*LMS J. Comput. Math.* 19 (1) (2016) 98–104
doi:10.1112/S1461157016000073 

# Oscillation of differential equations with non-monotone retarded arguments

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Abstract

Consider the first-order retarded differential equation

$$x'(t)+p(t)x(\tau(t))=0, \quad t \geqslant t_0,$$

where $p(t) \geqslant 0$ and $\tau(t)$ is a function of positive real numbers such that $\tau(t) \leqslant t$ for $t \geqslant t_0$, and $\lim_{t\to\infty} \tau(t)=\infty$. Under the assumption that the retarded argument is non-monotone, a new oscillation criterion, involving lim inf, is established when the well-known oscillation condition

$$\liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)\,ds > \frac{1}{e}$$

is not satisfied. An example illustrating the result is also given.

## 1. *Introduction*

Consider the retarded differential equation

$$x'(t)+p(t)x\left(\tau(t)\right)=0, \quad t \geqslant t_0, \tag{E}$$

where $p(t) \geqslant 0$ and $\tau(t)$ is a function of positive real numbers such that

$$\tau(t) \leqslant t \quad \text{for } t \geqslant t_0 \text{ and } \lim_{t\to\infty} \tau(t)=\infty. \tag{1.1}$$

By a solution of (E) we mean a continuously differentiable function defined on $[\tau(T_0),\infty]$ for some $T_0 \geqslant t_0$ and such that (E) is satisfied for $t \geqslant T_0$. Such a solution is called *oscillatory* if it has arbitrarily large zeros. Otherwise, it is called *non-oscillatory*.

The problem of establishing sufficient conditions for the oscillation of all solutions of equation (E) has been the subject of many investigations. See, for example, [**1**–**18**] and the references cited therein. The first systematic study for the oscillation of all solutions of equation (E) was made by Myshkis. In 1950 [**17**], he proved that every solution oscillates if

$$\limsup_{t\to\infty}[t-\tau(t)] < \infty \quad \text{and} \quad \liminf_{t\to\infty}[t-\tau(t)] \liminf_{t\to\infty} p(t) > \frac{1}{e}.$$

In 1972, Ladas *et al.* [**15**] proved that the same conclusion holds if, in addition, $\tau(t)$ is a non-decreasing function and

$$\limsup_{t\to\infty} \int_{\tau(t)}^{t} p(s)\,ds > 1. \tag{1.2}$$

In 1982, Koplatadze and Canturija [**13**] established the following result.

Received 5 May 2015; revised 21 July 2015.

*2010 Mathematics Subject Classification* 34K11 (primary), 34K06 (secondary).

If $\tau(t)$ is a non-monotone or non-decreasing function, and

$$\liminf_{t\to\infty}\int_{\tau(t)}^{t} p(s)\,ds > \frac{1}{e}, \tag{1.3}$$

then all solutions of (E) oscillate, while if

$$\limsup_{t\to\infty}\int_{\tau(t)}^{t} p(s)\,ds < \frac{1}{e}, \tag{1.4}$$

then the equation (E) has a non-oscillatory solution.

It is important to consider non-monotone arguments when solving differential equations rather than using a pure mathematics approach, which only approximates the natural phenomena described by equations of type (E). This is because there are always natural disturbances (for example, noise in communication systems) that affect all the parameters of the equation and therefore the fair (from a mathematical point of view) monotone argument almost always becomes non-monotone. In view of this, an interesting question that arises in the case where the argument $\tau(t)$ is non-monotone and (1.3) is not satisfied, is whether we can state an oscillation criterion involving $\liminf$.

In the present paper, we give a positive answer to the above question.

## 2. *Main Results*

In this section, we present a new sufficient condition for the oscillation of all solutions of (E), under the assumption that the argument $\tau(t)$ is non-monotone and (1.3) is not satisfied. Set

$$h(t) := \sup_{s\leqslant t}\tau(s), \quad t \geqslant 0. \tag{2.1}$$

Clearly, $h(t)$ is non-decreasing, and $\tau(t)\leqslant h(t)$ for all $t\geqslant 0$.

In 2011, Braverman and Karpuz [**3**] established the following theorem.

THEOREM 2.1. *Assume that* (1.1) *holds, and*

$$\limsup_{t\to\infty}\int_{h(t)}^{t} p(s)\exp\left\{\int_{\tau(s)}^{h(t)} p(\xi)\,d\xi\right\} ds > 1, \tag{2.2}$$

*where $h(t)$ is defined by* (2.1). *Then all solutions of* (E) *oscillate.*

THEOREM 2.2. *Assume that* (1.1) *holds, and*

$$\liminf_{t\to\infty}\int_{h(t)}^{t} p(s)\exp\left\{\int_{\tau(s)}^{h(t)} p(\xi)\,d\xi\right\} ds > \frac{1}{e}, \tag{2.3}$$

*where $h(t)$ is defined by* (2.1). *Then all solutions of* (E) *oscillate.*

*Proof.* Assume, for the sake of contradiction, that there exists a non-oscillatory solution $x(t)$ of (E). Since $-x(t)$ is also a solution of (E), we can confine our discussion only to the case where the solution $x(t)$ is eventually positive. Since $\tau(t)\to\infty$ as $t\to\infty$, there is a positive number $t_1\geqslant t_0$, such that $x(\tau(t))>0$ for all $t\geqslant t_1$. Thus, from (E),

$$x'(t) = -p(t)x(\tau(t)) \leqslant 0 \quad \text{for all } t\geqslant t_1,$$

which means that $x(t)$ is an eventually non-increasing function of positive numbers. Using the fact that $x(t)$ is non-increasing and $h(t) \leqslant t$ for $t \geqslant 0$, we have $x(h(t)) \geqslant x(t) > 0$ for all $t \geqslant t_1$.

By (E)

$$\frac{x'(t)}{x(t)} + p(t)\frac{x(\tau(t))}{x(t)} = 0 \quad \text{for all } t \geqslant t_1, \tag{2.4}$$

or

$$\ln \frac{x(t)}{x(h(t))} + \int_{h(t)}^{t} p(s)\frac{x(\tau(s))}{x(s)}\, ds = 0 \quad \text{for all } t \geqslant t_1.$$

Using the Grönwall inequality,

$$\ln \frac{x(t)}{x(h(t))} + \int_{h(t)}^{t} p(s)\frac{x(h(t))}{x(s)} \exp\left\{\int_{\tau(s)}^{h(t)} p(\xi)\, d\xi\right\} ds \leqslant 0.$$

Since $h(t) \leqslant s \leqslant t$, clearly $x(h(t))/x(s) \geqslant 1$, and the last inequality becomes

$$\ln\frac{x(t)}{x(h(t))} + \int_{h(t)}^{t} p(s) \exp\left\{\int_{\tau(s)}^{h(t)} p(\xi)\, d\xi\right\} ds \leqslant 0. \tag{2.5}$$

Also, from (2.3), it follows that there exists a constant $c > 0$ such that

$$\int_{h(t)}^{t} p(s) \exp\left\{\int_{\tau(s)}^{h(t)} p(\xi)\, d\xi\right\} ds \geqslant c > \frac{1}{e}, \quad t \geqslant t_2 > t_1. \tag{2.6}$$

Combining the inequalities (2.5) and (2.6), we obtain

$$\ln \frac{x(t)}{x(h(t))} + c \leqslant 0, \quad t \geqslant t_2.$$

Thus, we have

$$e^c x(t) \leqslant x(h(t)), \quad t \geqslant t_3,$$

or

$$(ec)x(t) \leqslant x(h(t)), \quad t \geqslant t_3.$$

Repeating the above procedure, it follows by induction that for any positive integer $k$,

$$(ec)^k x(t) \leqslant x(h(t)),$$

or

$$\frac{x(h(t))}{x(t)} \geqslant (ec)^k \quad \text{for sufficiently large } t, \tag{2.7}$$

where $ec > 1$.

Since $h(t) \geqslant \tau(t)$, by (E),

$$x'(t) + p(t)x(h(t)) \leqslant 0, \quad t \geqslant t_0. \tag{2.8}$$

Integrating (2.8) from $h(t)$ to $t$, and using the fact that the function $x(t)$ is non-increasing and the function $h(t)$ is non-decreasing, we obtain

$$x(t) - x(h(t)) + \int_{h(t)}^{t} p(s)x(h(s))\, ds \leqslant 0,$$

or

$$x(t) - x(h(t)) + x(h(t))\int_{h(t)}^{t} p(s)\, ds \leqslant 0.$$

https://doi.org/10.1112/S1461157016000073 Published online by Cambridge University Press

Thus

$$x(h(t))\left(1-\int_{h(t)}^{t} p(s)\,ds\right) \geqslant 0,$$

that is,

$$f(t) := \int_{h(t)}^{t} p(s)\,ds \leqslant 1. \tag{2.9}$$

Now, we claim that

$$\liminf_{t\to\infty} f(t) > 0. \tag{2.10}$$

If not, then

$$\liminf_{t\to\infty} f(t) = 0. \tag{2.11}$$

Since $f(t)$ is bounded, then there exists a sequence $\{t_k\}$ such that $\lim_{k\to\infty} t_k = \infty$ and $\lim_{k\to\infty} f(t_k) = 0$.

Observe that

$$\tau(s) < h(t) < s < t.$$

Then, we can write

$$\int_{\tau(s)}^{t} p(\xi)\,d\xi = \int_{\tau(s)}^{s} p(\xi)\,d\xi + \int_{s}^{t} p(\xi)\,d\xi = I_1 + I_2.$$

By (2.9), it is obvious that $I_2 \leqslant 1$ and $\liminf_{t\to\infty} I_2 = 0$.

But then, from the definition of $h(t)$, it is known (see [**16**]) that

$$\liminf_{t\to\infty} \int_{h(t)}^{t} p(\xi)\,d\xi = \liminf_{t\to\infty} \int_{\tau(t)}^{t} p(\xi)\,d\xi$$

and so

$$\lim_{k\to\infty} \int_{h(t_k)}^{t_k} p(\xi)\,d\xi = \lim_{k\to\infty} \int_{\tau(t_k)}^{t_k} p(\xi)\,d\xi = 0.$$

Thus, we have $\liminf_{t\to\infty} I_1 = 0$. Consequently,

$$\lim_{k\to\infty} \int_{\tau(s_k)}^{t_k} p(\xi)\,d\xi = 0$$

and, consequently,

$$\lim_{k\to\infty} \int_{\tau(s_k)}^{h(t_k)} p(\xi)\,d\xi = 0.$$

Therefore, we obtain

$$\lim_{k\to\infty} \int_{h(t_k)}^{t_k} p(s)\exp\left\{\int_{\tau(s_k)}^{h(t_k)} p(\xi)\,d\xi\right\} ds = \lim_{k\to\infty} \int_{h(t_k)}^{t_k} p(s)\,ds = 0,$$

which contradicts (2.3).

So, since (2.10) is satisfied, it follows that there exists a constant $d > 0$ such that

$$0 < d \leqslant \int_{h(t)}^{t} p(s)\,ds \leqslant 1. \tag{2.12}$$

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Thus, there exists a real number $t^* \in (h(t), t)$, for all $t \geqslant t_1$ such that

$$0 < \frac{d}{2} \leqslant \int_{h(t)}^{t^*} p(s)\, ds \leqslant 1 \tag{2.13}$$

and

$$0 < \frac{d}{2} \leqslant \int_{t^*}^{t} p(s)\, ds \leqslant 1. \tag{2.14}$$

Integrating (2.8) from $h(t)$ to $t^*$, and using the fact that the function $x(t)$ is non-increasing and the function $h(t)$ is non-decreasing, gives

$$x(t^*) - x(h(t)) + \int_{h(t)}^{t^*} p(s)x(h(s))\, ds \leqslant 0,$$

or

$$x(t^*) - x(h(t)) + x(h(t^*)) \int_{h(t)}^{t^*} p(s)\, ds \leqslant 0.$$

Thus, by (2.13),

$$-x(h(t)) + x(h(t^*))\frac{d}{2} \leqslant 0. \tag{2.15}$$

Integrating (2.8) from $t^*$ to $t$, and using the same arguments gives

$$x(t) - x(t^*) + \int_{t^*}^{t} p(s)x(h(s))\, ds \leqslant 0,$$

or

$$x(t) - x(t^*) + x(h(t)) \int_{t^*}^{t} p(s)\, ds \leqslant 0.$$

Thus, by (2.14),

$$-x(t^*) + x(h(t))\frac{d}{2} \leqslant 0. \tag{2.16}$$

Combining the inequalities (2.15) and (2.16), we obtain

$$x(t^*) \geqslant x(h(t))\frac{d}{2} \geqslant x(h(t^*))\left(\frac{d}{2}\right)^2,$$

or

$$\frac{x(h(t^*))}{x(t^*)} \leqslant \left(\frac{2}{d}\right)^2 < +\infty,$$

that is, $\liminf_{t\to\infty} x(h(t))/x(t)$ exists. This contradicts (2.7).

The proof of the theorem is complete. □

EXAMPLE 2.1. Consider the retarded differential equation

$$x'(t) + \frac{10}{11e}x(\tau(t)) = 0, \quad t \geqslant 0, \tag{2.17}$$

where

$$\tau(t) = \begin{cases} t-1 & \text{if } t \in [3k, 3k+1], \\ -3t+12k+3 & \text{if } t \in [3k+1, 3k+2], \\ 5t-12k-13 & \text{if } t \in [3k+2, 3k+3], \end{cases} \quad k \in \mathbb{N}_0.$$

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By (2.1), we see that

$$h(t) := \sup_{s\leqslant t} \tau(s) = \begin{cases} t-1 & \text{if } t\in[3k,3k+1], \\ 3k & \text{if } t\in[3k+1,3k+2.6], \\ 5t-12k-13 & \text{if } t\in[3k+2.6,3k+3], \end{cases} \quad k\in\mathbb{N}_0.$$

Observe that the function $f:\mathbb{R}_0\to\mathbb{R}_+$ defined as

$$f(t) = \int_{h(t)}^{t} p(s)\exp\left\{\int_{\tau(s)}^{h(t)} p(\xi)\,d\xi\right\} ds$$

attains its minimum at $t=3k$, $k\in\mathbb{N}_0$, which is equal to

$$\begin{aligned} f_{\min} &= \int_{h(3k)}^{3k} p(s)\exp\left\{\int_{\tau(s)}^{h(3k)} p(\xi)\,d\xi\right\} ds \\ &= \frac{10}{11e}\int_{3k-1}^{3k}\exp\left\{\frac{10}{11e}\int_{s-1}^{3k-1} d\xi\right\} ds = \frac{10}{11e}\int_{3k-1}^{3k}\exp\left(\frac{10}{11e}(3k-s)\right) ds \\ &= \exp\left(\frac{10}{11e}\right)-1 \simeq 0.397151967 \end{aligned}$$

and therefore

$$\liminf_{t\to\infty}\int_{h(t)}^{t} p(s)\exp\left\{\int_{\tau(s)}^{h(t)} p(\xi)\,d\xi\right\} ds \simeq 0.397151967 > \frac{1}{e},$$

that is, condition (2.3) of Theorem 2.2 is satisfied and thus all solutions of (2.17) oscillate.

Observe, however, that

$$\liminf_{t\to\infty}\int_{\tau(t)}^{t} p(s)\,ds = \liminf_{t\to\infty}\int_{t-1}^{t}\frac{10}{11e}\,ds = \frac{10}{11e} < \frac{1}{e},$$

that is, condition (1.3) is not satisfied.

## *References*


1. O. Arino, I. Győri and A. Jawhari, 'Oscillation criteria in delay equations', *J. Differential Equations* 53 (1984) 115–123.
2. L. Berezansky and E. Braverman, 'On some constants for oscillation and stability of delay equations', *Proc. Amer. Math. Soc.* 139 (2011) no. 11, 4017–4026.
3. E. Braverman and B. Karpuz, 'On oscillation of differential and difference equations with non-monotone delays', *Appl. Math. Comput.* 218 (2011) 3880–3887.
4. A. Elbert and I. P. Stavroulakis, 'Oscillations of first order differential equations with deviating arguments, University of Ioannina T. R. No 172, 1990', *Recent trends in differential equations*, World Scientific Series in Applicable Analysis 1 (World Scientific, River Edge, NJ, 1992) 163–178.
5. A. Elbert and I. P. Stavroulakis, 'Oscillation and non-oscillation criteria for delay differential equations', *Proc. Amer. Math. Soc.* 123 (1995) 1503–1510.
6. L. H. Erbe, Qingkai Kong and B. G. Zhang, *Oscillation theory for functional differential equations* (Marcel Dekker, New York, 1995).
7. L. H. Erbe and B. G. Zhang, 'Oscillation of first order linear differential equations with deviating arguments', *Differential Integral Equations* 1 (1988) 305–314.
8. N. Fukagai and T. Kusano, 'Oscillation theory of first order functional differential equations with deviating arguments', *Ann. Mat. Pura Appl.* 136 (1984) 95–117.
9. K. Gopalsamy, *Stability and oscillations in delay differential equations of population dynamics*, Mathematics and its Applications 74 (Kluwer, Dordrecht, 1992).

**10.** M. K. Grammatikopoulos, R. G. Koplatadze and I. P. Stavroulakis, 'On the oscillation of solutions of first order differential equations with retarded arguments', *Georgian Math. J.* 10 (2003) 63–76.
**11.** I. Győri and G. Ladas, *Oscillation theory of delay differential equations with applications* (Clarendon Press, Oxford, 1991).
**12.** B. R. Hunt and J. A. Yorke, 'When all solutions of $x' = \sum q_i(t)x(t - T_i(t))$ oscillate', *J. Differential Equations* 53 (1984) 139–145.
**13.** R. G. Koplatadze and T. A. Chanturija, 'Oscillating and monotone solutions of first-order differential equations with deviating arguments', *Differ. Uravn.* 8 (1982) 1463–1465 (in Russian).
**14.** G. Ladas, 'Sharp conditions for oscillations caused by delay', *Appl. Anal.* 9 (1979) 93–98.
**15.** G. Ladas, V. Laskhmikantham and J. S. Papadakis, 'Oscillations of higher-order retarded differential equations generated by retarded arguments', *Delay and functional differential equations and their applications* (Academic Press, New York, 1972) 219–231.
**16.** G. S. Ladde, V. Lakshmikantham and B. G. Zhang, *Oscillation theory of differential equations with deviating arguments*, Monographs and Textbooks in Pure and Applied Mathematics 110 (Marcel Dekker, Inc., New York, 1987).
**17.** A. D. Myshkis, 'Linear homogeneous differential equations of first order with deviating arguments', *Uspekhi Mat. Nauk* 5 (1950) 160–162 (in Russian).
**18.** X. H. Tang, 'Oscillation of first order delay differential equations with distributed delay', *J. Math. Anal. Appl.* 289 (2004) 367–378.

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