BULL. AUSTRAL. MATH. SOC. 16D90
VOL. 52 (1995) [189–194]

# MORITA DUALITY AND FINITELY GROUP-GRADED RINGS

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We give the relation between the (rigid) graded Morita duality and the Morita duality on a finitely group-graded ring and the relation between a left Morita ring and some of its matrix rings.

## 0. INTRODUCTION

The characterisations of Morita dualities can be found in [**6**]. (Rigid) graded Morita dualities are characterised in [**2**]. We use freely the same terminologies and notations on the Morita duality as in [**6**] and on the (rigid) graded Morita duality as in [**2**].

Throughout this paper, all rings are associative and have identity, all modules are unitary, $G$ is a finite group with an identity $e$, and $|G| = m$. $R$ is a graded ring of type $G$. $R$- mod $(R - gr)$ denotes the category of all (graded) left $R$-modules (of type $G$).

Let $M_G(R)$ denote the ring of $m$ by $m$ matrices over $R$ with rows and columns indexed by the elements of $G$. If $x \in M_G(R)$, we write $x_{g,h}$ for the entry in $(g, h)$-position of $x$. Then if $x, y \in M_G(R)$, the matrix product of $xy$ is given by

$$(xy)_{g,h} = \sum_{t \in G} x_{g,t} y_{t,h}$$

Following [**6**], we call the ring

$$RG = \{x \in M_G(R) \mid x_{g,h} \in R_{gh^{-1}}\}$$

is smash product of $R$ with $G$.

In this paper, we first prove that a graded ring $R$ has a (rigid) graded Morita duality on the left if and only if $R$ has a left Morita duality. Secondly, we probe that a graded ring $R$ has a left Morita duality if and only $M_n(R)_e(\overline{g})$ has a left Morita duality for every natural number $n$ and every $\overline{g} = (g_1, g_2, \dots, g_n) \in G^n$, where

$$M - n(R)_e(\overline{g}) = \left\{ \begin{pmatrix} r_{g_1 g_1^{-1}} & r_{g_1 g_2^{-1}} & \cdots & r_{g_1 g_n^{-1}} \\ r_{g_2 g_1^{-1}} & r_{g_2 g_2^{-1}} & \cdots & r_{g_2 g_n^{-1}} \\ \vdots & \vdots & \ddots & \vdots \\ r_{g_n g_1^{-1}} & r_{g_n g_2^{-1}} & \cdots & r_{g_n g_n^{-1}} \end{pmatrix} \middle| r_{g_i g_j^{-1}} \in R_{g_i g_j^{-1}} \right\}$$

Received 25 October 1994
This research is supported by the natural science foundation of Fujian Province (1994–1997)

Finally, we prove that a graded ring $R$ has a left Morita duality if and only if $R\{H\}$ has a left Morita duality for any subgroup $H$ of $G$, where

$$R\{H\} = \left\{ \alpha \in M_G(R) \mid \alpha_{x,y} \in R_{xHy^{-1}} = \bigoplus_{g \in xHy^{-1}} R_g \right\}.$$

## 1. Graded Morita duality and Morita duality

Let $xe_g$ denote the column vector with $x$ in the $g$-position and zero in the other positions and $e(g, h)$ denote the $m$ by $m$ matrix with the identity of $R$ in the $(g, h)$-position and the zero of $R$ in the other positions. For every graded left $R$-module $M = \bigoplus_{g \in G} {}_gM$, we denote $F(M) = \bigoplus_{g \in G} {}_gMe_g$ and define $r \cdot \widetilde{m} = \sum_g \left( \sum_h r_{g,h \cdot h}m \right) e_g$ for every $r \in R\#G$ and $\widetilde{m} = \sum_{g \in G} {}_gme_g \in F(M)$. So $F(M)$ is a left $R\#G$-module with this scalar multiplication and the column vector addition. Conversely, for every left $R\#G$-module $N$, we denote $G(N) = \bigoplus_{g \in G} (e(g \cdot g)N)$ and define $r \cdot n = \sum_{g \in G} r_g e(gh, h)n$ for every $r \in R$ and $n \in N$. Let ${}_gG(N) = e(g \cdot g)N$ for every $g \in G$, then $G(N)$ is a graded left $R$-module of type $G$ with this scalar multiplication and the original addition. We view every graded left $R$-homomorphism $f\colon M \to N$ as left $R\#G$-homomorphism

$$F(f)\colon F(M) \to F(N)$$

and view every left $R\#G$-homomorphism $g\colon U \to V$ as a graded left $R$-homomorphism $G(g)\colon G(V) \to G(V)$. Following [9], we know that $F\colon R - gr \to R\#G -$ mod and $G\colon R\#G -$ mod $\to R - gr$ are functors such that $FG = 1$ and $GF = 1$. It is clear that a graded left $R$-homomorphism $f\colon M \to N$ is monic (epic) if and only if $F(f)\colon F(M) \to F(N)$ is monic (epic) and a left $R\#G$-homomorphism $g\colon U \to V$ is monic (epic) if and only if $G(g)\colon G(U) \to G(V)$ is monic (epic). So the lattice of submodules of a left $R\#G$-module $U$ is isomorphic to the lattice of graded submodules of $G(U)$. Then we have the following.

**Lemma 1.1.** *Let $M$ be a left $R\#G$-module, then*

(1) *$M$ is injective if and only if $G(M)$ is $gr$-injective.*

(2) *$M$ is finitely cogenerated if and only if $G(M)$ is finitely $gr$-cogenerated.*

(3) *$M$ is cogenerator if and only if $\{(g)G(M) \mid g \in G\}$ is a set of cogenerators in $R - gr$.*

Definition 1.2: (1) Suppose $M$ is a left $R$-module $m_i \in M$, $M_i$ is a submodule of $M$, $i \in I$. A family $\{m_i, M_i\}_{i \in I}$ is called solvable in case there is an $m \in M$ such that $m - m_i \in M_i$ for all $i \in I$, it is called finitely solvable if $\{m_i, M_i\}_{i \in F}$ is solvable for any finite subset $F \subseteq I$, and the module $M$ is called linearly compact in case any finitely solvable family of $M$ is solvable.

(2) Let $M$ be a graded left $R$-module. A pair $(m, N)$ is called a homogeneous pair of degree $g$ if $N$ is a graded submodule of $M$ and $m \in_g M$. A homogeneous family of $M$ is a family of homogeneous pairs all of them with the same degree and the graded left $R$-module $M$ is called $gr$-linearly compact in case any finitely solvable homogeneous family of $M$ is solvable.

**Lemma 1.3.** *(1) ${}_RR$ is $gr$-linearly compact if ${}_{R\#G}R\#G$ is linearly compact.*

*(2) $G(M)$ is $gr$-linearly compact if a left $R\#G$-module $M$ is linearly compact.*

Proof: (1) Suppose that $\{m_i, M_i\}_{i\in T}$ is a finitely solvable homogeneous family of ${}_RR$ with the same degree $g$. Let $M_i^{\#} = \{\alpha \in M_G(R) \mid \alpha_{g,h} \in_{gh^{-1}} M_i\}$ and $m_i^{\#} = \sum_{h\in G} m_i e(gh, h)$, $i \in I$. For every finite subset $F \subseteq I$, $\{m_i, M_i\}_{i\in F}$ is solvable, so there is a $m_F \in M$ such that $m_F - m_i \in M_i$, $i \in F$, so ${}_gm_F - m_i \in_g M_i$, $i \in F$. Let $m_F^{\#} = \sum_{h\in G} {}_gm_F e(gh, h)$, then $m_F^{\#} \in R\#G$ such that $m_F^{\#} - m_i^{\#} \in M_i^{\#}$, $i \in F$. So $\{m_i^{\#}, M_i^{\#}\}_{i\in I}$ is finitely solvable. Since ${}_{R\#G}R\#G$ is linearly compact, there is an $r \in R\#G$ such that $r - m_i^{\#} \in M_i^{\#}$, $i \in I$, so $r_{g,e} - m_i \in_g M_i \subseteq M_i$, $i \in I$, so ${}_RR$ is $gr$-lineary compact.

(2) Suppose that $\{n_i, N_i\}_{i\in I}$ is a finitely solvable homogeneous family of $G(M)$ with the same degree $g(g \in G)$ and $n_i = e(g, g)m_i$, $m_i \in M$, $i \in I$. Let $M_i = F(N_i)$, $i \in I$. For any finite subset $F \subseteq I$, $\{n_i, N_i\}_{i\in F}$ is solvable. So there is an $n_F \in G(M)$ such that

$$n_F - n_i \in N_i, i \in F, \text{ so } {}_gn_F - n_i \in_g N_i, i \in F. \text{ Let } n_F = \sum_{h\in G} e(h, h)m^{(h)}, \text{ then}$$

$$e(g, g)m^{(g)} - e(g, g)m_i = e(g, g)\left(m^{(h)} - m_i\right) \in_g N_i, i \in F$$

Since $N_i = G(M_i)$, $i \in F$, ${}_gN_i = e(g, g)M_i$, $i \in F$, $m^{(g)} - m_i \in M_i$, $i \in F$. Therefore, $\{m_i, M_i\}_{i\in T}$ is finitely solvable. $M$ is linearly compact, so there is $m \in M$ such that $m - m_i \in M_i$, $i \in I$. Let $n = e(g, g)m$, then $n \in G(M)$ such that $n - n_i \in G(M_i) = N_i$, $i \in I$, so $G(M)$ is $gr$-linearly compact. □

**Theorem 1.4.** *A graded ring $R$ has a left Morita duality if and only if $R$ has a rigid graded Morita duality on the left.*

Proof: If $R$ has a rigid graded Morita duality on the left, then $R$ has a left Morita duality by [**3**, Proposition 4.3].

Conversely, if $R$ has a left Morita duality then $R\#G$ has a left Morita duality by [**8**] Theorem 3.9. Suppose that $R\#G$ has a left Morita duality induced by a left

$R\#G$-module $W$, then ${}_{R\#G}R\#G$ is linearly compact and $W$ is a linearly compact finitely cogenerated injective cogenerator by [7] Theorem 4.5. So $G(W)$ is a $gr$-finitely cogenerated $gr$-linearly compact left $R$-module such that $\{(g)G(W) \mid g \in G\}$ is a set of cogenerators of $R - gr$ by Lemma 1.1 and 1.3, and ${}_RR$ is $gr$-linearly compact by Lemma 1.3. Therefore $R$ has a rigid graded. Morita duality on the left by [3] Theorem 5.19. ∎

## 2. Morita rings and matrix rings

For any natural number $n$ and every $\overline{g} = (g_1, g_2, \dots, g_n) \in G^n$ and every $h \in G$, let

$$M_n(R)_h(\overline{g}) = \left\{ \begin{pmatrix} r_{g_1hg_1^{-1}} & r_{g_1hg_2^{-1}} & \cdots & r_{g_1hg_n^{-1}} \\ r_{g_2hg_1^{-1}} & r_{g_2hg_2^{-1}} & \cdots & r_{g_2hg_n^{-1}} \\ \vdots & \vdots & \ddots & vdots \\ r_{g_nhg_1^{-1}} & r_{g_nhg_2^{-1}} & \cdots & r_{g_nhg_n^{-1}} \end{pmatrix} \middle| r_{g_ihg_j^{-1}} \in R_{g_ihg_j^{-1}} \right\}$$

and $M_n(R)(\overline{g}) = \bigoplus_{h\in G} M_n(R)_h(\overline{g})$, then $M_n(R)_e(\overline{g})$ a ring with the matrix multiplication and the matrix addition and $M_n(R)(\overline{g})$ is a graded ring of type $G$, we have

**Theorem 2.1.** *If $R$ has a left Morita duality, then $M_n(R)_e(\overline{g})$ has a left Morita duality for every natural number $n$ and every $\overline{g} \in G^n$. Conversely, if $M_n(R)_e(\overline{g})$ has a left Morita duality for some natural number $n$ and every $\overline{g} \in G^n$, then $R$ has a left Morita duality.*

Proof: If $R$ has a left Morita duality, and let $E$ be the minimal injective cogenerator of $R_e -$ mod , then $R_e$ has a left Morita duality and $R_g$ and $\mathrm{Hom}_{R_g}(R_e, E)$ are linearly compact for every $g \in G \setminus \{e\}$ by [2] Theorem 2.3. Following [4], we know the matrix ring $M_n(R)_e(\overline{g})$ has a left Morita duality for every natural number $n$ and every $\overline{g} \in G^n$.

Conversely, if the matrix ring $M_n(R)_e(\overline{g})$ has a left Morita duality for some natural number $n$ and every $\overline{g} \in G^n$, then, following [4], $R_e$ has a left Morita duality, and $R_{g_ig_j^{-1}}$ and $\mathrm{Hom}_{R_e}\left(R_{g_ig_j^{-1}}, E\right)$ are linearly compact, $i, j = 1, 2, \dots, n$, so $R_e$ has a left Monta duality and $R_g$ and $\mathrm{Hom}_{R_g}(R_g, E)$ are linearly compact for every $g \in G \setminus \{e\}$. Following [2] Theorem 2.3, $R$ has a left Morita duality. ∎

**Theorem 2.2.** *If $R$ is a strongly graded ring and $M_n(R)(\overline{g})$ has a left Morita duality for some natural number $n$ and some $\overline{g} \in G^n$, then $R$ has a left Morita duality.*

Proof: If $R$ is a strongly graded ring, then $M_n(R)(\overline{g})$ is a strongly graded ring by [5] Theorem I.5.6. $M_n(R)_e(\overline{g})$ has a left Morita duality, so $M_n(R)(\overline{g})$ has a left

Morita duality by [2] Corollary 2.6. Since $M_n(R)(\bar{g})$ is equivalent to $R$, $R$ has a left Morita duality by [7] Corollary 4.6.

If $U$ is a nonempty subset of $G$, let $R_{(U)} = \sum_{x \in U} R_x$. Suppose $H$ is a subgroup of $G$, we define $R\{H\} \subseteq M_G(R)$ by

$$R\{H\} = \{\alpha \in M_G(R) \mid \alpha_{x,y} \in R_{xHy^{-1}}\}.$$

□

THEOREM 2.3. *If $R$ has a left Morita duality, then $R\{H\}$ has a left Morita duality for any subgroup $H$ of $G$. Conversely, if $R\{H\}$ has a left Morita duality for some subgroup $H$ of $G$, then $R$ has a left Morita duality.*

PROOF: $R$ has a left Morita duality, so $R\#G$ has a left Morita duality by [8] Theorem 3.9, so, $R\{H\}$ has a left Morita duality by [6] Lemma 1.2 and [2] Corollary 2.6 for every subgroup $H$ of $G$. Conversely, if $R\{H\}$ has a left Morita duality for some subgroup $H$ of $G$. Since $R\{H\}$ is a strongly graded ring by [6] Lemma 1.2, $R\#G$ has a left Morita duality by [2] Corollary 2.6. So $R$ has a left Morita duality by [8] Theorem 3.9. □

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