paper we extend this approach to prove Theorem 1.1 for PIDEs (1.4) when initial data $u_0 \in L^2$. Again our approach is based on an energy argument combined with repeated use of a time integral operator

$$\hat{\phi}(t) = \int_0^t \phi(s)\, ds, \tag{1.5}$$

instead of using the inverse of an associated discrete elliptic operator along with a semigroup theoretic approach as per Thomée and Zhang [21] or using energy arguments with parabolic-type duality techniques as per Pani and Sinha [16]. Essentially, our proof technique depends mainly on an energy argument which follows the standard pattern of error analysis related to PIDEs with smooth data. Therefore, as per Goswami and Pani [6], we believe that our approach unifies both these theories, one for smooth data and the other for nonsmooth data, under one umbrella. While the technique of using integration in time for nonsmooth data is not new, it has not been used to its full potential (see Goswami and Pani [6] for some comments on related papers [8, 14, 16]). Moreover, our superconvergence result, Theorem 4.4, is new in the context of PIDEs with nonsmooth data and, as a consequence, a maximum norm estimate is derived (Corollary 4.5). Further, superconvergence analysis can be used for better recovery of the gradient of the solution under a uniform mesh. Compared to the work of Goswami and Pani [6], the analysis of the present article becomes quite involved due to the presence of the integral term and the repeated use of the time integral operator under the integral term. For example, we need a careful analysis of the Ritz–Volterra projection and the related estimates using the time integral operator (1.5), especially for the time integral term in (1.1). The essential idea is to bring out the interaction of the time integral operator and the integral term and to use it judiciously to the advantage of optimal error estimates for the present problem with nonsmooth initial data. The present article is a refined version of our Oxford Center for Collaborative Applied Mathematics preprint [7].

Section 2 deals with some a priori estimates and regularity results for the exact solution. The Ritz–Volterra projection is introduced in Section 3 and related estimates are carried out. Section 4 focuses on optimal L^2-error estimates, when nonsmooth initial data $u_0 \in L^2(\Omega)$, and concludes with a superconvergence result which is then used to derive the maximum norm estimate for PIDEs (1.4) defined on a two-dimensional spatial domain.

Throughout this article, we denote by C a generic positive constant, which may vary from context to context.

2. A priori estimates

In this section, we derive some a priori bounds which are needed in our subsequent error analysis.

For our future use, we assume that the principal part of $A(t)$ is uniformly elliptic and the coefficient $a_0 \geq 0$. Further, we assume that all the coefficients of $A(t)$ and $B(t,s)$ are smooth and that their derivatives are bounded in their domains of definitions. Based on the assumptions on the coefficients, it is straightforward to show that the bilinear form $\mathcal{A}(t;\cdot,\cdot)$ is coercive, that is, there is a positive constant ρ_1 independent of t such that

$$\mathcal{A}(t;\phi,\phi) \geq \rho_1 \|\phi\|_1^2, \quad \phi \in H_0^1. \tag{2.1}$$

Also, the domain being a convex polygon or polyhedron, there is a positive constant ρ_2 independent of t such that

$$\|\phi\|_2 \leq \rho_2 \|A(t)\phi\|, \quad \phi \in H_0^1 \cap H^2. \tag{2.2}$$

Finally, there are positive constants ρ_3 and ρ_4 independent of t such that

$$|\mathcal{A}(t;\phi,\psi)| \leq \rho_3 \|\phi\|_1 \|\psi\|_1, \quad \phi,\psi \in H_0^1,$$
$$|\mathcal{B}(t,s;\phi(s),\psi)| \leq \rho_4 \|\phi(s)\|_1 \|\psi\|_1, \quad \phi(s).\psi \in H_0^1.$$

We now define the bilinear form $\mathcal{A}_t(t;\cdot,\cdot) : H_0^1 \times H_0^1 \to \mathbb{R}$ by

$$\mathcal{A}_t(t;\phi,\psi) = \int_\Omega \Big(\sum_{i,j=1}^{d} \frac{\partial}{\partial t}(a_{ij}(x,t)) \frac{\partial \phi}{\partial x_i} \frac{\partial \psi}{\partial x_j} + \frac{\partial}{\partial t}(a_0(x,t))\phi\psi \Big)\, dx, \quad \phi,\psi \in H_0^1.$$

https://doi.org/10.1017/S1446181114000030 Published online by Cambridge University Press

As the coefficients and their derivatives are bounded both in time and space, we conclude that there exists a positive constant ρ_5 independent of t such that

$$|\mathcal{A}_t(t;\phi,\psi)| \le \rho_5\|\phi\|_1\|\psi\|_1, \quad \phi,\psi \in H_0^1.$$

We present below some a priori estimates and regularity results for the solution of (1.1), when $u_0 \in L^2$. For a proof, we refer the reader to Pani and Sinha [16].

LEMMA 2.1. *Let u be a solution of the PIDE (1.1) and $u_0 \in L^2$. Then the following estimates hold for $t \in J$:*

$$t\|u(t)\|_1^2 + \int_0^t s\|u_s(s)\|^2\,ds \le C\|u_0\|^2, \quad t^2\|u_t(t)\|^2 + \int_0^t s^2\|u_s(s)\|_1^2\,ds \le C\|u_0\|^2,$$

$$\|\hat{u}(t)\|_2 \le C\|u_0\|, \quad t\|u(t)\|_2 \le C\|u_0\|.$$

Next we discuss the estimates for $\|u_t\|_1$ and $\|u_t\|_2$, again when $u_0 \in L^2$.

LEMMA 2.2. *Let u be a solution of the PIDE (1.1) and $u_0 \in L^2$. Then the following estimate holds for $k \in \{1,2\}$ and $t \in J$:*

$$\|u_t\|_k \le Ct^{-(1+k/2)}\|u_0\|.$$

PROOF. Differentiate (1.1) with respect to time to obtain

$$u_{tt} + A(t)u_t + A_t(t)u = B(t,t)u(t) + \int_0^t B_t(t,s)u(s)\,ds. \tag{2.3}$$

Multiply (2.3) by $t^3A(t)u_t$, integrate over Ω and rewrite the resulting equation as

$$\begin{aligned}(u_{tt}, t^3A(t)u_t) + (A(t)u_t, t^3A(t)u_t) = -(A_t(t)u, t^3A(t)u_t) + (B(t,t)u(t), t^3A(t)u_t)\\ + \int_0^t (B_t(t,s)u(s), t^3A(t)u_t)\,ds.\end{aligned} \tag{2.4}$$

Observe that

$$\begin{aligned}\frac{d}{dt}(u_t, t^3A(t)u_t) &= \frac{d}{dt}t^3\mathcal{A}(t;u_t,u_t)\\ &= 3t^2\mathcal{A}(t;u_t,u_t) + t^3\mathcal{A}_t(t;u_t,u_t) + 2t^3\mathcal{A}(t;u_{tt},u_t).\end{aligned} \tag{2.5}$$

Using integration by parts, rewrite the last term of (2.4) as

$$\begin{aligned}\int_0^t (B_t(t,s)u(s), t^3A(t)u_t)\,ds = t^3(B_t(t,t)\hat{u}(t), A(t)u_t)\\ - t^3\int_0^t (B_{ts}(t,s)\hat{u}(s), A(t)u_t)\,ds.\end{aligned} \tag{2.6}$$

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On substituting (2.5) and (2.6) in (2.4), we arrive at

$$\frac{1}{2}\frac{d}{dt}(t^3\mathcal{A}(t;u_t,u_t)) + t^3\|A(t)u_t\|^2$$
$$= \frac{3}{2}t^2\mathcal{A}(t;u_t,u_t) + \frac{1}{2}t^3\mathcal{A}_t(t;u_t,u_t) - t^3(A_t(t)u, A(t)u_t) + t^3(B(t,t)u(t), A(t)u_t)$$
$$+ t^3(B_t(t,t)\hat{u}(t), A(t)u_t) + t^3\int_0^t (B_{ts}(t,s)\hat{u}(s), A(t)u_t)\,ds.$$

Integrate the above equation with respect to time from 0 to t, and use (2.1) and (2.2). The smoothness of the coefficients of $\mathcal{A}(t)$ and $B(t,s)$ along with Young's inequality yields

$$t^3\|u_t\|_1^2 + \int_0^t s^3\|u_s(s)\|_2^2\,ds \le C\int_0^t s^2(\|u_s(s)\|_1^2 + \|u(s)\|_2^2)\,ds + C\int_0^t s^3\|\hat{u}(s)\|_2^2\,ds.$$

After applying Lemma 2.1 we obtain

$$t^3\|u_t\|_1^2 + \int_0^t s^3\|u_s(s)\|_2^2\,ds \le C\|u_0\|^2.$$

Now multiply (2.3) by t^3u_{tt} and integrate over Ω to obtain

$$t^3\|u_{tt}\|^2 + \mathcal{A}(t;u_t,t^3u_{tt}) + \mathcal{A}_t(t;u,t^3u_{tt}) = \mathcal{B}(t,t;u(t),t^3u_{tt}) + \int_0^t \mathcal{B}_t(t,s;u(s),t^3u_{tt})\,ds.$$

Note that

$$\frac{d}{dt}(t^3\mathcal{A}(t;u_t,u_t)) = 3t^2\mathcal{A}(t;u_t,u_t) + t^3\mathcal{A}_t(t;u_t,u_t) + 2t^3\mathcal{A}(t;u_t,u_{tt})$$

and

$$\int_0^t \mathcal{B}_t(t,s;u(s),t^3u_{tt})\,ds = t^3\mathcal{B}_t(t,t,\hat{u}(t),u_{tt}) - t^3\int_0^t \mathcal{B}_{ts}(t,s;\hat{u}(s),u_{tt})\,ds.$$

Hence,

$$t^3\|u_{tt}\|^2 + \frac{1}{2}\frac{d}{d}(t^3\mathcal{A}(t;u_t,u_t)) = \frac{3}{2}t^2\mathcal{A}(t;u_t,u_t) + \frac{1}{2}t^3\mathcal{A}_t(t;u_t,u_t) - t^3\mathcal{A}_t(t;u,u_{tt})$$
$$+ t^3\mathcal{B}(t,t;u,u_{tt}) + t^3\mathcal{B}_t(t,t,\hat{u}(t),u_{tt})$$
$$- t^3\int_0^t \mathcal{B}_{ts}(t,s;\hat{u}(s),u_{tt})\,ds.$$

Use the Cauchy–Schwarz inequality along with Young's inequality and then integrate with respect to time from 0 to t to obtain

$$t^3\mathcal{A}(t;u_t,u_t) + \int_0^t s^3\|u_{ss}(s)\|^2\,ds \le C\int_0^t \{s^2(\|u_s(s)\|_1^2 + \|u(s)\|_2^2) + \|\hat{u}(s)\|_2^2\}\,ds.$$

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Using (2.1) and Lemma 2.1, we find

$$t^3\|u_t\|_1^2 + \int_0^t s^3\|u_{ss}(s)\|^2\,ds \le C\|u_0\|^2. \tag{2.7}$$

Now differentiate (2.3) with respect to time to obtain

$$u_{ttt} + A(t)u_{tt} + 2A_t(t)u_t - A_{tt}u = B(t,t)u_t + 2B(t,t)u + \int_0^t B_{tt}(t,s)u(s)\,ds. \tag{2.8}$$

Multiply (2.8) by t^4u_{tt} and integrate over Ω to rewrite it as

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(t^4\|u_{tt}\|^2) + t^4\mathcal{A}(t;u_{tt},u_{tt}) &= 2t^3\|u_{tt}\|^2 - 2t^4\mathcal{A}_t(t;u_t,u_{tt}) - t^4\mathcal{A}_{tt}(t;u,u_{tt})\\ &\quad + t^4\mathcal{B}(t,t;u,u_{tt}) + 2t^4\mathcal{B}_t(t,t;u_t,u_{tt})\\ &\quad + t^4\int_0^t \mathcal{B}_{tt}(t,s;u(s),u_{tt})\,ds\\ &\le 2t^3\|u_{tt}\|^2 + \varepsilon\|u_{tt}\|_1^2\\ &\quad + C(\varepsilon)t^4\Big(\|u_t\|_1^2 + \|u\|_1^2 + \int_0 4\|u(s)\|_1^2\,ds\Big).\end{aligned}$$

Use (2.1) and choose $\varepsilon = \rho_1/2$. Finally, integrate and use (2.7) with Lemma 2.1 to conclude that

$$t^4\|u_{tt}\|^2 + \int_0^t s^4\|u_{ss}(s)\|_1^2\,ds \le C\|u_0\|^2. \tag{2.9}$$

Rewrite (2.3) as

$$\begin{aligned}A(t)u_t &= u_{tt} - A_t(t)u - B(t,t)u - \int_0^t B_t(t,s)u(s)\,ds\\ &= u_{tt} - A_t(t)u - B(t,t)u - B_t(t,t)\hat{u}(t) + \int_0^t B_t(t,s)\hat{u}(s)\,ds.\end{aligned}$$

Using elliptic regularity (2.2), we arrive at

$$\|u_t\|_2^2 \le C\Big(\|u_{tt}\|^2 + \|u\|_2^2 + \|\hat{u}\|_2^2 + \int_0^t \|\hat{u}(s)\|_2^2\,ds\Big).$$

Multiply by t^4, and use (2.9) and Lemma 2.1 to obtain

$$\|u_t\|_2 \le Ct^{-2}\|u_0\|,$$

completing the proof. □

3. The Ritz–Volterra projection

In this section we discuss the Ritz–Volterra projection and the related error estimates which are useful for the proof of our main theorem.

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Following Lin et al. [2, 3, 10], define the Ritz–Volterra projection $W_h : (0, T] \to S_h$ satisfying

$$\mathcal{A}(t; (u - W_h u)(t), \phi_h) = \int_0^t \mathcal{B}(t, s; (u - W_h u)(s), \phi_h)\, ds \quad \text{for all } \phi_h \in S_h. \tag{3.1}$$

We refer to Cannon and Lin [2] and Lin et al. [10] to see that the Ritz–Volterra projection is well defined. We also use the Ritz projection $R_h = R_h(t) : H_0^1 \to S_h$ defined by

$$\mathcal{A}(t; u - R_h u, \phi_h) = 0, \quad \text{for all } \phi_h \in S_h, u \in H_0^1. \tag{3.2}$$

With $\theta = u - R_h u$, we discuss below some estimates for θ. For a proof, we refer the reader to Luskin and Rannacher [12].

Lemma 3.1. *For θ as defined above and $u \in H_0^1 \cap H^2$ with $u_0 \in L^2$, there is a positive constant C independent of h such that the following estimates hold for $k \in \{1, 2\}$, $j \in \{0, 1\}$ and for $t > 0$:*

$$\|\theta(t)\|_j \le Ch^{k-j}\|u(t)\|_k \le Ch^{k-j}t^{-k/2}\|u_0\|,$$
$$\|\theta_t(t)\|_j \le Ch^{k-j}\{\|u(t)\|_k + \|u_t(t)\|_k\} \le Ch^{k-j}t^{-(1+k/2)}\|u_0\|.$$

Next, we present an estimate of $\hat{\theta} := \int_0^t \theta(s)\, ds$. For a proof, we refer the reader to Goswami and Pani [6, Lemma 3.2].

Lemma 3.2. *For θ as defined above and $u \in H_0^1 \cap H^2$ with $u_0 \in L^2$, there exists a positive constant C independent of h such that, for $k \in \{1, 2\}$ and $j \in \{0, 1\}$,*

$$\|\hat{\theta}\|_j \le Ch^{k-j}\|u_0\|.$$

In the rest of this section, we prove estimates of $\eta = u - W_h u$. Using the Ritz projection, we set $\eta = \theta - \rho$, where $\theta = u - R_h u$ and $\rho = W_h u - R_h u$. Hence, we now rewrite (3.1) using (3.2) as

$$\mathcal{A}(t; \rho, \phi_h) = \int_0^t \mathcal{B}(t, s; \rho(s), \phi_h)\, ds - \int_0^t \mathcal{B}(t, s; \theta(s), \phi_h)\, ds \quad \text{for all } \phi_h \in S_h. \tag{3.3}$$

Using integration by parts in time we again rewrite (3.3) as

$$\begin{aligned}\mathcal{A}(t; \rho, \phi_h) &= \mathcal{B}(t, t; \hat{\rho}(t), \phi_h) - \int_0^t \mathcal{B}_s(t, s; \hat{\rho}(s), \phi_h)\, ds \\ &\quad - \mathcal{B}(t, t; \hat{\theta}(t), \phi_h) + \int_0^t \mathcal{B}_s(t, s; \hat{\theta}(s), \phi_h)\, ds \quad \text{for all } \phi_h \in S_h.\end{aligned} \tag{3.4}$$

Below, we discuss estimates of η and $\hat{\eta}$.

Lemma 3.3. *For η as defined above and $u(t) \in H_0^1 \cap H^2$, $t > 0$, with $u_0 \in L^2$, there exists a positive constant C independent of h such that the following estimates hold for $k \in \{1, 2\}$, $j \in \{0, 1\}$, and $t > 0$:*

$$\|\eta(t)\|_j \le Ch^{k-j}t^{-k/2}\|u_0\|, \quad \|\hat{\eta}(t)\|_j \le Ch^{k-j}\|u_0\|. \tag{3.5}$$

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PROOF. Set $\phi_h = \rho$ in (3.4) to obtain

$$\begin{aligned}\mathcal{A}&(t;\rho,\rho)\\ &= \mathcal{B}(t,t;\hat\rho,\rho) - \int_0^t \mathcal{B}_s(t,s;\hat\rho(s),\rho)\,ds - \mathcal{B}(t,t;\hat\theta,\rho) + \int_0^t \mathcal{B}_s(t,s;\hat\theta(s),\rho)\,ds.\end{aligned}$$

Using the coercivity of $\mathcal{A}$ and the Cauchy–Schwarz inequality yields

$$\|\rho\|_1 \le C\Big(\|\hat\rho\|_1 + \int_0^t \|\hat\rho(s)\|_1\,ds + \|\hat\theta\|_1 + \int_0^t \|\hat\theta(s)\|_1\,ds\Big). \tag{3.6}$$

To find $\|\hat\rho\|_1$, we integrate (3.4) and obtain

$$\begin{aligned}\mathcal{A}(t;\hat\rho,\phi_h) &- \int_0^t \mathcal{A}_s(s;\hat\rho(s),\phi_h)\,ds\\ &= \int_0^t \mathcal{B}(s,s;\hat\rho(s),\phi_h)\,ds - \int_0^t\int_0^s \mathcal{B}_\tau(s,\tau;\hat\rho(\tau),\phi_h)\,d\tau\,ds\\ &\quad - \int_0^t \mathcal{B}(s,s;\hat\theta(s),\phi_h)\,ds + \int_0^t\int_0^s \mathcal{B}_\tau(s,\tau;\hat\theta(\tau),\phi_h)\,d\tau\,ds.\end{aligned}$$

Choose $\phi_h = \hat\rho$ and apply the Cauchy–Schwarz inequality and then the coercivity of $\mathcal{A}$ to obtain

$$\|\hat\rho\|_1 \le C\Big(\int_0^t \|\hat\theta(s)\|_1\,ds + \int_0^t \|\hat\rho(s)\|_1\,ds\Big).$$

Now an application of Lemma 3.2 yields

$$\|\hat\rho\|_1 \le Ch^{k-1}\|u_0\| + C\int_0^t \|\hat\rho(s)\|_1\,ds.$$

Apply Gronwall's Lemma to arrive at

$$\|\hat\rho\|_1 \le Ch^{k-1}\|u_0\|. \tag{3.7}$$

Using Lemma 3.2 and the triangle inequality, we obtain

$$\|\hat\eta\|_1 \le \|\hat\theta\|_1 + \|\hat\rho\|_1 \le Ch^{k-1}\|u_0\|. \tag{3.8}$$

Now substitute the estimate of $\|\hat\rho\|_1$ from (3.7) in (3.6) and use Lemma 3.2 to obtain

$$\|\rho\|_1 \le Ch^{k-1}\|u_0\|.$$

Again use the triangle inequality and Lemma 3.2 to obtain

$$\|\eta\|_1 \le \|\theta\|_1 + \|\rho\|_1 \le Ch^{k-1}t^{-k/2}\|u_0\|. \tag{3.9}$$

To estimate $\hat\eta$ in the L^2 norm, we now appeal to the Aubin–Nitsche duality arguments and hence consider the following auxiliary problem:

$$A(t)\phi = \hat\eta \quad \text{in } \Omega, \tag{3.10}$$

$$\phi = 0 \quad \text{on } \partial\Omega, \tag{3.11}$$

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where $\phi \in H^2 \cap H_0^1$ satisfies the regularity condition

$$\|\phi\|_2 \leq C\|\hat{\eta}\|.$$

Note that

$$\|\hat{\eta}\|^2 = \mathcal{A}(t;\phi,\hat{\eta}) = \mathcal{A}(t;\hat{\eta},\phi-\chi) + \mathcal{A}(t;\hat{\eta},\chi) \quad \text{for some } \chi \in S_h. \tag{3.12}$$

On integrating (3.1) and using the fact that $d(\hat{\eta}(t))/dt = \eta(t)$, we obtain

$$\mathcal{A}(t;\hat{\eta},\phi_h) - \int_0^t \mathcal{A}_s(s;\hat{\eta}(s),\phi_h)\,ds$$
$$= \int_0^t \mathcal{B}(s,s;\hat{\eta}(s),\phi_h)\,d - \int_0^t \int_0^s \mathcal{B}_\tau(s,\tau;\hat{\eta}(\tau),\phi_h)\,d\tau\,ds. \tag{3.13}$$

On substituting (3.13) with $\phi_h = \chi$ in (3.12) we obtain

$$\|\hat{\eta}\|^2 = \mathcal{A}(t;\hat{\eta},\phi-\chi) + \int_0^t \mathcal{A}_s(s;\hat{\eta}(s),\chi)\,ds + \int_0^t \mathcal{B}(s,s;\hat{\eta}(s),\chi)\,ds$$
$$- \int_0^t \int_0^s \mathcal{B}_\tau(s,\tau;\hat{\eta}(\tau),\chi)\,d\tau\,ds$$
$$= \mathcal{A}(t;\hat{\eta},\phi-\chi) - \int_0^t \mathcal{A}_s(s;\hat{\eta}(s),\phi-\chi)\,ds - \int_0^t \mathcal{B}(s,s;\hat{\eta}(s),\phi-\chi)\,ds$$
$$+ \int_0^t \int_0^s \mathcal{B}_\tau(s,\tau;\hat{\eta}(\tau),\phi-\chi)\,d\tau\,ds + \int_0^t (\hat{\eta}(s),A_s^*(s)\phi)\,ds$$
$$+ \int_0^t (\hat{\eta}(s),B^*(s,s)\phi)\,ds - \int_0^t \int_0^s (\hat{\eta}(\tau),B_\tau^*(s,\tau)\phi)\,d\tau\,ds$$
$$\leq C\Big(\|\hat{\eta}\|_1 + \int_0^t \|\hat{\eta}(s)\|_1\,ds\Big)\|\phi-\chi\|_1 + C\Big(\int_0^t \|\hat{\eta}(s)\|\,ds\Big)\|\phi\|_2.$$

Here A_t^*, $B^*(t,t)$ and $B_s^*(t,s)$ are the formal adjoints of A_t, $B(t,t)$ and $B_s(t,s)$, respectively. Using the approximation property (1.3) for S_h,(3.8) and (3.9), we obtain

$$\|\hat{\eta}\| \leq Ch^2\|u_0\| + \int_0^t \|\hat{\eta}(s)\|\,ds.$$

Now use Gronwall's Lemma to obtain the desired estimate for $\|\hat{\eta}\|$.

To find the estimate of $\|\eta\|$, we again consider the auxiliary problem (3.10) and (3.11) by replacing the function $\hat{\eta}$ by η. Then we proceed similarly as for the estimate of $\|\hat{\eta}\|$, and obtain

$$\|\eta\| \leq Ch\Big(\|\eta\|_1 + \|\hat{\eta}\|_1 + \int_0^t \|\hat{\eta}(s)\|_1\,ds\Big) + C\Big(\|\hat{\eta}\| + + \int_0^t \|\hat{\eta}(s)\|\,ds\Big).$$

Using (3.5) with (3.9) yields the desired estimate of $\|\eta\|$ and completes the proof. □

In the following lemma we discuss the estimate of $\|\eta_t\|$.

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LEMMA 3.4. *For η as defined by (3.1) and $u_0 \in L^2$, let both $u(t)$ and $u_t(t)$ be in $H_0^1 \cap H^2$ for $t \in J$. Then there is a positive constant C independent of h such that*

$$\|\eta_t\| \le Ch^2 t^{-2} \|u_0\| \quad \textit{for } t > 0.$$

PROOF. To find the estimate of $\|\eta_t\|$, we first obtain an estimate for $\|\rho_t\|_1$. Differentiate (3.3) with respect to time to arrive at

$$\mathcal{A}(t; \rho_t, \phi_h) + \mathcal{A}_t(t; \rho, \phi_h) = \mathcal{B}(t, t; \rho, \phi_h) + \int_0^t \mathcal{B}_t(t, s; \rho(s), \phi_h)\, ds \\ - \mathcal{B}(t, t; \theta, \phi_h) - \int_0^t \mathcal{B}_t(t, s; \theta(s), \phi_h)\, ds. \tag{3.14}$$

Using integration by parts in time, rewrite (3.14) as

$$\begin{aligned}\mathcal{A}(t; \rho_t, \phi_h) &+ \mathcal{A}_t(t; \rho, \phi_h) \\ &= \mathcal{B}(t, t; \rho, \phi_h) + \mathcal{B}_t(t, t; \hat{\rho}, \phi_h) - \int_0^t \mathcal{B}_{ts}(t, s; \hat{\rho}(s), \phi_h)\, ds \\ &\quad - \mathcal{B}(t, t; \theta, \phi_h) - \mathcal{B}_t(t, t; \hat{\theta}(s), \phi_h) + \int_0^t \mathcal{B}_{ts}(t, s; \hat{\theta}(s), \phi_h)\, ds.\end{aligned} \tag{3.15}$$

Choose $\phi_h = \rho_t$ in (3.15) to obtain

$$\mathcal{A}(t; \rho_t, \rho_t) = -\mathcal{A}_t(t; \rho, \rho_t) + \mathcal{B}(t, t; \rho, \rho_t) + \mathcal{B}_t(t, t; \hat{\rho}, \rho_t) - \int_0^t \mathcal{B}_{ts}(t, s; \hat{\rho}(s), \rho_t)\, ds \\ - \mathcal{B}(t, t; \theta, \rho_t) - \mathcal{B}_t(t, t; \hat{\theta}, \rho_t) + \int_0^t \mathcal{B}_{ts}(t, s; \hat{\theta}(s), \rho_t)\, ds.$$

Using (2.1) and the smoothness of the coefficients of $\mathcal{A}(t)$ and $B(t, s)$, we find

$$\|\rho_t\|_1 \le C\Big(\|\rho\|_1 + \|\hat{\rho}\|_1 + \|\theta\|_1 + \|\hat{\theta}\|_1 + \int_0^t (\|\hat{\rho}(s)\|_1 + \|\hat{\theta}(s)\|_1)\, ds\Big).$$

Use of (2.1), Lemma 3.1, Lemma 3.2 and (3.7) yields

$$\|\rho_t\|_1 \le Cht^{-1/2} \|u_0\|.$$

Hence the triangle inequality and Lemma 3.1 yield

$$\|\eta_t\|_1 \le \|\theta_t\|_1 + \|\rho_t\|_1 \le Cht^{-2} \|u_0\|. \tag{3.16}$$

For the L^2 estimate, we again consider the auxiliary problem (3.10) and (3.11), now replacing the right-hand side of (3.10) by η_t. Note that ϕ now satisfies the regularity condition

$$\|\phi\|_2 \le C\|\eta_t\|. \tag{3.17}$$

Observe that, for $\chi \in S_h$,

$$\begin{aligned}\|\eta_t\|^2 &= \mathcal{A}(t; \eta_t, \phi - \chi) + \mathcal{A}(t; \eta_t, \chi) \\ &= \mathcal{A}(t; \eta_t, \phi - \chi) - \mathcal{A}_t(t; \eta, \chi) + \mathcal{B}(t, t, \eta, \chi) + \int_0^t \mathcal{B}_t(t, s; \eta(s), \chi)\, ds.\end{aligned}$$

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Here we have differentiated (3.1) with respect to time and then substituted the value of $\mathcal{A}(t;\eta_t,\chi)$. Integrating by parts in time, we obtain

$$\|\eta_t\|^2 = \mathcal{A}(t;\eta_t,\phi-\chi) + \mathcal{A}_t(t;\eta_t,\phi-\chi) - \mathcal{A}_t(t;\eta_t,\phi) - \mathcal{B}(t,t,\eta,\phi-\chi) - \mathcal{B}(t,t,\eta,\phi) - \mathcal{B}_t(t,t;\hat{\eta}(s),\phi-\chi) + \int_0^t \mathcal{B}_{ts}(t,s;\hat{\eta}(s),\phi-\chi)\,ds + \mathcal{B}_t(t,s;\hat{\eta}(t),\phi) - \int_0^t \mathcal{B}_{ts}(t,s;\hat{\eta}(s),\phi)\,ds.$$

Using the smoothness of the coefficients of $\mathcal{A}(t)$ and $B(t,s)$, we obtain

$$\|\eta_t\|^2 \le C\Big(\|\eta_t\|_1 + \|\eta\|_1 + \|\hat{\eta}\|_1 + \int_0^t \|\hat{\eta}(s)\|_1\,ds\Big)\|\phi-\chi\|_1 + C\Big(\|\eta\| + \|\hat{\eta}\| + \int_0^t \|\hat{\eta}(s)\|_1\,ds\Big)\|\phi\|_2.$$

Using the approximation property (1.3), Lemma 3.3, (3.16) and the regularity result (3.17), it follows that

$$\|\eta_t\| \le Ch^2t^{-2}\|u_0\|.$$

This completes the proof. □

4. Semidiscrete error estimates for nonsmooth data

In this section we discuss the proof of our main theorem, Theorem 1.1. Observe that $e = u - u_h$ satisfies the following equation:

$$(e_t,\phi_h) + \mathcal{A}(t;e,\phi_h) = \int_0^t \mathcal{B}(t,s;e(s),\phi_h)\,ds \quad \text{for all } \phi_h \in S_h, t > 0. \tag{4.1}$$

Using the Ritz–Volterra projection W_hu of u, we rewrite

$$e = u - u_h = (u - W_hu) - (u_h - W_hu) =: \eta - \xi.$$

Using equation (3.1), equation (4.1) can be written as

$$(\xi_t,\phi_h) + \mathcal{A}(t;\xi,\phi_h) = (\eta_t,\phi_h) + \int_0^t \mathcal{B}(t,s;\xi(s),\phi_h)\,ds \quad \text{for all } \phi_h \in S_h. \tag{4.2}$$

We now sketch the proof of our main theorem.

Proof of Theorem 1.1. Choose $\phi_h = t^3\xi(t)$ in (4.2) to find that

$$\frac{1}{2}\frac{d}{dt}(t^3\|\xi\|^2) + \mathcal{A}(t;\xi,t^3\xi) = \Big(\frac{3}{2}t^2\|\xi\|^2 + t^3(\eta_t,\xi)\Big) + t^3\int_0^t \mathcal{B}(t,s;\xi(s),\xi)\,ds.$$

Integrate with respect to time. Then the coercivity property (2.1) for $\mathcal{A}(t;\cdot,\cdot)$ yields

$$t^3\|\xi\|^2 + 2\rho_1\int_0^t s^3\|\xi(s)\|_1^2\,ds \le \int_0^t (3s^2\|\xi(s)\|^2 + 2s^3(\eta_s(s),\xi(s)))ds + 2\int_0^t\int_0^s s^3\mathcal{B}(s,\tau;\xi(\tau),\xi(s))\,d\tau\,ds =: I_1 + I_2. \tag{4.3}$$

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Rewrite the I_1 term as

$$I_1 \leq C \int_0^t s^2 \|\xi(s)\|^2 \, ds + C \int_0^t s^4 \|\eta_s(s)\|^2 \, ds. \tag{4.4}$$

From Lemma 3.4 we obtain

$$\|\eta_t\|^2 \leq C h^4 t^{-4} \|u_0\|^2,$$

and hence

$$\int_0^t s^4 \|\eta_s(s)\|^2 ds \leq C h^4 t \|u_0\|^2. \tag{4.5}$$

For the estimate of I_1, if we can obtain the estimate of the first term on the right-hand side of (4.4) as

$$\int_0^t s^2 \|\xi(s)\|^2 \, ds \leq C h^4 t \|u_0\|^2, \tag{4.6}$$

then substituting (4.5) and (4.6) in (4.4) yields

$$I_1 \leq C h^4 t \|u_0\|^2.$$

For I_2, integrating by parts we rewrite

$$\begin{aligned} I_2 &= 2 \int_0^t s^3 \mathcal{B}(s, s; \hat{\xi}(s), \xi(s)) \, ds - 2 \int_0^t \int_0^s s^3 \mathcal{B}_\tau(s, \tau; \hat{\xi}(\tau), \xi(s)) \, d\tau \, ds \\ &= 2 \int_0^t s^3 \mathcal{B}(s, s; \hat{\xi}(s), \xi(s)) \, ds - 2 \int_0^t s^3 \mathcal{B}_s(s, s; \hat{\hat{\xi}}(s), \xi(s)) \, ds \\ &\quad + 2 \int_0^t \int_0^s s^3 \mathcal{B}_{\tau\tau}(s, \tau; \hat{\hat{\xi}}(\tau), \xi(s)) \, d\tau \, ds, \end{aligned}$$

and hence we find

$$|I_2| \leq \rho_1 \int_0^t s^3 \|\xi(s)\|_1^2 \, ds + C \int_0^t (s \|\hat{\xi}(s)\|_1^2 + \|\hat{\hat{\xi}}(s)\|_1^2) \, ds.$$

Further, for the second term on the right-hand side of I_2, if we have an estimate, say,

$$\int_0^t (s \|\hat{\xi}(s)\|_1^2 + \|\hat{\hat{\xi}}(s)\|_1^2) \, ds \leq C h^4 t \|u_0\|^2, \tag{4.7}$$

then using (4.7) in I_2 yields

$$|I_2| \leq \rho_1 \int_0^t s^3 \|\xi(s)\|_1^2 \, ds + C h^4 t \|u_0\|^2.$$

Substituting the estimates of I_1 and I_2 in (4.3), we obtain

$$t^3 \|\xi\|^2 + \int_0^t s^3 \|\xi(s)\|_1^2 \, ds \leq C t h^4 \|u_0\|^2, \tag{4.8}$$

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and hence

$$\|\xi\| \le Ch^2 t^{-1}\|u_0\|. \tag{4.9}$$

Now, from Lemma 3.3 with $k = 2$ and $j = 0$ and (4.9), we conclude that

$$\|e\| \le Ch^2 t^{-1}\|u_0\|,$$

and this completes the proof of Theorem 1.1. □

It remains to obtain the estimates (4.6) and (4.7), which we do now.

With $u_h(0) = P_h u_0$, integrate (4.1) twice. Use (3.13) and its integrated version to obtain

$$\begin{aligned}(\xi, \phi_h) + \mathcal{A}(t; \hat{\xi}, \phi_h) &- \int_0^t \mathcal{A}_s(s; \hat{\xi}(s), \phi_h)\, ds \\ &= (\eta, \phi_h) + \int_0^t \mathcal{B}(s, s; \hat{\xi}(s), \phi_h)\, ds \\ &\quad - \int_0^t \int_0^s \mathcal{B}_\tau(s, \tau; \hat{\xi}(\tau), \phi_h)\, d\tau\, ds \quad \text{for all } \phi_h \in S_h \end{aligned} \tag{4.10}$$

and

$$\begin{aligned}(\hat{\xi}, \phi_h) + \mathcal{A}(t; \hat{\hat{\xi}}, \phi_h) - 2\int_0^t \mathcal{A}_s(s; \hat{\hat{\xi}}(s), \phi_h)\, ds + \int_0^t \int_0^s \mathcal{A}_{\tau\tau}(\tau; \hat{\hat{\xi}}(\tau), \phi_h)\, d\tau\, ds \\ = (\hat{\eta}, \phi_h) + \int_0^t \mathcal{B}(s, s; \hat{\hat{\xi}}(s), \phi_h)\, ds - 2\int_0^t \int_0^s \mathcal{B}_\tau(\tau, \tau; \hat{\hat{\xi}}(\tau), \phi_h)\, d\tau\, ds \\ + \int_0^t \int_0^s \int_0^\tau \mathcal{B}_{\tau\tau'}(\tau, \tau'; \hat{\hat{\xi}}(\tau'), \phi_h)\, d\tau'\, d\tau\, ds \quad \text{for all } \phi_h \in S_h, \end{aligned} \tag{4.11}$$

respectively. Below we prove two lemmas involving estimates of ξ.

Lemma 4.1. *Let $\hat{\hat{\xi}}$ satisfy (4.11). Then there exists a positive constant C independent of h such that the following estimates hold for $t > 0$:*

$$\|\hat{\hat{\xi}}\|^2 + \int_0^t \|\hat{\hat{\xi}}(s)\|_1^2\, ds \le Cth^4\|u_0\|^2, \tag{4.12}$$

$$\|\hat{\hat{\xi}}\|_1^2 + \int_0^t \|\hat{\xi}(s)\|^2\, ds \le Cth^4\|u_0\|^2. \tag{4.13}$$

Proof. Choose $\phi_h = \hat{\hat{\xi}}(t)$ in (4.11) to obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\hat{\hat{\xi}}\|^2 + \mathcal{A}(t; \hat{\hat{\xi}}, \hat{\hat{\xi}}) &= 2\int_0^t \mathcal{A}_s(s; \hat{\hat{\xi}}(s), \hat{\hat{\xi}})\, ds - \int_0^t \int_0^s \mathcal{A}_{\tau\tau}(\tau; \hat{\hat{\xi}}(\tau), \hat{\hat{\xi}})\, d\tau\, ds \\ &\quad + (\hat{\eta}, \hat{\hat{\xi}}) + \int_0^t \mathcal{B}(s, s; \hat{\hat{\xi}}(s), \hat{\hat{\xi}})\, ds \\ &\quad - 2\int_0^t \int_0^s \mathcal{B}_\tau(\tau, \tau; \hat{\hat{\xi}}(\tau), \hat{\hat{\xi}})\, d\tau\, ds \\ &\quad + \int_0^t \int_0^s \int_0^\tau \mathcal{B}_{\tau\tau'}(\tau, \tau'; \hat{\hat{\xi}}(\tau'), \hat{\hat{\xi}})\, d\tau'\, d\tau\, ds.\end{aligned}$$

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Apply the Cauchy–Schwarz inequality along with Young's inequality and then integrate the resulting inequality to arrive at

$$\|\hat{\hat{\xi}}\|^2 + \int_0^t \|\hat{\hat{\xi}}(s)\|_1^2\,ds \le C\int_0^t \|\hat{\eta}(s)\|^2\,ds + C\int_0^t\Big(\|\hat{\hat{\xi}}(s)\|^2 + \int_0^s \|\hat{\hat{\xi}}(\tau)\|_1^2\,d\tau\Big)\,ds.$$

Use Lemma 3.3 and then apply Gronwall's Lemma to obtain the estimate (4.12).

To estimate (4.13), set $\phi_h = \hat{\xi}(t)$ in (4.11) to obtain

$$\begin{aligned}\|\hat{\xi}\|^2 + \mathcal{A}(t;\hat{\hat{\xi}},\hat{\xi}) &= 2\int_0^t \mathcal{A}_s(s;\hat{\hat{\xi}}(s),\hat{\xi})\,ds - \int_0^t\int_0^s \mathcal{A}_{\tau\tau}(\tau;\hat{\hat{\xi}}(\tau),\hat{\xi})\,d\tau\,ds\\ &\quad + (\hat{\eta},\hat{\xi}) - \int_0^t \mathcal{B}(s,s;\hat{\hat{\xi}}(s),\hat{\xi})\,ds\\ &\quad + 2\int_0^t\int_0^s \mathcal{B}_\tau(\tau,\tau;\hat{\hat{\xi}}(\tau),\hat{\xi})\,d\tau\,ds\\ &\quad + \int_0^t\int_0^s\int_0^\tau \mathcal{B}_{\tau\tau'}(\tau,\tau';\hat{\hat{\xi}}(\tau'),\hat{\xi})\,d\tau'\,d\tau\,ds. \end{aligned}\tag{4.14}$$

Since

$$\frac{d}{dt}\mathcal{A}(t;\hat{\hat{\xi}},\hat{\hat{\xi}}) = \mathcal{A}_t(t;\hat{\hat{\xi}},\hat{\hat{\xi}}) + 2\mathcal{A}(t;\hat{\hat{\xi}},\hat{\xi}),$$

we rewrite (4.14) as

$$\begin{aligned}\|\hat{\xi}\|^2 + \frac{1}{2}\frac{d}{dt}\mathcal{A}(t;\hat{\hat{\xi}},\hat{\hat{\xi}}) &= \frac{1}{2}\mathcal{A}_t(t;\hat{\hat{\xi}},\hat{\hat{\xi}}) + (\hat{\eta},\hat{\xi}) - 2\mathcal{A}(t;\hat{\hat{\xi}},\hat{\hat{\xi}}) + 2\frac{d}{dt}\Big(\int_0^t \mathcal{A}(s;\hat{\hat{\xi}}(s),\hat{\hat{\xi}})\,ds\Big)\\ &\quad + \mathcal{B}(t,t;\hat{\hat{\xi}},\hat{\hat{\xi}}) - \frac{d}{dt}\Big(\int_0^t \mathcal{B}(s,s;\hat{\hat{\xi}}(s),\hat{\hat{\xi}})\,ds\Big)\\ &\quad + \int_0^t \mathcal{A}_{ss}(s;\hat{\hat{\xi}}(s),\hat{\hat{\xi}})\,ds - 2\int_0^t \mathcal{B}_s(s,s;\hat{\hat{\xi}}(s),\hat{\hat{\xi}})\,ds\\ &\quad - \frac{d}{dt}\Big(\int_0^t\int_0^s (\mathcal{A}_{\tau\tau}(\tau;\hat{\hat{\xi}}(\tau),\hat{\hat{\xi}}) - 2\mathcal{B}_\tau(\tau,\tau;\hat{\hat{\xi}}(\tau),\hat{\hat{\xi}}))\,d\tau\,ds\Big)\\ &\quad + \frac{d}{dt}\Big(\int_0^t\int_0^s\int_0^\tau \mathcal{B}_{\tau\tau'}(\tau,\tau';\hat{\hat{\xi}}(\tau'),\hat{\hat{\xi}})\,d\tau'\,d\tau\,ds\Big)\\ &\quad - \int_0^t\int_0^s \mathcal{B}_{s\tau}(s,\tau;\hat{\hat{\xi}}(\tau),\hat{\hat{\xi}})\,d\tau\,ds. \end{aligned}\tag{4.15}$$

Integrate (4.15) with respect to time and use the coercivity property (2.1) of $\mathcal{A}(t;\cdot,\cdot)$ with the smoothness of the coefficients of $A(t)$ and $B(t,s)$. Then an application of the Cauchy–Schwarz inequality with Young's inequality yields

$$\int_0^t \|\hat{\xi}(s)\|^2\,ds + \|\hat{\hat{\xi}}(t)\|_1^2 \le C\int_0^t \|\hat{\eta}\|^2\,ds + C\int_0^t \|\hat{\hat{\xi}}(s)\|_1^2\,ds.$$

Using (4.12) and Lemma 3.3, we obtain

$$\|\hat{\hat{\xi}}\|_1^2 + \int_0^t \|\hat{\xi}(s)\|^2\,ds \le Cth^4\|u_0\|^2,$$

and this completes the proof. □

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Lemma 4.2. *Let $\hat{\xi}$ satisfy equation (4.10). Then there exists a positive constant C independent of h such that, for $t > 0$,*

$$t\|\hat{\xi}\|^2 + \int_0^t s\|\hat{\xi}(s)\|_1^2\,ds \le Cth^4\|u_0\|^2, \tag{4.16}$$

$$t^2\|\hat{\xi}\|_1^2 + \int_0^t s^2\|\xi(s)\|^2\,ds \le Cth^4\|u_0\|^2. \tag{4.17}$$

Proof. Choose $\phi_h = t\hat{\xi}(t)$ in (4.10) to obtain

$$\frac{1}{2}\frac{d}{dt}(t\|\hat{\xi}\|^2) + t\mathcal{A}(t;\hat{\xi},\hat{\xi}) = \frac{1}{2}\|\hat{\xi}\|^2 + t\int_0^t \mathcal{A}_s(s;\hat{\xi}(s),\hat{\xi})\,ds + t(\eta,\hat{\xi}) + t\int_0^t \mathcal{B}(s,s;\hat{\xi}(s),\hat{\xi})\,ds - t\int_0^t\int_0^s \mathcal{B}_\tau(s,\tau;\hat{\xi}(\tau),\hat{\xi})\,d\tau\,ds.$$

Then integration by parts with respect to time yields

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(t\|\hat{\xi}\|^2) + t\mathcal{A}(t;\hat{\xi},\hat{\xi}) &= \frac{1}{2}\|\hat{\xi}\|^2 + t\mathcal{A}_t(t;\hat{\hat{\xi}}(t),\hat{\xi}) - t\int_0^t \mathcal{A}_{ss}(s;\hat{\hat{\xi}}(s),\hat{\xi})\,ds + t(\eta,\hat{\xi})\\ &\quad + t\mathcal{B}(t,t;\hat{\hat{\xi}}(t),\hat{\xi}) - 2t\int_0^t \mathcal{B}_s(s,s;\hat{\hat{\xi}}(s),\hat{\xi})\,ds\\ &\quad + t\int_0^t\int_0^s \mathcal{B}_{\tau\tau}(s,\tau;\hat{\hat{\xi}}(\tau),\hat{\xi})\,d\tau\,ds.\end{aligned} \tag{4.18}$$

Now integrate (4.18) with respect to time to obtain

$$\begin{aligned}\frac{1}{2}t\|\hat{\xi}\|^2 + \int_0^t s\mathcal{A}(s;\hat{\xi}(s),\hat{\xi}(s))\,ds &= \frac{1}{2}\int_0^t \|\hat{\xi}(s)\|^2\,ds + \int_0^t s\mathcal{A}_s(s;\hat{\hat{\xi}}(s),\hat{\xi}(s))\,ds\\ &\quad - \int_0^t\int_0^s s\mathcal{A}_{\tau\tau}(\tau;\hat{\hat{\xi}}(\tau),\hat{\xi}(s))\,d\tau\,ds\\ &\quad + \int_0^t s(\eta,\hat{\xi})\,ds + \int_0^t s\mathcal{B}(s,s;\hat{\hat{\xi}}(s),\hat{\xi}(s))\,ds\\ &\quad - 2\int_0^t\int_0^s s\mathcal{B}_\tau(\tau,\tau;\hat{\hat{\xi}}(\tau),\hat{\xi}(s))\,ds\\ &\quad + \int_0^t\int_0^s\int_0^\tau s\mathcal{B}_{\tau'\tau'}(\tau,\tau';\hat{\hat{\xi}}(\tau'),\hat{\xi}(s))\,d\tau'\,d\tau\,ds.\end{aligned}$$

Therefore, using the coercivity property (2.1), the Cauchy–Schwarz inequality and Young's inequality, it follows that

$$t\|\hat{\xi}\|^2 + \int_0^t s\|\hat{\xi}(s)\|_1^2\,ds \le C\int_0^t (\|\hat{\xi}(s)\|^2 + s^2\|\eta(s)\|^2 + \|\hat{\hat{\xi}}(s)\|_1^2)\,ds.$$

Now use Lemma 3.3 and (4.13) to arrive at (4.16).

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In order to estimate (4.17), set $\phi_h = t^2\xi(t)$ in (4.10) to obtain

$$t^2\|\xi\|^2 + t^2\mathcal{A}(t;\hat{\xi},\xi) - t^2\int_0^t \mathcal{A}_s(s;\hat{\xi}(s),\xi)\,ds = t^2(\eta,\xi) + \int_0^t \mathcal{B}(s,s;\hat{\xi}(s),t^2\xi)\,ds.$$

Note that

$$\frac{1}{2}\frac{d}{dt}(t^2\mathcal{A}(t;\hat{\xi},\hat{\xi})) = t\mathcal{A}(t;\hat{\xi},\hat{\xi}) + \frac{t^2}{2}\mathcal{A}_t(t;\hat{\xi},\hat{\xi}) + t^2\mathcal{A}(t;\hat{\xi},\xi),$$

and hence

$$\begin{aligned} t^2\|\xi\|^2 + \frac{1}{2}\frac{d}{dt}(t^2\mathcal{A}(\hat{\xi},\hat{\xi})) = t\mathcal{A}(t;\hat{\xi},\hat{\xi}) &+ \frac{t^2}{2}\mathcal{A}_t(t;\hat{\xi},\hat{\xi}) + t^2\int_0^t \mathcal{A}_s(s;\hat{\xi}(s),\xi)\,ds \\ &+ t^2(\eta,\xi) + t^2\int_0^t \mathcal{B}(s,s;\hat{\xi}(s),\xi)\,ds \\ &- t^2\int_0^t\int_0^s \mathcal{B}_\tau(s,\tau;\hat{\xi}(\tau),\xi)\,d\tau\,ds. \end{aligned}$$

Integrate the above equation with respect to time from 0 to t and then rewrite the resulting equation as

$$\begin{aligned} &\int_0^t s^2\|\xi(s)\|^2\,ds + \frac{1}{2}t^2\mathcal{A}(t;\hat{\xi},\hat{\xi}) \\ &\quad= \int_0^t \Big(s\mathcal{A}(s;\hat{\xi}(s),\hat{\xi}(s)) + \frac{s^2}{2}\mathcal{A}_s(s;\hat{\xi}(s),\hat{\xi}(s)) \\ &\qquad + s^2(\eta(s),\xi(s)) - s^2\mathcal{A}_s(s;\hat{\xi}(s),\hat{\xi}(s)) - s^2\mathcal{B}(s,s;\hat{\xi}(s),\hat{\xi}(s))\Big)\,ds \\ &\qquad + t^2\int_0^t (\mathcal{A}_s(s;\hat{\xi}(s),\hat{\xi}(t)) + \mathcal{B}(s,s;\hat{\xi}(s),\hat{\xi}(t)))\,ds \\ &\qquad - 2\int_0^t\int_0^s s(\mathcal{A}_\tau(\tau;\hat{\xi}(\tau),\hat{\xi}(s)) + \mathcal{B}(\tau,\tau;\hat{\xi}(\tau),\hat{\xi}(s)))\,d\tau\,ds \\ &\qquad - t^2\int_0^t\int_0^s \mathcal{B}_\tau(s,\tau;\hat{\xi}(\tau),\hat{\xi}(t))\,d\tau\,ds \\ &\qquad + \int_0^t\int_0^s s^2\mathcal{B}_\tau(s,\tau;\hat{\xi}(\tau),\hat{\xi}(s))\,d\tau\,ds \\ &\qquad + 2\int_0^t\int_0^s\int_0^\tau s\mathcal{B}_{\tau'}(\tau,\tau';\hat{\xi}(\tau'),\hat{\xi}(s))\,d\tau'\,d\tau\,ds \\ &\quad=: I_1 + I_2 + I_3 + I_4 + I_5 + I_6. \end{aligned} \tag{4.19}$$

To estimate I_1 on the right-hand side of (4.19), we obtain

$$|I_1| \le \frac{1}{2}\int_0^t s^2\|\xi(s)\|^2\,ds + \frac{1}{2}\int_0^t s^2\|\eta(s)\|^2\,ds + C\int_0^t \|\hat{\xi}(s)\|_1^2\,ds.$$

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For I_2, use integration by parts in time to rewrite

$$I_2 = t^2(\mathcal{A}_t(t;\hat{\hat{\xi}}(t),\hat{\xi}(t)) + \mathcal{B}(t,t;\hat{\hat{\xi}}(t),\hat{\xi}(t))) - t^2\int_0^t (\mathcal{A}_{ss}(s;\hat{\hat{\xi}}(s),\hat{\xi}(t)) + \mathcal{B}_s(s,s;\hat{\hat{\xi}}(s),\hat{\xi}(t))\,ds$$

and hence obtain

$$|I_2| \leq \frac{\rho_1}{4}t^2\|\hat{\xi}\|_1^2 + C\Big(\|\hat{\hat{\xi}}(t)\|_1^2 + \int_0^t \|\hat{\hat{\xi}}(s)\|_1^2\,ds\Big).$$

For I_4, again use integration by parts in time to obtain

$$I_4 = -t^2\int_0^t \mathcal{B}_s(s,s;\hat{\hat{\xi}}(s),\hat{\xi}(t))\,ds + t^2\int_0^t\int_0^s \mathcal{B}_{\tau\tau}(s,\tau;\hat{\hat{\xi}}(\tau),\hat{\xi}(t))\,d\tau\,ds,$$

and hence

$$|I_4| \leq \frac{\rho_1}{4}t^2\|\hat{\xi}\|_1^2 + C\int_0^t \|\hat{\hat{\xi}}(s)\|_1^2\,ds.$$

Similarly, rewrite I_3, I_5 and I_6 as

$$\begin{aligned}
I_3 &= -2\int_0^t s(\mathcal{A}_s(s;\hat{\hat{\xi}}(s),\hat{\xi}(s)) + \mathcal{B}(s,s;\hat{\hat{\xi}}(s),\hat{\xi}(s)))\,ds \\
&\quad + 2\int_0^t\int_0^s s(\mathcal{A}_{\tau\tau}(\tau;\hat{\hat{\xi}}(\tau),\hat{\xi}(s)) + \mathcal{B}_\tau(\tau,\tau;\hat{\hat{\xi}}(\tau),\hat{\xi}(s)))\,d\tau\,ds, \\
I_5 &= \int_0^t s^2\mathcal{B}_s(s,s;\hat{\hat{\xi}}(s),\hat{\xi}(s))\,ds - \int_0^t\int_0^s s^2\mathcal{B}_{\tau\tau}(s,\tau;\hat{\hat{\xi}}(\tau),\hat{\xi}(s))\,d\tau\,ds, \\
I_6 &= 2\int_0^t\int_0^s s\mathcal{B}_\tau(\tau,\tau;\hat{\hat{\xi}}(\tau),\hat{\xi}(s))\,d\tau\,ds \\
&\quad - 2\int_0^t\int_0^s\int_0^\tau s\mathcal{B}_{\tau'\tau'}(\tau,\tau';\hat{\hat{\xi}}(\tau'),\hat{\xi}(s))\,d\tau'\,d\tau\,ds.
\end{aligned}$$

Thus

$$|I_3| + |I_5| + |I_6| \leq C\int_0^t s\|\hat{\xi}(s)\|_1^2\,ds + C\int_0^t \|\hat{\hat{\xi}}(s)\|_1^2\,ds.$$

Substituting the estimates of $I_1,\ldots,I_6$ in (4.19) and using the coercivity property (2.1) yields

$$\int_0^t s^2\|\xi(s)\|^2\,ds + t^2\|\hat{\xi}\|_1^2 \leq C\int_0^t (s\|\hat{\xi}(s)\|_1^2 + s^2\|\eta(s)\|^2 + \|\hat{\hat{\xi}}(s)\|_1^2)\,ds + C\|\hat{\hat{\xi}}(t)\|_1^2.$$

From (4.12), (4.13) and (4.16), we obtain the estimate (4.17), and this completes the proof. □

Observe that the estimate (4.6) is derived in Lemma 4.2 and estimate (4.7) is obtained from Lemmas 4.1 and 4.2. This completes the proof of Theorem 1.1.

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Remark 4.3. For a completely discrete scheme based on the backward Euler method, we obtain from Pani and Sinha [15, Lemma 3.10] that at each time level t_n,

$$\|U^n - u_h(t_n)\| \le Ckt_n^{-1}\Big(1 + \log\frac{1}{k}\Big)\|u_0\|, \tag{4.20}$$

where U^n denotes the backward Euler approximation at t_n. Note that at each time level t_n, we find from Theorem 1.1 that

$$\|u(t_n) - u_h(t_n)\| \le Ch^2 t_n^{-1}\|u_0\|. \tag{4.21}$$

Combining (4.20) and (4.21), we therefore arrive at the following final completely discrete error estimate:

$$\|u(t_n) - U^n\| \le Ct_n^{-1}\Big(h^2 + k\Big(1 + \log\frac{1}{k}\Big)\Big)\|u_0\|.$$

Below we discuss a superconvergence result for ξ in the H^1 norm.

Theorem 4.4. *There is a positive constant C independent of h such that, for $t \in (0, T]$, the following superconvergence result holds:*

$$\|\xi(t)\|_1 \le Ch^2 t^{-3/2}\|u_0\|.$$

Proof. Setting $\phi_h = t^4\xi_t$ in (4.2), we obtain

$$t^4\|\xi_t\|^2 + t^4\mathcal{A}(t; \xi, \xi_t) = t^4(\eta_t, \xi_t) + \int_0^t \mathcal{B}(t, s; \xi(s), t^4\xi_t)\, ds.$$

Observe that

$$\frac{d}{dt}(t^4\mathcal{A}(t; \xi, \xi)) = 4t^3\mathcal{A}(t; \xi, \xi) + 2t^4\mathcal{A}(t; \xi, \xi_t) + t^4\mathcal{A}_t(t; \xi, \xi)$$

and

$$\frac{d}{dt}(t^4\mathcal{B}(t, s; \xi(s), \xi)) = 4t^3\mathcal{B}(t, s; \xi(s), \xi) + t^4\mathcal{B}(t, s; \xi, \xi_t) + t^4\mathcal{B}_t(t, s; \xi(s), \xi).$$

Therefore,

$$\begin{aligned} t^4\|\xi_t\|^2 + \frac{1}{2}\frac{d}{dt}\{t^4\mathcal{A}(t; \xi, \xi)\} = t^4(\eta_t, \xi_t) + 2t^3\mathcal{A}(t; \xi, \xi) + \frac{1}{2}t^4\mathcal{A}_t(t; \xi, \xi)& \\ + \frac{d}{dt}\Big(\int_0^t t^4\mathcal{B}(t, s; \xi(s), \xi)\, ds\Big) - t^4\mathcal{B}(t, t; \xi, \xi)& \\ - \int_0^t (4t^3\mathcal{B}(t, s; \xi(s), \xi) - t^4\mathcal{B}_t(t, s; \xi(s), \xi))\, ds.& \end{aligned}$$

Integrate the above equation with respect to time and then use the smoothness of the coefficients of $\mathcal{A}(t)$ and $B(t, s)$ with the Cauchy–Schwarz inequality and Young's inequality to obtain

$$t^4\mathcal{A}(t; \xi, \xi) + \int_0^t s^4\|\xi_s\|^2\, ds \le 2\int_0^t s^4\|\eta_s\|^2\, ds + C\int_0^t s^3(1 + s)\|\xi(s)\|_1^2\, ds + \frac{\rho_1}{2}\|\xi\|_1^2.$$

https://doi.org/10.1017/S1446181114000030 Published online by Cambridge University Press

Using the coercivity property (2.1), we find

$$t^4\|\xi\|_1^2 + \int_0^t s^4\|\xi_s\|^2\,ds \le C\int_0^t s^4\|\eta_s\|^2\,ds + C\int_0^t s^3\|\xi\|_1^2\,ds.$$

We conclude, using Lemma 3.4 and (4.8), that

$$t^4\|\xi\|_1^2 + \int_0^t s^4\|\xi_s\|^2\,ds \le Ch^4 t\|u_0\|^2,$$

and hence

$$\|\xi\|_1 \le Ch^2 t^{-3/2}\|u_0\|. \tag{4.22}$$

This completes the proof. □

As a consequence of Theorem 4.4, we now obtain the following maximum norm estimate.

Corollary 4.5. *Assume that the triangulation is quasiuniform and $d = 2$. Then there exists a positive constant C such that, for $t \in (0, T]$,*

$$\|(u - u_h)(t)\|_{L^\infty} \le Ch^2(t^{-3/2}|\log h|^{1/2}\|u_0\| + t^{-1}|\log h|^2\|u_0\|_{L^\infty}).$$

Proof. Since $d = 2$, and the triangulation is quasiuniform, we note from the subspace Sobolev inequality [1, 20] for elements in S_h that, for $t \in (0, T]$,

$$\|\chi\|_{L^\infty} \le C|\log h|^{1/2}\|\chi\|_1 \quad \text{for all } \chi \in S_h, \tag{4.23}$$

where $\|\cdot\|_\infty$ denotes the L^∞-norm. Now, from (4.22), we arrive using (4.23) at

$$\|\xi\|_\infty \le Ch^2|\log h|^{1/2}t^{-3/2}\|u_0\|. \tag{4.24}$$

From a paper of Lin [9], we note that for $u \in W^{2,\infty} \cap H^2 \cap H_0^1$,

$$\|\eta\|_\infty = \|u - W_h u\|_\infty \le Ch^2|\log h|^{1/2}t^{-1}\|u_0\|_\infty. \tag{4.25}$$

From (4.24) and (4.25), we conclude using the triangle inequality that

$$\|e\|_\infty \le \|\eta\|_\infty + \|\xi\|_\infty \le Ch^2|\log h|^{1/2}(t^{-3/2}\|u_0\| + t^{-1}\|u_0\|_\infty),$$

and this completes the proof. □

Remark 4.6. The superconvergence analysis can be used for better recovery of the gradient of the solution under a uniform mesh.

Acknowledgements

The first author would like to thank CSIR, Government of India, as well as INCTMat/CAPES (http://inctmat.impa.br) for financial support. The second author gratefully acknowledges the research support of the Department of Science and Technology, Government of India, under DST-CNPq Indo-Brazil Project-DST/INT/Brazil/RPO-05/2007 (Grant No. 490795/2007-2). The third author would like to acknowledge the financial support of MHRD, India. This publication is also based on work supported in part by Award No. KUK-C1-013-04, made by King Abdullah University of Science and Technology (KAUST).

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