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# ON THE CHARACTERS OF AFFINE KAC–MOODY GROUPS

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Let $G$ be an affine Kac–Moody group over $\mathbb{C}$, and $V^\omega$ an integrable simple quotient of a Verma module for $\mathfrak{g}$. Let $G^{\min}$ be the subgroup of $G$ generated by the maximal algebraic torus T, and the real root subgroups.

It is shown that $\delta\in\Phi_+^{\mathrm{im}}$ (the least positive imaginary root) gives a character $\delta\in\mathrm{Hom}(G,\mathbb{C}^*)$ such that the pointwise character $\chi^\omega$ of $V^\omega$ may be defined on $G^{\min}\cap G^{>1}$.

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## 0. Introduction

A Kac–Moody group $G$ over $\mathbb{C}$, is associated to a pair $(A,\mathfrak{h}_{\mathbb{Z}})$ where $A$ is a generalized, indecomposable, Cartan $n\times n$ matrix of rank $l$, and $\mathfrak{h}_{\mathbb{Z}}$ is a free $\mathbb{Z}$-module such that $n-l=\mathrm{rank}\ \mathfrak{h}_{\mathbb{Z}}-n$. Then $G$ has a $(B,N)$ pair forming a Tits system with Weyl group $W=N/(B\cap N)$ (see also [**12**, **4**, **9**]).

The Lie algebra $\mathfrak{g}$ of $G$ has a root space decomposition, and it is required that the roots $\Phi\subseteq\mathrm{Hom}(\mathfrak{h}_{\mathbb{Z}},\mathbb{Z})=:\mathfrak{h}_{\mathbb{Z}}^*$. We have $\Phi=\Phi^{\mathrm{re}}\cup\Phi^{\mathrm{im}}$, where $\Phi^{\mathrm{re}}$ is the $W$ orbit of the simple roots and $\Phi^{\mathrm{im}}=\Phi\backslash\Phi^{\mathrm{re}}$.

If $G$ is affine (that is $A$ is symmetrizable, positive semidefinite) then there is an analytic construction as a loop group [**3**]. Take a central extension $S^1\to\tilde{L}K_{(0)}\to LK_{(0)}$ of the loop group of a compact, connected almost simple Lie group $K_{(0)}$ by the circle $S^1$ (this is obtained [**8**] from a closed, left invariant integral 2-form on $LK_{(0)}$, if $K_0$ is simply connected). Imbed $K_{(0)}$ in a group of finite dimensional unitary matrices, and let $L_{\mathrm{pol}}K_{(0)}$ be the dense subgroup of $LK_{(0)}$ consisting of $\gamma$: $S^1\to K_{(0)}$ with each matrix entry of $\gamma(z)$ a finite Laurent polynomial in $z$. The loop algebra $L\mathfrak{k}_{(0)}=\mathbb{C}[z,z^{-1}]\otimes_{\mathbb{R}}\mathfrak{k}_{(0)}$ has a derivation $d$ by $z\frac{d}{dz}\otimes 1$, and on $\tilde{L}\mathfrak{k}_{(0)}, d(c)=0$. In 3 the untwisted affine Lie algebra is $\mathfrak{g}=\mathbb{C}d\oplus\mathbb{C}c\oplus L\mathfrak{k}_{(0)\mathbb{C}}$. There is a subgroup $S^1$ of the group of diffeomorphisms of the circle, having Lie algebra $\mathbb{R}d$ as a subalgebra of the Virasoro algebra. Set $G_1=(S^1\ltimes\tilde{L}_{\mathrm{pol}}K_{(0)})_{\mathbb{C}}$. The Lie algebra $\mathfrak{g}$ decomposes (restricting the adjoint representation of $G_1$) by characters of $S^1\otimes T_{(0)}$, where $T_{(0)}$ is a maximal torus of $K_{(0)}$. The Weyl group $W=W_0\ltimes\Upsilon$ where $W_0$ is the Weyl group of $(K_{(0)},T_{(0)})$ and $\Upsilon$ is the cocharacter lattice $\mathrm{Hom}(S^1,T_{(0)})$. The "twisted" loop groups are obtained by the outer automorphisms of $\mathfrak{k}_{(0)\mathbb{C}}$ of orders 2, 3.

An algebraic construction (as in [**6**, **9**]) for general $G$ is used here. This is obtained as a subgroup of $\mathrm{GL}(V)$ where $V$ is the direct sum of the "integrable" simple quotients $V^\omega$

of Verma modules for $\mathfrak{g}$. And see **[12]** for the Chevalley–Demazure, and Tits group functor on the category of rings.

To briefly describe a correspondence between the analytic approach and the algebraic of (1.3), (3.1):

Let $B_1^-$ be the group of polynomial maps $\gamma$: $\{z\in\mathbb{C}; |z|\leqq 1\}\to K_{(0)\mathbb{C}}$ restricted to $S^1$, with $\gamma(0)\in B_{(0)}^-$ where $B_{(0)}^-$ is the opposite Borel subgroup to $B_{(0)}\leqq K_{(0)\mathbb{C}}$, the latter associated to a choice of positive roots $\Phi_{0+}$ for $(K_{(0)}, T_{(0)})$. Let $U_{(0)\alpha}$ be the root subgroup in $K_{(0)\mathbb{C}}$ of $\alpha\in\Phi_{0+}$, and define $U_{\alpha_i}=\{\gamma_g\in B_1;\ \operatorname{Im}\gamma_g=\{g\},\ g\in U_{(0)\alpha_i}\}$, $i\neq 0$, $U_{\alpha_0}=\{\gamma\in B_1; \gamma^{(1)}(0)\in U_{(0)-\theta}, \gamma^{(s)}(0)=0, s\neq 1\}$, $\theta\in\Phi_{0+}$ the highest root. Let $B^-=S^1\ltimes\tilde{B}_1^-$. Over a completion of $G_1/B^-$ there is a holomorphic $G_1$ vector bundle $G_1\times_{B^-}\mathbb{C}_\omega, \omega$ a character of $B^-$ which is trivial on $U^-$. The Borel–Weil theorem for compact Lie groups has a generalization to loop groups (see for example **[8]**). In particular the $G_1$-space of holomorphic sections $H^0(\omega)$ is $\mathfrak{g}$ equivalent to $V^\omega$. The group $G$ in Section 3 is the homomorphic image in $\mathrm{GL}(V)$ of $G_1$ (and see **[8**, p. 144] for the Bruhat decomposition of $G_1$).

In this paper, for $G$ affine, we give the subdomain of $G^{\min}$ on which a pointwise character $\chi^\omega$ of the representation $(V^\omega, R), \omega\in\mathscr{I}\mathrm{nt}_+\cap\mathfrak{h}_{\mathbb{Z}}^*$ can be defined. Here $G^{\min}$ is the subgroup of $G$ generated by the algebraic torus $T=\mathfrak{h}_{\mathbb{Z}}\otimes_{\mathbb{Z}}\mathbb{C}^*$ and the root subgroups $U_\alpha$, $\alpha\in\Phi^{\mathrm{re}}$. We show that this domain is given by $G^{>1}=\{g\in G; |\delta(g)|>1\}$ where $\delta\in\Phi_+^{\mathrm{im}}$ is the least positive imaginary root trivially extended to $\delta\in\operatorname{Hom}(G,\mathbb{C}^*)$. The proof holds for twisted $G$, and the present approach does not exploit the topology as a loop group. The subdomain in $T$ on which $\chi^\omega$ behaves well analytically is known in general (**[4]**, and also **[10]** for $N$ the normalizer of $T$ in $G$). Then to prove that $G^{>1}$ is the set of elements of $G$ acting as Hilbert–Schmidt operators on $V^\omega$, we use that (1) $V^\omega$ is a pre-Hilbert space with $K$ acting as unitary operators (2) the complex Iwasawa decomposition $G=KB$, and (3) a Levi subgroup $L_1$ of $G$ of finite type has a $K_1TK_1$ decomposition, $K_1=L_1\cap K$. These elements $g\in G^{>1}$ have a trace which is denoted $\chi^\omega(g)$, and $\chi^\omega$ is shown to be $G$-conjugation invariant there. A corollary to this result is an affirmative answer to the remark in [**8**, *p.* 275].

## 1. Notation and preliminary results

**1.1.** Let $G$ be a Kac–Moody group associated to the root datum $(\mathfrak{h}_{\mathbb{Z}}, \Delta^\vee, \Delta)$. That is (see also (1.2). (1.3)) from a general Cartan $n\times n$ matrix $A$ of rank $l$ we take a free $\mathbb{Z}$-module $\mathfrak{h}_{\mathbb{Z}}$ of finite rank and $\mathbb{Z}$ independent subsets $\Delta^\vee=\{h_1,\ldots,h_n\}\subseteq\mathfrak{h}_{\mathbb{Z}}$ "the simple coroots", $\Delta=\{\alpha_1,\ldots,\alpha_n\}\subseteq\mathfrak{h}_{\mathbb{Z}}^*=\operatorname{Hom}(\mathfrak{h}_{\mathbb{Z}},\mathbb{Z})$ "the simple roots" with $\alpha_j(h_i)=a_{ij}, \forall i, j$ and $n-l=\operatorname{rank}\mathfrak{h}_{\mathbb{Z}}-n$.

The Weyl group $W$ of $(\mathfrak{h}_{\mathbb{Z}},\Delta^\vee,\Delta)$ is a Coxeter group generated by reflections $r_i$: $\mathfrak{h}_{\mathbb{Z}}\to\mathfrak{h}_{\mathbb{Z}}$, $r_i(h)=h-\alpha_i(h)h_i, h\in\mathfrak{h}_{\mathbb{Z}}$ and acts (faithfully) contragrediently on $\mathfrak{h}_{\mathbb{Z}}^*$.

There is a Lie algebra $\mathfrak{g}=\mathfrak{g}(A)$ with bracket [ ] and adjoint representation ad, generated by $\mathfrak{h}=\mathfrak{h}_{\mathbb{Z}}\otimes_{\mathbb{Z}}\mathbb{C}, e_i, f_i, i=1,\ldots,n$ with relations $[h,h']=0$, $[h,e_i]=\alpha_i(h)e_i$, $[h,f_i]=-\alpha_i(h)f_i$, $[e_i,f_j]=\delta_{ij}h_j$, $(\operatorname{ad}e_i)^{-a_{ij}+1}(e_j)=0$, $(\operatorname{ad}f_i)^{-a_{ij}+1}(f_j)=0$, $\forall h,h'\in\mathfrak{h}, i,$

$j \in \{1, \ldots, n\}$. Also by taking the factor Lie algebra, we may assume that the $\mathfrak{h}$ radical of $\mathfrak{g}$ is zero; that is every ideal of $\mathfrak{g}$ which intersects $\mathfrak{h}$ trivially is zero.

Then $\mathfrak{g}$ is $\mathbb{Z}\Delta$-graded and has a triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ over $\mathbb{C}$. If $A$ is indecomposable, then $\mathfrak{g}$ is simple if and only if $\det A \neq 0$. The root space decomposition is $\mathfrak{g} = \sum^{\oplus}_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha$ where $\mathfrak{g}_\alpha = \{x \in \mathfrak{g}; [hx] = \alpha(h)x, \forall h \in \mathfrak{h}\}$ with roots $\Phi = \{\alpha \in \mathfrak{h}^*; \mathfrak{g}_\alpha \neq 0\}$. The Cartan subalgebra $\mathfrak{h} = \mathfrak{g}_0$. We have $\mathfrak{g}_{\alpha_i} = \mathbb{C}e_i, \mathfrak{g}_{-\alpha_i} = \mathbb{C}f_i$ and $\mathfrak{n}_\pm = \sum^{\oplus}_{\alpha \in \Phi_+} \mathfrak{g}_{\pm\alpha}$ where $\Phi_+ = \Phi \cap \mathbb{N}\Delta, \Phi_- = -\Phi_+$.

The root system $\Phi$ is invariant under $W$. The multiplicity of the root $\alpha$, mult $\alpha$ is dim $\mathfrak{g}_\alpha = \dim \mathfrak{g}_{w(\alpha)}, w \in W$. Let $\Phi^{re} = W.\Delta$ the real roots, $\Phi^{im} = \Phi \backslash \Phi^{re}$ the imaginary roots. Then mult $\alpha = 1$, $\forall \alpha \in \Phi^{re}$. The set of positive imaginary roots $\Phi^{im}_+$ is $W$-invariant.

If $A$ is symmetrizable (see also (2.1)) then $\mathfrak{g}$ carries a symmetric nondegenerate $\mathbb{C}$-bilinear form $(\,,)$, which is infinitesimally invariant under the adjoint representation ad. This restricts to a nondegenerate form on $\mathfrak{h}$, and gives an isomorphism $\nu: \mathfrak{h} \to \mathfrak{h}^*, \nu(h)(h') = (h, h'), \forall h, h' \in \mathfrak{h}$.

**1.2.** The universal enveloping algebra $u(\mathfrak{g})$ is $\mathbb{Z}\Delta$-graded. Let $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}_+$, a standard Borel subalgebra. The line $\mathbb{C}_\omega, \omega \in \mathfrak{h}^*$ is a $u(\mathfrak{b})$-module by $x.1 = 0$, $x \in \mathfrak{n}_+, h.1 = \omega(h)1, h \in \mathfrak{h}$. Then define the Verma module $M^\omega = u(\mathfrak{g}) \otimes_{u(\mathfrak{b})} \mathbb{C}_\omega$ with $u(\mathfrak{g})$ acting on the left. If $M'$ is the maximal $\mathfrak{g}$-submodule not containing $1 \otimes \mathbb{C}_\omega$, then $V^\omega = M^\omega/M'$ is simple. In particular $V^\omega = \sum^{\oplus}_{\lambda \in \mathfrak{h}^*} V_\lambda$ an $\mathfrak{h}$-diagonalization into finite dimensional weight spaces. Denote the set of weights by $P^\omega := P(V^\omega)$. This is partially ordered by the natural filtration of $u(\mathfrak{g})$, with the highest weight $\omega$ minimal. If $\alpha = \sum_i c_i\alpha_i \in \mathbb{N}\Delta$, the height $\mathrm{ht}(\alpha) = \sum_i c_i$. The support $\mathrm{supp}(\alpha) = \{i; c_i \neq 0\}$ is connected as a subdiagram of the Coxeter–Dynkin diagram of $W$, if $\alpha \in \Phi_+$. And if $\lambda = \omega - \sum_i c_i\alpha_i \in \omega - \mathbb{N}\Delta$, the depth $\mathrm{dep}(\lambda) := \sum_i c_i$.

Define for root datum $(\mathfrak{h}_\mathbb{Z}, \Delta^\vee, \Delta)$, $\mathscr{I}\mathrm{nt} = \{\lambda \in \mathfrak{h}^*; \lambda(h_i) \in \mathbb{Z}, i = 1, \ldots, n\}$ "the lattice of integral forms", $\mathscr{I}\mathrm{nt}_+ = \{\lambda \in \mathfrak{h}^*; \lambda(h_i) \in \mathbb{N}, i = 1, \ldots, n\}$ "the dominant integral forms", $\mathscr{I}\mathrm{nt}_{++} = \{\lambda \in \mathscr{I}\mathrm{nt}_+; \lambda(h_i) \neq 0, i = 1, \ldots, n\}$ "the strictly dominant forms". Therefore $\Phi \subseteq \mathscr{I}\mathrm{nt}$. The "fundamental weights" are $\{\omega_i; i = 1, \ldots, n\}$ which on restriction are dual to $\Delta^\vee \otimes 1$. For $\omega \in \mathscr{I}\mathrm{nt}_+, P^\omega$ is $W$-invariant, and the multiplicity $\mathrm{mult}_\omega(\lambda) = \mathrm{mult}_\omega(w\lambda)$, $\forall w \in W$, $\forall \lambda \in P^\omega$. The root datum is "simply connected" if $\omega_i \in \mathfrak{h}^*_\mathbb{Z} \subseteq \mathfrak{h}^*, \forall i$, [**4, 10**].

**1.3.** Let the conjugate linear involution $\underline{\omega}_0$ on $\mathfrak{g}$ be given by $\underline{\omega}_0(e_i) = -f_i$, $\underline{\omega}_0(f_i) = -e_i, i \in \{1, \ldots, n\}$, $\underline{\omega}_0(h) = -h, h \in \mathfrak{h}_\mathbb{R} := \mathfrak{h}_\mathbb{Z} \otimes_\mathbb{Z} \mathbb{R}$. If $A$ is symmetrizable there is a hermitian form $(\,,)_0$ on $\mathfrak{g}$ by $(x, y)_0 = -(x, \underline{\omega}_0(y)), x, y \in \mathfrak{g}$.

Define $V = \sum^{\oplus}_{\omega \in \mathscr{I}\mathrm{nt}_+ \cap \mathfrak{h}^*_\mathbb{Z}} V^\omega$, then for each $i \in \{1, \ldots, n\}$ the one parameter subgroups $U_i := \{\exp ce_i; c \in \mathbb{C}\}, \underline{\omega}_0(U_i) = \{\exp cf_i; c \in \mathbb{C}\}$ generate a subgroup $G_i \leq \mathrm{GL}(V)$ isomorphic to $\mathrm{SL}(2, \mathbb{C})$. The algebraic torus $T := \mathfrak{h}_\mathbb{Z} \otimes_\mathbb{Z} \mathbb{C}^*$ has character group $\mathfrak{h}^*_\mathbb{Z}$. With $\mathfrak{n}^{(i)} = \sum^{\oplus}_{\alpha \in \Phi_+, \mathrm{ht}(\alpha) > i} \mathfrak{g}_\alpha$ let $U^{(i)}$ be the unipotent algebraic group with Lie algebra $\mathfrak{n}_+/\mathfrak{n}^{(i)}, i \in \mathbb{N}$. Let $U = \lim_{\leftarrow} U^{(i)}$ the inverse limit, and $B = TU$ a semidirect product. Finally $G \leq \mathrm{GL}(V)$ is defined to be the group generated by $B$ and $G_i, i = 1, \ldots, n$. The involution $\underline{\omega}_0$ lifts to $G$. There are monomorphisms $\phi_i: G_i \to G$ with $\phi_i\{\begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}; c \in \mathbb{C}\} = U_i, i \in \{1, \ldots, n\}$, see [**6**].

Let $v_\omega$ be the highest weight vector of $V^\omega, \omega \in \mathscr{I}\mathrm{nt}_+ \cap \mathfrak{h}^*_\mathbb{Z}$. Now $B = \{g \in G; g\sum_\omega \mathbb{C}v_\omega = \sum_\omega \mathbb{C}v_\omega\}$ (the Borel subgroup with Lie algebra $\mathfrak{b}$). We may regard the maximal torus

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$T=B\cap\omega_0(B)$. Also let $N=N_G(T)$ the normalizer of $T$ in $G$. With $n_i:=\phi_i\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)=(\exp e_i)(\exp -f_i)(\exp e_i)$, $i=1,\dots,n$ and $N_{(1)}:=\langle n_i; i=1,\dots,n\rangle$, there is an exact sequence $1\to T_{(2)}\to N_{(1)}\to W$, where $T_{(2)}:=\langle n_i^2; i=1,\dots,n\rangle$, $n_i\mapsto r_i$. Then $N=\langle T,N_{(1)}\rangle$, $T_{(2)}=N_{(1)}\cap T=\{t\in G'\cap T; t^2=1\}\simeq\mathbb{Z}_2^n$, and $W\to N/T, r_i\mapsto n_iT$ is an isomorphism.

For any $\alpha\in\Phi_+^{re}$ let $i\in\{1,\dots,n\}, w\in W$ be such that $w(\alpha_i)=\alpha$ and define root subgroup $U_\alpha=nU_in^{-1}, n\in N, nT=w$. Each such $U_\alpha, \alpha\in\Phi_+^{re}$ is normalized by $T$ with $tu_i(c)t^{-1}=u_i(\alpha_i(t)c), t\in T, c\in\mathbb{C}$ where $u_i(c):=\phi_i\left(\begin{smallmatrix}1&c\\0&1\end{smallmatrix}\right)$ **[5, 9]**.

Let $U^{\min}=\langle U_\alpha; \alpha\in\Phi_+^{re}\rangle$ and $B^{\min}=TU^{\min}\leqq B$. And $G^{\min}:=\langle T, G_i; i=1,\dots,n\rangle\leqq G$.

The group $G$ acts on $V^\omega$ by representation $R$, and also $G^{\min}$ acts on $\mathfrak{g}$ by the adjoint representation Ad. In fact if $(V,\phi)$ is a representation of $\mathfrak{g}$ such that the action of $\mathfrak{h}$ lifts to $T$ and $e_\alpha, f_i$ act locally finitely on $V$, $e_\alpha\in\mathfrak{g}_\alpha\hookrightarrow\mathfrak{n}_+/\mathfrak{n}^{(j)}, \mathrm{ht}(\alpha)\leqq j, \forall\alpha\in\Phi_+, \forall i, j$, then there is $(V,\mathrm{R})$ of $G$ satisfying (with $\exp:\mathfrak{g}_f\to G^{\min}$ the exponential mapping, having domain $\mathfrak{g}_f=\{y\in\mathfrak{g}; y$ acts locally finitely on $\mathfrak{g}$ by ad$\}$), $\mathrm{R}(\exp x)=\exp\phi(x), x\in\mathfrak{g}_f$. Thus $\phi=d\mathrm{R}$ the differential of R, $\mathrm{ad}=d(\mathrm{Ad})$. And $d\mathrm{R}(\mathrm{Ad}(g)x)=\mathrm{R}(g)d\mathrm{R}(x)\mathrm{R}(g)^{-1}$, $g\in G^{\min}$, $x\in\mathfrak{g}_f$.

We note that $R(n)V_\lambda=V_{w\lambda}$ and $\mathrm{Ad}(n)\mathfrak{g}_\alpha=\mathfrak{g}_{w(\alpha)}$, $\forall\lambda\in P^\omega$, $\forall\alpha\in\Phi$ where $n\in N, nT=w\in W$.

The group $G$ is said to have Lie algebra $\mathfrak{g}$ and is associated to the root datum $(\mathfrak{h}_{\mathbb{Z}},\Delta^\vee,\Delta)$.

The properties of a Tits system are satisfied. The group $G$ has $(B,N)$ pair with Coxeter group $W$. The Bruhat decomposition of $G$ into a disjoint union of double cosets of $B$ in $G$ is $G=\bigcup_{w\in W}BwB$; that is there is a bijection between the double cosets $BnB$ and $W$ under the natural epimorphism $N\to W$. Also to multiply double cosets

$$(BsB)(BwB)=BswB \text{ if } l(sw)=l(w)+1$$

$$=BwB\cup BswB \text{ if } l(sw)=l(w)-1$$

$w\in W, s=r_i, i\in\{1,\dots,n\}$, where $l(.)$ is the length function on $W$, **[12]**.

**1.4.** Let $K=G^{\omega_0}$ the subgroup of fixed points of $\omega_0$; this is called the "unitary form". The complex Iwasawa decomposition $G=KB$ holds **[5]**. Moreover $G^{\min}=KB^{\min}$.

From now on, unless stated otherwise, the superscript "min" will be omitted.

**Proposition 1.** *Let $\alpha\in\Phi$ be such that the orbit $W.\alpha=\{\alpha\}$.*
*Then $\alpha\in\Phi^{im}$ with $\alpha$ isotropic ($(\alpha,\alpha)=0$). And $\alpha$ as an element of the character group $\mathfrak{h}_{\mathbb{Z}}^*$ extends trivially to $\alpha\in\mathrm{Hom}(G,\mathbb{C}^*)$*

**Proof.** Let $\alpha\in\Phi_+$ with $w\alpha=\alpha, \forall w\in W$. As $w\alpha=\alpha_i\Rightarrow\alpha=\alpha_i\Rightarrow r_i\alpha=-\alpha_i=\alpha$, have $\alpha\in\Phi^{im}$. Also $\alpha=\sum_{i\in\mathrm{supp}\,\alpha}c_i\alpha_i\Rightarrow(\alpha,\alpha)=\sum_{i\in\mathrm{supp}\,\alpha}c_i(\alpha,\alpha_i)=0$. The support of $\alpha$, $\mathrm{supp}\,\alpha$, is connected of affine type (see (2.1)).

In fact **[4]** conversely, $\alpha\in\mathbb{N}\Delta$, $\mathrm{supp}\,\alpha$ connected and affine $\Rightarrow\alpha\in\Phi_+^{im}$ and $\alpha$ is isotropic with $w\alpha=\alpha, \forall w\in W$.

Let $G'$ be the derived group of $G$. Decompose $T=T_0T_1$ with $T_1\cap G'=\{1\}$ and $G=T_1G'$ a semidirect product. Define for $w\in W, \Phi(w)=\{\alpha\in\Phi_+; w^{-1}\alpha\in\Phi_-\}$ and $U_w=\prod_{\beta\in\Phi(w)}U_\beta$. There is a bijection $U_w\times B\to BnB=:C(w)$, where $nT=w$, by $(u,b)\mapsto unb$.

As $\alpha$ is zero on $\Delta^\vee$ define $\alpha(G')=1$. We have $U<G'$, $U\lhd B$, $G_i\leqq G'\ \forall i$, $N_{(1)}<G'$. Now

$W$ acts on $T$ by $t \mapsto ntn^{-1}$. Therefore with $unb = un'u_1 t \in C(w)$, $n' \in N_{(1)}$, $u_1 \in U$, $t \in T$ put $\alpha(unb) = \alpha(t)$. To check that $\alpha$ is a homomorphism take $g_1 \in C(w_1)$, $g_2 \in C(w_2)$; now $g_1 = x_1 t_1$, $g_2 = x_2 t_2$, $x_j \in Bn_jB \cap G'$, $j = 1, 2, t_1, t_2 \in T_1$ gives $g_1 g_2 = x_1(t_1 x_2 t_1^{-1}) t_1 t_2$.
Alternatively, after $T = T_0 T_1$ one could observe that $(t_1 g_1)(t_2 g_2) = (t_1 t_2)(t_2^{-1} g_1 t_2 g_2)$.
□

**Proposition 2.** *Let $L_1 = \langle T, U_\alpha, \omega_0(U_\alpha); \alpha \in \Phi_{1+} \rangle$ be a Levi subgroup of $G$ of finite type $\Phi_{1+} \subseteq \Phi_+$.*
*Then $L_1 = K_1 T K_1$ where $K_1 = L_1' \cap B$.*

**Proof.** It is clear from (1.3) that $L_1 = K_1 B_1, B_1 = L_1 \cap B$.
A real finite dimensional semisimple Lie algebra $\mathfrak{g}_0$ has Cartan subalgebras $\mathfrak{h}_0$, the set of which having finitely many conjugacy classes under the adjoint group $\operatorname{Int}\mathfrak{g}_0 = \operatorname{Ad} G_0$, ($G_0$ connected with Lie algebra $\mathfrak{g}_0$). If $\mathfrak{g}_0 = \mathfrak{k}_1 \oplus \mathfrak{p}$ is a Cartan decomposition with involution $\theta$, then under the action of the inner automorphisms $\operatorname{Int}\mathfrak{g}_0$ we can assume that $\mathfrak{h}_0$ is $\theta$ stable. There are two extreme conjugacy classes; writing $\mathfrak{h} = \mathfrak{a}_{\mathfrak{k}_1} \oplus \mathfrak{a}_\mathfrak{p}$ these are the fundamental class, when $\mathfrak{a}_{\mathfrak{k}_1}$ is maximal abelian in $\mathfrak{k}_1$, and the split class, when $\mathfrak{a}_\mathfrak{p}$ is maximal abelian in $\mathfrak{p}$. The pair $(\mathfrak{g}_0, \mathfrak{h}_0)$ gives root system $\Phi_0$, and with the split class $(\mathfrak{g}_0, \mathfrak{a}_1)$ the restricted root system $\Psi_0$. There is **[13]** the real Iwasawa decomposition $\mathfrak{g}_0 = \mathfrak{k}_1 \oplus \mathfrak{a}_1 \oplus \mathfrak{n}_1$ which is globally $G_0 = K_1 A_1 N_1, A_1 = \exp\mathfrak{a}_1$. Also $\mathfrak{g}_0$ has one conjugacy class of Cartan subalgebras $\Leftrightarrow \mathfrak{a}_{\mathfrak{k}_1}$ is maximal abelian in $\mathfrak{k}_1$, (here $\mathfrak{h}_0 = \mathfrak{a}_{\mathfrak{k}_1} \oplus \mathfrak{a}_1$). Since any two maximal abelian subalgebras in $\mathfrak{p}$ are conjugate under $K_1, \mathfrak{g}_0 = \mathfrak{k}_1 \oplus \bigcup_{k \in K_1} \operatorname{Ad}(k)\mathfrak{a}_1$, and so $G_0 = K_1 A_1 K_1$.
In our situation $K_1$ is maximal compact in $G_0 := K_{1\mathbb{C}} \leqq L_1$ and $\mathfrak{p} = \sqrt{-1}\mathfrak{k}_1$. Then (the centralizer of $\mathfrak{a}_1$ in $\mathfrak{k}_1$)$\mathfrak{m}_0 := Z_{\mathfrak{k}_1}(\mathfrak{a}_1) = \sqrt{-1}\mathfrak{a}_1$ is a Cartan subalgebra of $\mathfrak{k}_1$. Thus $\Psi_0 = \Phi_0$. And if $M_0 := Z_{K_1}(\mathfrak{a}_1)$, then $B_0 = M_0 A_1(\theta(N_1))$ is a complex Lie subgroup of $G_0$, as $\mathfrak{b}_0 = \mathfrak{m}_0 \oplus \mathfrak{a}_1 \oplus \theta\mathfrak{n}_1 = \mathfrak{m}_{0\mathbb{C}} \oplus \sum_{\alpha \in \Phi_{0+}}^{\oplus} \mathfrak{g}_{-\alpha}$, and is closed.
The complex torus $T$ has Lie algebra $\mathfrak{h}$. And $T = T_0 T_1$ with $T_0 \leqq G_0$ having Lie algebra $\mathfrak{h}_0 = \mathfrak{m}_{0\mathbb{C}}$, $\mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{h}_1$. Then $\mathfrak{l}_1 = \mathfrak{h}_1 \oplus \mathfrak{g}_0$ with $\mathfrak{g}_0 = [\mathfrak{l}_1 \mathfrak{l}_1] \trianglelefteq \mathfrak{l}_1$, and $G_0 = K_1 T_0 K_1 \trianglelefteq L_1, L_1 = T_1 \, G_0$ a semidirect product.
Let $T_{1\mathbb{R}}$ be the 'real points' that is $\mathfrak{h}_{1\mathbb{R}} = \{h \in \mathfrak{h}_1; \alpha(h) \in \sqrt{-1}\mathbb{R} \ \forall \alpha \in \Phi_1\}$; here $T_1$ may not be central (see (3.1)). Now $\mathfrak{k}_1 = \sqrt{-1}\mathfrak{a}_1 \oplus \sum_{\alpha \in \Phi_1}^{\oplus} \mathbb{R}u_\alpha$ where $u_\alpha = (e_\alpha - e^\alpha) + \sqrt{-1}(e_\alpha + e^\alpha)$ with $e_\alpha \in \mathfrak{g}_\alpha, -e^\alpha := \theta(e_\alpha) \in \mathfrak{g}_{-\alpha}, \alpha \in \Phi_{1+}$. We have $[hu_\alpha] = -\alpha(h)\sqrt{-1}u^\alpha$, $\forall h \in \mathfrak{h}_{1\mathbb{R}}$ and so, since $\operatorname{Ad}(\exp x) = e^{\operatorname{ad} x}$, $\forall x \in \mathfrak{k}_1$ and each point of $K_1$ lies on a one parameter subgroup, then $\operatorname{Ad}(k)\mathfrak{h}_{1\mathbb{R}} \subseteq \mathfrak{h}_{1\mathbb{R}} + \mathfrak{k}_1, \forall k \in K_1$. Thus $\mathfrak{k}_1 + \bigcup_{k \in K_1} \operatorname{Ad}(k)(\mathfrak{h}_{1\mathbb{R}} \oplus \mathfrak{h}_0) = \mathfrak{k}_1 \oplus \mathfrak{h}_{1\mathbb{R}} \oplus \mathfrak{p} \leqq \mathfrak{l}_1$ over $\mathbb{R}$. Next as $T_{1\mathbb{R}}$ is contained in the normalizer of $K_1$ in $L_1$ it follows that $K_1 T_0 T_{1\mathbb{R}} K_1 \leqq L_1$.
Hence over $\mathbb{C}, L_1 = K_1 T K_1$. □

**Note.** For any subset $J \subseteq I = \{1, \ldots, n\}$ let $W_J = \langle r_i; i \in J \rangle \leqq W$, and $N_J = \langle n_i; i \in J \rangle \leqq N$. The conjugates in $G$ of $P_J = BN_JB$ are called the parabolic subgroups of $G$. Such a group **[1, 11]** has a Levi decomposition $P_J = L_J \ltimes U_{(J)}$ where $L_J$ is the Kac–Moody group associated to the root datum $(\mathfrak{h}_\mathbb{Z}, \Delta_J^\vee, \Delta_J)$ with $\Delta_J^\vee = \{h_i; i \in J\}, \Delta_J = \{\alpha_i; i \in J\}$. The parabolic subgroup $P_J$ is said to be of finite type if $W_J$ is finite.

The type of $G$ is defined according to the type of $A$ (with $A$ indecomposable, see (2.1)). We say (with $A$ possibly not symmetrizable) that $G$ is of type (3) if the orbits of $W$ acting on $\Phi_+^{\mathrm{im}}$ are not all singleton sets. The group $G$ is type (1) $\Leftrightarrow W$ is finite $\Leftrightarrow G$ is the homomorphic image of an almost simple, complex Lie group (with fundamental group $\mathfrak{h}_{\mathbb{Z}}/\mathbb{Z}\Delta^\vee$).

**Proposition 3.** *Let $G$ be of type (2) or (3). For each $\alpha\in\Phi^{\mathrm{re}}$ denote by $V_m, m\in\mathbb{N}$ the standard simple $G_\alpha=\phi_\alpha(\mathrm{SL}(2,\mathbb{C}))$ module; then $\{m\in\mathbb{N}; V_m\leqq_{G\alpha} V^\omega\}$ is unbounded.*

**Proof.** By $W$ conjugacy it suffices to prove this for a simple root $\alpha_i, i\in\{1,\dots,n\}$. We have for type (1), (2), or (3) that $P^\omega=(\omega+\mathbb{Z}\Delta)\cap$ convex hull $(W.\omega)$, [4].

*Type (2).* The simple roots are (see (3.1)) labelled $\{\alpha_0,\alpha_1,\dots,\alpha_l\}$. Let $\delta\in\mathfrak{h}^*$ be the positive imaginary root of least height. Then $\mathrm{supp}(\delta)=\{0,1,\dots,l\}$ and $\Phi_+^{\mathrm{im}}=\{n\delta; n\in\mathbb{N}\}$. Define maximal weights $\max(\omega)=\{\lambda\in P^\omega; \lambda+\delta\in\mathfrak{h}^*\backslash P^\omega\}$. Then $P^\omega=\bigcup_{\lambda\in\max(\omega)}\{\lambda-n\delta; n\in\mathbb{N}\}$. The weight system lies in the paraboloid whose boundary intersects $P^\omega$ in the orbit $W.\omega$. Also $\max(\omega)$ consits of the highest weights of simple subquotients of $V^\omega$ under the action of Levi subgroups of $G$ of finite type.

*Type (3).* There exists a unique $\alpha\in\Phi_+^{\mathrm{im}}$ of minimal height with $\mathrm{supp}(\alpha)=\{1,\dots,n\}$ and $\alpha(h_i)<0, \forall i$. For $0\neq v\in V_{\omega-\alpha}$ the mapping $\mathfrak{n}_-\to V^\omega$, $y\mapsto y.v$ is injective. As $\{j\alpha; j\in\mathbb{N}\}\subseteq\Phi_+^{\mathrm{im}}$ we now have that $\{\omega-k\alpha; k\in\mathbb{N}\}\subseteq P^\omega$, and $\forall i\{(\omega-k\alpha)(h_i); k\in\mathbb{N}\}$ is unbounded in $\mathbb{N}$. □

**Proposition 4.** *Let $L_J$ be a Levi subgroup of $G$ of finite type. Denote by $V_J^\lambda$ the simple $L_J$ module with highest weight $\lambda$. Then $\forall\omega\in\mathscr{I}\mathrm{nt}_+\cap\mathfrak{h}_{\mathbb{Z}}^*$, the set $\{\lambda\in P^\omega; V_J^\lambda\leqq_{L_J}V^\omega\}$ is infinite.*

**Proof.** Any $\lambda\in P^\omega$ can be uniquely written $\lambda=\omega-\sum_{i\in I\backslash J}c_i\alpha_i-\sum_{i\in J}c_i\alpha_i$ where $c_i\in\mathbb{N}, \forall i$. Define $\mathrm{dep}_J(\lambda)=\sum_{i\in I\backslash J}c_i$. Then $V_{(m)}:=\sum^{\oplus}_{\mathrm{dep}_J(\lambda)\leq m}V_\lambda, m\in\mathbb{N}$ is a finite dimensional $P_J$ submodule of $V^\omega$. And $V^\omega$ is completely reducible as an $L_J$-module. Thus $\{\mathrm{dep}_J(\lambda); \lambda\in P^\omega\}$ and $i\in J$, $\{m\in\mathbb{N}; \exists\lambda\in P^\omega$ with $c_i\geqq m\}$ are unbounded.

The result is now a consequence of Proposition 3. □

If $i,j\in\{1,\dots,n\}$ with $m=a_{ij}a_{ji}\geqq 2$ then label the $(i,j)$ edge in the Coxeter graph $\overset{\alpha_i}{\bigcirc}\overset{m}{—}\overset{\alpha_j}{\bigcirc}$. It can be seen as in the examples affine $\bigcirc\overset{4}{—}\bigcirc$ $\omega=\omega_1$, and hyperbolic $\bigcirc\overset{4}{—}\bigcirc—\bigcirc$, $\omega=\omega_1$ that in Proposition 4 the multiplicity $\dim\mathrm{Hom}_{L_J}(V^\omega, V_J^\lambda)=0$ or $\infty$.

## 2. Hilbert space structure and trace class operators

**2.1.** Let $A$ be a symmetrizable Cartan matrix; so there is a positive rational matrix $D$ with $D^{-1}A$ symmetric. Then there are three types:

(1) $A$ has rank $n$ and $D^{-1}A$ has signature $n$

(2) $A$ has corank 1 and $D^{-1}A$ has signature $n-1$

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(3) The signature of $D^{-1}A$ is less than the rank of $A$,

of finite, affine and indefinite type respectively.

The simple quotient $V^{\omega}, \omega \in \mathscr{I}\mathrm{nt}_+$ is, [6], a pre-Hilbert space via a contravariant, $K$-invariant, positive definite hermitian form $\langle , \rangle$ which is unique with norm $\|v_\omega\| = 1$. Order the weights $P^{\omega}$ by the depth, with $\omega$ minimal. Then $V_\lambda \perp V_\mu, \lambda \neq \mu, \lambda, \mu \in P^{\omega}$; and the completion also denoted by $V^{\omega}$ is separable. We fix an orthonormal basis $\{z_i\}_{i \in \mathbb{N}}$ of $V^{\omega}$ where $z_i$ is of weight $\lambda_i, z_0 = \omega$ and $\mathrm{dep}_\omega(\lambda_i) \geqq \mathrm{dep}_\omega(\lambda_j), i \geqq j$.

In the representation $(V^{\omega}, \mathrm{R})$ of $G = G(A)$ we will say that an operator $\mathrm{R}(g), g \in G$ is traceable if the complex series $\sum_{i=0}^{\infty} \langle \mathrm{R}(g) z_i, z_i \rangle$ is convergent; then this value is written $\mathrm{trace}_\omega \mathrm{R}(g)$.

**2.2.** As in a general separable Hilbert space, let $\mathbb{B}\mathrm{d}(V)$, $\mathbb{F}\mathrm{r}(V)$, $\mathbb{K}\mathrm{p}(V)$, $\mathbb{S}\mathrm{t}(V)$ and $\mathbb{T}\mathrm{r}(V)$ be the set of bounded linear, finite rank, compact, Hilbert–Schmidt and traceable (with absolute convergence) operators on $V$. That is $\mathbb{S}\mathrm{t}(V) = \{T \in \mathrm{End}(V); \|T\|_2 < \infty\}$ where $\|T\|_2 = \sum_i \|Tz_i\|^2$ (the $T \in \mathbb{S}\mathrm{t}(V)$ are $l^2$). And $\mathbb{T}\mathrm{r}(V) = \{T \in \mathrm{End}(V); \sum_i |\langle Tz_i, z_i \rangle| < \infty\}$. In fact $\mathbb{S}\mathrm{t}(V) \subseteq \mathbb{B}\mathrm{d}(V)$ and $(\mathbb{S}\mathrm{t}(V), \|\cdot\|_2)$ is a Banach * algebra. A $T \in \mathbb{T}\mathrm{r}(V)$ may not be bounded. For $T \in \mathbb{S}\mathrm{t}(V)$, the Hilbert–Schmidt norm is independent of the complete orthonormal basis. Then $\mathbb{K}\mathrm{p}(V)$ is the unique maximal ideal in $\mathbb{B}\mathrm{d}(V)$ which is closed in the operator norm; and $\mathbb{F}\mathrm{r}(V)$ is the unique minimal ideal in $\mathbb{B}\mathrm{d}(V)$. The ideal $\mathbb{S}\mathrm{t}(V)$ is not closed. In fact $\mathbb{K}\mathrm{p}(V) = \overline{\mathbb{F}\mathrm{r}(V)}$.

One says that $T \in \mathbb{B}\mathrm{d}(V)$ is $l^1$ if $\sum_i \|Tz_i\| < \infty$; in fact $T$ is $l^1 \Leftrightarrow T \in \mathbb{S}\mathrm{t}(V)^2$. Then $\mathbb{S}\mathrm{t}(V)^2 \subseteq \mathbb{B}\mathrm{d}(V) \cap \mathbb{T}\mathrm{r}(V)$ and $\mathrm{trace}(T), T \in \mathbb{S}\mathrm{t}(V)^2$ is independent of the orthonormal basis. Also $\mathrm{trace}(ST) = \mathrm{trace}(TS)$ for $S \in \mathbb{B}\mathrm{d}(V), T \in \mathbb{S}\mathrm{t}(V)^2$. These give a chain of (two sided) ideals

$$\{0\} \subseteq \mathbb{F}\mathrm{r}(V) \subseteq \mathbb{S}\mathrm{t}(V)^2 \subseteq \mathbb{S}\mathrm{t}(V) \subseteq \mathbb{K}\mathrm{p}(V) \subseteq \mathbb{B}\mathrm{d}(V)$$

A $T \in \mathrm{End}(V)$ is said to be closed if its graph is closed in $V \times V$; and closeable if $\overline{\mathrm{graph}(T)}$ is a graph. If $T$ is closeable then there is a unique $\bar{T} \in \mathrm{End}(V)$ with $\mathrm{graph}(\bar{T}) = \overline{\mathrm{graph}(T)}$; the domain being $\mathrm{dom}(\bar{T}) = \{x \in V; \exists$ sequence $(x_n)$ in $\mathrm{dom}(T)$ with $x_n \to x$ and $(Tx_n)$ convergent$\}$, and $\bar{T}x = \lim Tx_n$. A $T \in \mathrm{End}(V)$ is said to be hermitian if it is a formal adjoint of itself, and symmetric if it is hermitian and densely defined.

**2.3.** Subsets of $G$ are defined

$$G^{\mathrm{b}} = \{g \in G; \mathrm{R}(g) \in \mathbb{B}\mathrm{d}(V^{\omega}), \forall \omega \in \mathscr{I}\mathrm{nt}_+ \cap \mathfrak{h}_{\mathbb{Z}}^*\}$$

$$G^{\mathrm{tr}} = \{g \in G; \mathrm{R}(g) \text{ is traceable on } V^{\omega}, \forall \omega \in \mathscr{I}\mathrm{nt}_+ \cap \mathfrak{h}_{\mathbb{Z}}^*\}$$

Thus $G^{\mathrm{b}} = \bigcap_{\omega \in \mathscr{I}\mathrm{nt}_+ \cap \mathfrak{h}_{\mathbb{Z}}^*} \mathrm{R}^{-1}(\mathrm{R}(G) \cap \mathbb{B}\mathrm{d}(V^{\omega}))$, and $\bigcap_{\omega \in \mathscr{I}\mathrm{nt}_+ \cap \mathfrak{h}_{\mathbb{Z}}^*} \mathrm{R}^{-1}(\mathrm{R}(G) \cap \mathbb{T}\mathrm{r}(V^{\omega})) \subseteq G^{\mathrm{tr}}$. Also define $G^{\mathrm{hs}}$ the set of "Hilbert-Schmidt" elements, $G^{\mathrm{cpt}}$ the set of "compact" elements, $G^{\mathrm{fr}}$ the set of "finite rank" elements, giving $G^{\mathrm{fr}} \subseteq (G^{\mathrm{hs}})^2 \subseteq G^{\mathrm{hs}} \subseteq G^{\mathrm{cpt}} \subseteq G^{\mathrm{b}}$. And $G^{\mathrm{sym}}$ the set "symmetric" elements, $G^{\mathrm{cl}}$ the set of "closeable" elements.

**Lemma 1.** (i) *$KG^{s}K=G^{s}$ where $G^{s}$ is the semigroup $G^{b}$, $G^{cpt}$, $G^{hs}$ or $(G^{hs})^{2}$.*
(ii) *$G^{fr}=\emptyset$ if $A$ is not of type* (i).

**Proof.** (i) This follows from $R(K)\subseteq U(V^{\omega}), \forall\omega$ (the unitary group).

(ii) The Iwasawa decomposition $G=KB$ gives $G^{fr}=KB^{fr}$. Further $R(b)V_{\lambda}\subseteq \sum^{\oplus}_{\mu\in P^{\omega},\mu\leq\lambda} V_{\mu}$ with $\langle R(b)V_{\lambda}, V_{\lambda}\rangle\neq 0, \forall b\in B, \forall\lambda\in P^{\omega}$. Hence $B^{fr}=\emptyset$. □

**Proposition 5.** *Let $A$ be of type* (2) *or* (3). *Then*

$$U\cap G^{b}=\{1\}$$

**Proof.** For each $\alpha_i\in\Delta$ there is the Levi subgroup $L_{\{i\}}$ of the parabolic subgroup $P_{\{i\}}$ of $G$ (see (1.4)) $i\in\{1,\ldots,n\}$. And if $\alpha\in\Phi^{re}$ with $w\in W, w(\alpha_i)=\alpha$ and $n\in N$, $n\mapsto w\in N/T$, we have $L_{\alpha}=nL_{\{i\}}n^{-1}=\langle T, U_{\alpha}, \underline{\omega}_0(U_{\alpha})\rangle$. The derived group $L'_{\alpha}\simeq \mathrm{SL}(2,\mathbb{C})$. The simple $G$-module $V^{\omega}, \omega\in\mathscr{I}\mathrm{nt}_{+}\cap\mathfrak{h}^{*}_{\mathbb{Z}}$ is semisimple under $L_{\alpha}$, which is such that this decomposition under $\phi_{\alpha}(\mathrm{SU}(2))$ is a complete orthogonal direct sum.

Also recall that $\mathrm{GL}(2,\mathbb{C})$ acts on $\bigvee^{m}(\mathbb{C}^2)$, the symmetric polynomials of degree $m$ in $X, Y$, by $(g.p)\binom{X}{Y}=p(g^{t}\binom{X}{Y})$. The standard basis vectors are

$$Z=\left(\frac{1}{a!b!}\right)^{\frac{1}{2}}X^{a}Y^{b}, a+b=m,$$

and the unipotent element $u=u(c)=\begin{pmatrix}1 & c\\ 0 & 1\end{pmatrix}, c\in\mathbb{C}$, acts as

$$u.Z=\sum_{r=0}^{b}c^{r}\binom{b}{r}\left(\frac{(a+r)!(b-r)!}{a!b!}\right)^{\frac{1}{2}}\frac{X^{a+r}Y^{b-r}}{((a+r)!(b-r)!)^{\frac{1}{2}}}.$$

The superdiagonal entries are with $r=1$, $c(a+1)^{\frac{1}{2}}b^{\frac{1}{2}}$ which with $a=0$, $b=m$ is $cm^{\frac{1}{2}}$. Label the weight vectors

$$z_0=Y^{m}, z_1=\left(\frac{1}{(m-1)!}\right)^{\frac{1}{2}}XY^{m-1},\ldots,z_m=X^{m}$$

with weights $-m, 2-m,\ldots,m$ under $h_i$. This $\{\bigvee^{m}(\mathbb{C}^2); m\in\mathbb{N}\}$ is a complete set of simple finite dimensional $\mathrm{SL}(2,\mathbb{C})$-modules.

Let $u=u_1\cdots u_k\in U$, $u_j=u_j(c_j)$ with $c_j\neq 0$ some $j$ and for each $j\in\{1,\ldots,k\}$ we have $u_j\in U_{\beta_j}, \beta_j\in\Phi^{re}_{+}$. There is $w_1\in W$ with $w_1(\beta_1)=\alpha_i$ for some $i\in\{1,\ldots,n\}$. Let $K_i=\phi_i(\mathrm{SU}(2))$, $i\in\{1,\ldots,n\}$. As a product of fundamental reflections $w_1=r_{i_1}\cdots r_{i_t}$ say, so taking conjugates $u$, $n_{i_t}un_{i_t}^{-1}$, $n_{i_{t-1}}n_{i_t}un_{i_t}^{-1}n_{i_{t-1}}^{-1},\ldots$, where $n_{i_j}\mapsto r_{i_j}\in N/T$, $n_{i_j}=\phi_{i_j}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\in K_{i_j}$, and using the fact that $\forall i'$, $(r_{i'}(\alpha)\in\Phi^{re}_{+}$, $\forall\alpha\in\Phi^{re}_{+}, \alpha\neq\alpha_{i'})$, we stop this sequence when a conjugate of $u$ contains a term in the product belonging to a simple root subgroup. Therefore we may as well start with $u=u_1\cdots u_k$ such that $\beta_j=\alpha_i$ some $i,j$.

Let such $\{z_0,\ldots,z_m\}$ with weights $\{\lambda_0,\lambda_1,\ldots,\lambda_m\}$ refer to a simple module in the $L_{\alpha_i}$ decomposition of $V^\omega$. We have

$$\mathrm{R}(u)z_0=\mathrm{R}(u_{(2)})(z_0+z) \text{ if } \beta_k\neq\alpha_i$$

$$=\mathrm{R}(u_{(2)})(z_0+c_km^{\frac{1}{2}}z_1+z) \text{ if } \beta_k=\alpha_i$$

where $u_{(k')}=u_1\cdots u_{k-(k'-1)}$ and $z$ is a sum of weight vectors (or zero) with weights $\lambda_0+r\beta_k$, and $r\geqq 1$ if $\beta_k\neq\alpha_i$ or $r\geqq 2$, $\lambda_1=\lambda_0+\alpha_i$ if $\beta_k=\alpha_i$. Next $\mathrm{R}(u)z_0=R(u_{(3)})R(u_{k-1})R(u_k)z_0$ etc. to obtain finally

$$\mathrm{R}(u)z_0=z_0+\sum_{j,\beta_j=\alpha_i}c_jm^{\frac{1}{2}}z_1+z' \text{ and } \langle R(u)z_0,z_1\rangle=\sum_{\beta_j=\alpha_i}c_jm^{\frac{1}{2}}.$$

The result now follows immediately from Proposition 3 if $\sum_{\beta_j=\alpha_i}c_j\neq 0$.

Otherwise proceed as follows. The $L_{\beta_j}$ decomposition of $V^\omega$ is such that

$$\mathrm{R}(u_j(c_j))z_{\mathrm{a}}=\sum_{r=0}^{\mathrm{b}}c_j^{\mathrm{r}}\binom{a+r}{r}^{\frac{1}{2}}\binom{b}{r}^{\frac{1}{2}}z_{a+r}, a+b=m.$$

Again under $L_{\alpha_i}$, the matrix elements $m_{l'l''}(u)$ of $\mathrm{R}(u)$ are polynomials in the $c_j, j\in\{1,\ldots,k\}$ with positive integer coefficients. Then, and using convex properties of $P^\omega$ described in (1.4), one sees that $u\in G^{\mathrm{b}}\Rightarrow\forall l',l''$ each polynomial in $m_{l'l''}(u)$ which involves and is homogeneous in the $c_j,\beta_j=\alpha_i$ must be zero. Thus $u\in G^{\mathrm{b}}\Rightarrow u=u^{(i)}$ (obtained from $u$ be deleting the $u_j\in U_{\alpha_i}$). Continuing, up to conjugation by $N\cap K$ the element $u^{(i)}$ has $u_j\in U_{\alpha_{i'}}$ for some $i',j$. Finally, $u\in G^{\mathrm{b}}\Rightarrow u=1$.

To make the previous section more precise we include the following auxiliary results:

Let [ ] denote the group commutator, that is $[x,y]=x^{-1}y^{-1}xy, x,y\in G$. Define inductively $[x_1,\ldots,x_m]:=[[x_1,\ldots,x_{m-1}],x_m], m>2, x_i\in G$. Sometimes we denote $x^y:=y^{-1}xy$, therefore $x^y=x[x,y], x,y\in G$.

Here $A$ need not be symmetrizable. Recall that $U=U^{\min}$.

**Lemma 5a.** *If $W\ni w=r_{i_1}\cdots r_{i_s}$ is a reduced expression (where $r_{i_j}=r_{\alpha_{i_j}}$) then*

$$\Phi(w)=\{\alpha_{i_1},r_{i_1}(\alpha_{i_2}),\ldots,r_{i_1}\cdots r_{i_{s-1}}(\alpha_{i_s})\}.$$

*In particular $l(w)=|\Phi(w)|$.*

*For $w\in W$, let $U_w=\prod_{\beta\in\Phi(w)}U_\beta$; this expresses an element uniquely as a product.*

**Proof.** See for example [1]. ▽

**Lemma 5b.** *Let $w_1,w_2\in W,\Phi(w_1)\cap\Phi(w_2)=\emptyset$. Then*

$w\in W, \Psi\subseteq\Phi_+, w\Psi\subseteq\Phi_+ \Rightarrow (w(\Psi\cap\Phi(w_1))\subseteq\Phi(ww_1)$ *and* $w\Psi\cap\Phi(ww_1)\cap\Phi(ww_2)=\emptyset)$. ∇

Let $W_{(0)}=\{w\in W; \Phi(w)\cup-\Phi(w)$ is a closed subsystem of roots in $\Phi\}$ and $N_{(0)}=\{n\in N; n\mapsto w\in W_{(0)}\}$.

**Lemma 5c.** *Let* $w\in W_{(0)}$, $\beta,\beta'\in\Phi(w)$, $\gamma\in\Phi_+^{re}\backslash\Phi(w)$. *Then*

$$[U_\beta, U_\gamma]\leqq U\cap nUn^{-1}, N\ni n\mapsto w$$

$$[[U_\beta, U_\gamma], U_{\beta'}]\leqq U\cap nUn^{-1}.$$

**Proof.** Take $w=r_{i_1}\cdots r_{i_m}$ a reduced expression. First consider $\beta=\alpha_{i_1}$. The $+\alpha_{i_1}$ chain of roots through $\gamma$ is $C_{i_1,\gamma}=\Phi\cap\{\gamma+s\alpha_{i_1}; s\in\mathbb{N}\}$. Using the $L_{\alpha_{i_1}}$ decomposition of $\mathfrak{g}$ we see that $C_{i_1,\gamma}$ is finite and "unbroken". Let $x_{i_1}(c)=\exp ce_{i_1}, x_\gamma(c)=\exp ce_\gamma$ (where $e_\gamma=\mathrm{Ad}(n')e_{\alpha_{i'}}, n'\mapsto w'$, $w'^{-1}(\gamma)=\alpha_{i'}$ for chosen $w', \alpha_{i'}$). We have

$$\begin{aligned}[x_\gamma(c), x_{i_1}(c_1)] &= x_\gamma(-c)x_{i_1}(-c_1)x_\gamma(c)x_{i_1}(c_1)\\ &= x_\gamma(-c)\exp \mathrm{Ad}(x_{i_1}(-c_1))(ce_\gamma)\\ &= x_\gamma(-c)\exp(e^{-c_1\,\mathrm{ad}\,e_{i_1}}(ce_\gamma))\\ &= x_\gamma(-c)\exp\Big(c(e_\gamma - c_1[e_{i_1}e_\gamma]+\frac{c_1^2}{2}[e_{i_1}[e_{i_1}e_\gamma]]\\ &\qquad -\frac{c_1^3}{3!}[e_{i_1}[e_{i_1}[e_{i_1}e_\gamma]]]+\dots\Big)\end{aligned}$$

a finite series (†)

$$(=1 \text{ if } \gamma+\alpha_{i_1}\in\mathscr{I}\mathrm{nt}\backslash\Phi).$$

Next $\Phi(w)$ is a system of positive roots for a semisimple Lie subalgebra of $\mathfrak{g}$, with Cartan subalgebra contained in $\mathfrak{h}$. Also mult $\alpha=1, \forall\alpha\in\Phi^{re}$. It follows that

$$\gamma+s\alpha_{i_1}\in\Phi(w)\Rightarrow 0\neq\underbrace{[f_{i_1}\dots[f_{i_1}[f_{i_1}}_{s\text{ times}}e_{\gamma+s\alpha_{i_1}}]]\dots]\in\Phi(w)\Rightarrow\gamma\in\Phi(w)$$

We conclude that $C_{i_1,\gamma}\cap\Phi_+\subseteq\Phi_+\backslash\Phi(w)$. Hence $n^{-1}[x_\gamma(c), x_{i_1}(c_1)]n\in U$.

Secondly, with any $\beta\in\Phi(w)$, use induction on $l(w)$. Suppose $l(w)=1$, $w=r_{i_1}$. Therefore $\beta=\alpha_{i_1}$. We want to show $[U\alpha_{i_1}, U_\gamma]\leqq U\cap n_{i_1}Un_{i_1}^{-1}$, which follows from (†). Suppose $l(w)=m>1$. Again by the first part we need only consider $\beta\neq\alpha_{i_1}$. Therefore

$r_{i_1}(\beta)\in\Phi(r_{i_1}w)$, $r_{i_1}(\gamma)\in\Phi_+^{\mathrm{re}}\backslash\Phi(r_{i_1}w)$. Thus if we have the assertion for length$=m-1$, it follows that

$$n_i[U_\beta U_\gamma]n_i^{-1}=[U_{r_{i_1}(\beta)}U_{r_{i_1}(\gamma)}]\leqq U\cap n_{i_1}nUn^{-1}n_{i_1}^{-1}$$

which on conjugation by $n_{i_1}$ gives the result.

From (†) with $\alpha_{i_1}$ replaced by $\beta$ and using the commutator formula $[xy,z]=[xz]^y[yz]$, $x$, $y$, $z\in G$, we see by a similar argument that

$$\{(\gamma+s\beta)+s'\beta';s,s'\in\mathbb{N}\}\cap\Phi_+\subseteq\Phi_+\backslash\Phi(w)$$

and

$$[[x_\gamma(c),x_\beta(c_1)],x_{\beta'}(c')]\in U\cap nUn^{-1},\forall c,c_1,c'\in\mathbb{C}$$

as required. ▽

**Proposition 5d.** $U=U_w\ltimes(U\cap nUn^{-1})$, $N\ni n\mapsto w\in W_{(0)},\forall w\in W_{(0)}$.

**Proof.** A $u\in U$ can be expressed $u=u_0u_1\cdots u_k$ where $u_0\in U_w$ and $\forall j(u_j\in U_{w_j}$, $\Phi(w)\cap\Phi(w_j)=\emptyset$ or $\exists\beta_j\in\Phi(w)$, $\gamma_j\in\Phi_+^{\mathrm{re}}\backslash\Phi(w)$ with $u_j\in[U_{\beta_j}U_{\gamma_j}]^{U_w})$.

Let $j\in\{1,\ldots,k\}$ with $u_j\in U_{w_j}$ and $\Phi(w)\cap\Phi(w_j)=\emptyset$. Then $U_\beta\leqq W_{w_j}\Rightarrow w^{-1}\beta=\alpha\in\Phi_+^{\mathrm{re}}\Rightarrow\beta=w\alpha\Rightarrow U_\beta=nU_\alpha n^{-1}$. Thus also using Lemmas 5a, 5c, we have $U=U_w(U\cap nUn^{-1})$. Also $U_w\cap nUn^{-1}=\{1\}$, and with $u\in U\cap nUn^{-1}$, $v\in U_\alpha$, $\alpha\in\Phi(w)$ it follows that $u^v=u_1^v\cdots u_k^v=u_1[u_1v]\cdots u_k[u_kv]\in U\cap nUn^{-1}$. Hence $U\cap mUm^{-1}\lhd U$, $\forall m\in N_{(0)}$. ▽

**Lemma 5e.** *Let* $U_{(0)}=\bigcap_{n\in N_{(0)}}U\cap nUn^{-1}$ *and* $U_{(00)}=\bigcap_{n\in N}U\cap nUn^{-1}$. *Then* $U_{(0)}\lhd U$, $U_{(00)}\leqq U'$ *and any* $u\in U_{(00)}$ *can be expressed* $u=u_1\cdots u_k$ *with each* $u_j$ *of the form* $x=[x_1,\ldots,x_m]$, $x_{j'}\in U_{\beta_{j'}}$, $\beta_{j'}\in\Phi_+^{\mathrm{re}}$ or $x^{-1}$ and $\forall j$, $u_j\in U_{(0)}$.

**Proof.** Let $u\in U_{(00)}$. First write $u=u'u''$ with $u''\in U'$. Now $u'=v_1\cdots v_{k'}$ a product of elements of $U$ each lying in real root subgroups. Similarly $u''$ can be so expressed. If there are $i,j$ with $v_j\in U_{\alpha_i}$, then using Proposition 5d, we can reexpress $u=v'v''$ where $v''\in U'$ and $v'$ is the product of $\leqq k'-1$ elements of $U$ lying in real root subgroups. Otherwise, there is a sequence $(i_1,i_2,\ldots,i_{m'})$ and an $i$ such that $\{u'^n,u''^n\}\subseteq U$ and the $\alpha_i$ root subgroup contains an element occurring in $u'^n$ (see the first part of the proof), where $N\cap K\ni n\mapsto w=r_{i_1}r_{i_2}\cdots r_{i_{m'}}$. Now $u^n\in U_{(00)}$, also if $k'=1$ we must have $v_1=1$. Hence by induction on $k',u\in U'$. And $U_{(0)}$ the intersection of normal subgroups, is therefore normal in $U$.

Although $U$ is not locally nilpotent in type (2) or (3), the lower central series gives that $u=v_1\cdots v_{k'}$ with each $v_{j'}$ of the form $x=[x_1,\ldots,x_m]$ or $x^{-1}$ as in the statement of the result. Next

$$\exists n\in N,n^{-1}xn\in G\backslash U\Rightarrow\exists j,n^{-1}x_jn\in G\backslash U\Rightarrow\beta_j\in\Phi(w),n\mapsto w\in W.$$

If $\exists j', j'' \in \{1,\ldots,m\}$, $m \geqq 2$ with $\beta_{j'} \in \Phi(w)$, $\beta_{j''} \in \Phi_+^{\mathrm{re}} \backslash \Phi(w)$ for $w \in W_{(0)}$, then Lemma 5c, Proposition 5d and induction on $m$ give $n^{-1}xn \in U$. Thus for $n \in N_{(0)}$, $n^{-1}xn \in G \backslash U \Leftrightarrow \forall j$, $\beta_j \in \Phi(w)$. Set $I = \{1,\ldots,k'\}$, $I_1 = \{j \in I;\ \exists n \in N_{(0)}, n^{-1}v_j n \in G \backslash U\}$. Then $j \in I_1, j' \in I \backslash I_1 \Rightarrow [v_j v_{j'}] \in U_{(0)}$ and can be written in the required form. Finally using $U \cap \underline{\omega}_0(U) = \{1\}$ we see that the result follows.

Note that $x \in U_{(00)} \Rightarrow \sum_{j=1}^m \mathbb{Z}\beta_j \cap \Phi^{\mathrm{im}} \neq \emptyset$. ▽

If $w = w_1 w_2 \in W$ where $l(w) = l(w_1) + l(w_2)$, then $\Phi(w_1) \subseteq \Phi(w)$ and (†) (with $\alpha_{i_1}$ replaced by $\beta \in \Phi(w)$, $\gamma \in \Phi_+^{\mathrm{re}} \backslash \Phi(w)$) give that $w \in W_{(0)}$ implies $U \cap nUn^{-1} \leqq U \cap n'Un'^{-1}$, $N \ni n \mapsto w$, $N \ni n' \mapsto w_1$.

Let $w \in W$, and $u = u_0 u_1 \cdots u_k$ with $u_0 \in U_w$, and $u_j, j \neq 0$ as in the proof of Proposition 5d. From Lemma 5a we can further write uniquely $u_0 = u_{01} \cdots u_{0m}$ with $u_{0s} \in Ur_{i_1} \ldots r_{i_{s-1}}(\alpha_{i_s})$, $m = l(w)$. Suppose that $u \in G^{\mathrm{b}}$. Then as in the first part of the proof we see that $u_{01} = 1$. Next let $u_{02} = \cdots = u_{0,s-1} = 1$ and put $w_1 = r_{i_1} \cdots r_{i_{s-1}}, s \leqq m$. Now $w_1 \Phi(w_1^{-1} w) \subseteq \Phi(w)$, Lemma 5b and (†) give that $w \in W_{(0)}$, $N \cap K \ni n' \mapsto w_1$, $u^{n'} \in G^{\mathrm{b}} \Rightarrow u_{0s} = 1$. Thus $u_0 = 1$. And as this holds $\forall w \in W_{(0)}$, we have shown $U \cap G^{\mathrm{b}} \subseteq U_{(0)}$. Note that in general one has $U = U_w(U \cap nUn^{-1})$ for any $w \in W$. In fact for $w \in W$, use induction on $l(w)$. Suppose $u^{n'} \in U$. Then as $u_0^{n'} \in U$ we have $(u_1 \ldots u_k)^{n'} \in U$ giving $u_{0s} = 1$. Therefore $u_0 = 1$ and $u^n = u^{(nn_{i_m}^{-1} n_{i_m}} \in U$. Thus $U \cap G^b \subseteq U_{(00)}$.

Let $u \in U_{(00)} \cap G^{\mathrm{b}}$. From Lemma 5e we write $u = u_1 u', u_1 = x = [x_1,\ldots,x_m]$ or $u_1 = x^{-1}$. And show $u_1 = 1$. This is by induction on $m$.

If $m = 2$, $x = [x_{\beta_1}(c_1), x_{\beta_2}(c_2)]$ and refer to (†). We can assume $\beta_1 + \beta_2 \in \Phi_+$. Note as before that $\lambda, \mu \in P^\omega$, $\langle \mathrm{R}(x) V_\lambda, V_\mu \rangle \neq \{0\} \Rightarrow \mu = \lambda + s_1\beta_1 + s_2\beta_2$, $s_1$, $s_2 \in \mathbb{N} \backslash \{0\}$. Consider $C_{\beta_2,\beta_1} \cap \Phi_+^{\mathrm{re}}$ and recall $W\Phi_+^{\mathrm{im}} = \Phi_+^{\mathrm{im}}$. If the $+\beta_2$ chain of roots through $\beta_1$ contains at least two real roots then $\exists w \in W, \exists s \in \mathbb{N} \backslash \{0\}$, $w^{-1}(C_{\beta_2,\beta_1} \backslash \{\beta_1\}) \subseteq \Phi_+$ and $w^{-1}(\beta_1 + s\beta_2) = \alpha_i \in \Delta$. (Also $w^{-1}\beta_1 \in \Phi_+$ if $\Phi(rw_i)$ is a system of positive roots). Otherwise $C_{\beta_2,\beta_1} \cap \Phi_+^{\mathrm{re}} = \{\beta_1\}$ which is false. Thus also using $u' \in U'$, we have $u_1 = 1$.

For the induction step, $x = [[x_1,\ldots,x_{m-1}]x_m] = [x_1,\ldots,x_{m-1}]^{-1}[x_1,\ldots,x_{m-1}]^{x_m}$. Firstly, suppose $\exists w \in W_{(0)}$ with $u_2 := [x_1,\ldots,x_{m-1}] \in U_w$. Now (see Lemma 5a)

$$u_2 = yz, y, z \in U_w, [yz, x_m] = [yx_m]^z [zx_m] = [yx_m][yx_m z][zx_m],$$

and therefore by a second induction on the "length" of an element in $U_w$, one sees that $u \in U_{(00)} \cap G^{\mathrm{b}} \Rightarrow [yx_m] = 1$, $[zx_m] = 1$. Secondly, suppose $u_2 := [x_1,\ldots,x_{m-1}] \in U_{(0)}$. Then $u = u_2^{-1}(u_2^{x_m} u') \in G^{\mathrm{b}} \Rightarrow u_2 = 1$. And argue similarly if $u = x^{-1}$.

Hence if follows that $U_{(00)} \cap G^{\mathrm{b}} = \{1\}$, which completes the proof of the proposition. □

## 3. Characters of affine Kac–Moody groups

It is the aim of this section to give the subdomain in $G$ on which a (pointwise) character of $V^\omega, \omega \in \mathscr{I}\mathrm{nt}_+ \cap \mathfrak{h}_{\mathbb{Z}}^*$ can be defined.

**3.1.** Let $A$ be a type (2) affine Cartan matrix. Index the simple roots by $\{0,1,\ldots,l\}$

where $A_0$, of finite type (1), is obtained by deleting the 0 vertex in the Coxeter–Dynkin diagram of $A$. Here $\{h_0, h_1, \ldots, h_l, d\} \subseteq \mathfrak{h}_{\mathbb{Z}}$ where $\alpha_i(d)=0$, $i \in \{1, \ldots, l\}$, $\alpha_0(d)=1$ and rank $\mathfrak{h}_{\mathbb{Z}} = l+2$ (see (1.1),(2.1)). Let the components of the least positive imaginary root $\delta \in \sum_{i=0}^{l} \mathbb{N}\alpha_i$ be $\delta = (a_0, a_1, \ldots, a_l)$. That is $\underline{a} \in \mathbb{N}^{l+1}$ is of least height and $\mathbb{R}\underline{a}$ is the kernel of the quadratic form on $\mathbb{R}^{l+1}$ associated to $A$. In the dual $A^\vee = A^t$ write $\delta^\vee = (a_0^\vee, a_1^\vee, \ldots, a_l^\vee)$, (so in each case $a_0^\vee = 1$ [4]); then the affine Kac–Moody Lie algebra $\mathfrak{g} = \mathfrak{g}(A)$ has a 1-dim centre containing the canonical central element $c = \sum_{i=0}^{l} a_i^\vee h_i$.

In general a real root $\alpha \in \Phi_+^{re}$ has coroot $\alpha^\vee \in \mathbb{N}\Delta^\vee$ by $W$. The reflection $r_\alpha = w r_i w^{-1}$ if $w\alpha_i = \alpha$. For symmetrizable $A$, the $W$ invariant form (,) on $\mathfrak{g}$ (see (1.1)) is chosen such that $v(\Delta^\vee) = \Delta D$. And for $A$ affine take $D = \operatorname{diag}(a_0 a_0^{\vee\,-1}, a_1 a_1^{\vee\,-1}, \ldots, a_l a_l^{\vee\,-1})$. Define $\Delta_0 = \Delta \backslash \{\alpha_0\}$, $\mathfrak{h}_{0\mathbb{Z}} = \mathfrak{h}_{\mathbb{Z}} \cap \mathbb{Q}\Delta_0^\vee$, $W_0 = \langle r_i; i \neq 0 \rangle \leqq W$, $\Phi_0 = W_0 \Delta_0$ and $\mathfrak{g}_0 = \mathfrak{g}(A_0)$. Denote by $\theta \in \Phi_{0+}$ the highest root; then $h_0 = c - a_0\theta^\vee$, $\delta = a_0\alpha_0 + \theta$. Let $\Upsilon$ be the translation subgroup of $W$ generated by $w r_0 r_\theta w^{-1}$, $w \in W_0$. Then $\Upsilon \lhd W$ and $W = W_0 \propto \Upsilon$.

**3.2. Lemma 2.** (i) *The "derivation element" $d$ acts semisimply on $V^\omega$ with finite dimensional eigenspaces.*

(ii) *The character $\delta \in \operatorname{Hom}(T, \mathbb{C}^*)$ extends trivially to $\delta \in \operatorname{Hom}(G, \mathbb{C}^*)$.*

**Proof.** (i) If $\lambda = \omega - \sum_i c_i \alpha_i \in P^\omega$ we have $d.V_\lambda = (\omega(d) - c_0)V_\lambda$. The parabolic subgroup $P_J$, $J = \{1, \ldots, l\}$ of $G$ is of finite type. Thus (see (1.4)) $V_{(m)} = \sum^{\oplus}_{\operatorname{dep}_J(\lambda) \leqq m} V_\lambda$ is finite dimensional $\forall m \in \mathbb{N}$.

(ii) This is a corollary to (1.4) Proposition 1. □

Let $G_0$ be the almost simple, complex Lie group with root datum $(\mathfrak{h}_{0\mathbb{Z}}, \Delta_0^\vee, \Delta_0)$. Thus $\mathfrak{h}_{0\mathbb{Z}}^*$ is the character group of $T_0 = T \cap G_0 \leqq G_0$ a maximal (algebraic) torus, and $\mathfrak{h}_{0\mathbb{Z}}/\mathbb{Z}\Delta_0^\vee$ is the fundamental group. There is a homomorphic image of $G_0$ as a subgroup of $G$. Now $T = Z T_0 T_1$ where $Z = \{\exp ac; c \in \mathbb{C}\}$ is contained in the centre of $G$ and $T_1 = \{\exp(a/a_0)d; a \in \mathbb{C}\}$. Thus $\delta$ is trivial on $ZT_0$ and $\delta(t) = e^a$, $t \in T_1$. Denote $T_c = T \cap K$.

**Lemma 3.** $K \subseteq \operatorname{Ker}|\delta|$.

**Proof.** This is because $K' = \langle K_i; i = 0, 1, \ldots, l \rangle$, $K_i = \phi_i(\mathrm{SU}(2))$ and $\underline{K_i} = \bigcup_{k \in K_i} k_i T k^{-1}$ with ${}_iT = T \cap K_i \simeq \mathrm{U}(1)$. Then $ZK' \subseteq \operatorname{Ker}\delta$. Also $T_c \cap T_1 = \{\exp\sqrt{-1}\pi a d; a \in \mathbb{R}\}$ and $G = T_1 \ltimes G'$. □

**3.3.** In general the set of functions $\{f: \mathfrak{h}^* \to \mathbb{Z};\ \operatorname{supp} f \subseteq \bigcup_{j=1}^{m} \lambda_j - \mathbb{N}\Delta,\ \lambda_j \in \mathfrak{h}^*\}$ becomes a commutative associative algebra $E$, with unit, under convolution. Introduce $e^\lambda \in E$, $\lambda \in \mathfrak{h}^*$ by $e^\lambda(\mu) = \delta_{\lambda\mu}$. The formal character $\chi^\omega$ of $V^\omega$, $\omega \in \mathscr{I}\mathrm{nt}_+ \cap \mathfrak{h}_{\mathbb{Z}}^*$ is given by $\chi^\omega = \sum_{\lambda \in P^\omega} (\dim V_\lambda) e^\lambda \in E$, which can be expressed as the "Weyl–Kac" formula. The exact sequence $0 \to \mathfrak{h}_{\mathbb{Z}} \xrightarrow{\iota} \mathfrak{h} \xrightarrow{\exp} T \to 1$ where $\iota(h) = h \otimes 1$ and $\exp(h \otimes a) = h \otimes e^{2\pi\sqrt{-1}a}$, $h \in \mathfrak{h}_{\mathbb{Z}}$, $a \in \mathbb{C}$, gives to $\lambda \in \mathfrak{h}_{\mathbb{Z}}^*$ the character of $T$, $e^\lambda(t) = e^{2\pi\sqrt{-1}a\lambda(h)}$, $t = \exp(h \otimes a)$. Then, analytically, the

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region of absolute convergence of $\chi^{\omega}$ (an open, convex, $W$-invariant set in $\mathfrak{h}$) has been found in [4].

Returning to $A$ affine, define for a subgroup (or subset) $H$ of $G$, $H^{>1}=\{h\in H; |\delta(h)|>1\}$ and similarly $H^{<1}$, $H^{=1}$. Also $H^{\neq 1}=H^{<1}\cup H^{>1}$.

**Theorem 1.** (i) $T^{\mathrm{tr}}=T^{>1}=(T^{\mathrm{hs}})^2$,

(ii) $T^{\mathrm{b}}=ZT_{\mathrm{c}}\cup T^{>1}$.

**Proof.** Using the estimate $\mathrm{multi}_{\omega}\lambda\leqq K(\omega-\lambda)$ (where $K(\cdot)$ is the Kostant partition function) and $\mathrm{mult}\,\alpha=1$, $\alpha\in\Phi^{\mathrm{re}}$, $\mathrm{mult}\,y=l, \gamma\in\Phi^{\mathrm{im}}$ one sees that the region of absolute convergence of $\chi^{\omega}$ is given by the interior of the "Tits cone", $\{h\in\mathfrak{h}; \mathrm{Re}\,\delta(h)>0\}$ where $\chi^{\omega}$ defines a holomorphic function (see [4, p. 138]).

Let $b_{\lambda}^{\omega}=\sum_{m=0}^{\infty}\mathrm{mult}_{\omega}(\lambda-m\delta)\mathrm{e}^{-m\delta}$, and $W_{\lambda}$ the stabilizer of $\lambda$ in $W$. Notice that $W_{\lambda}\cap\Upsilon=\{1\}, \lambda\in P^{\omega}$. Then the formal character splits into a sum over the orbits of $\Upsilon$ on $\max(\omega)$ as

$$\chi^{\omega}=\sum_{\lambda\in\max(\omega)}\mathrm{e}^{\lambda}b_{\lambda}^{\omega}=\sum_{\substack{\lambda\in\max(\omega)\\ \lambda\bmod\Upsilon}}\left(\sum_{\tau\in\Upsilon}\mathrm{e}^{\tau(\lambda)}\right)b_{\lambda}^{\omega}.$$

The powers of the translation element $\tau_{v(\theta^{\vee})}=r_{\alpha_0}r_{\theta}$ are given by (see [4, p. 74])

$$\tau_{v(\theta^{\vee})}(\lambda)=\lambda+\lambda(c)v(\theta^{\vee})-(\lambda(\theta^{\vee})+\tfrac{1}{2}|\theta^{\vee}|^2\lambda(c))\delta$$

$$\tau_{v(\theta^{\vee})}^{m}(\lambda)=\lambda+m\lambda(c)v(\theta^{\vee})-(m\lambda(\theta^{\vee})+\frac{m}{2}|\theta^{\vee}|^2\,\lambda(c)$$

$$+\tfrac{1}{2}m(m-1)\lambda(c)v(\theta^{\vee})(\theta^{\vee}))\delta, m\in\mathbb{Z}, \lambda\in\mathfrak{h}^*.$$

Here $a_0v(\theta^{\vee})=\theta$ the highest root of $\Phi_{0_+}$. We know $w\delta=\delta$, $\forall w\in W$. Also $\delta(d)=a_0, \theta(c)=0=\theta(d)$.

Let $t\in T$ with $|\delta(t)|\leqq 1$ so $t=\exp h$, $\mathrm{Re}\,\delta(h)\leqq 0$. Consider the translations $w\tau_{v(\theta^{\vee})}^{m}w^{-1}(\lambda)$ with $w\in W_0$ chosen so that $w^{-1}(h\bmod\mathbb{C}c+\mathbb{C}d+\sqrt{-1}\mathfrak{h}_{0\mathbb{R}})$ lies in the fundamental chamber for $(\mathfrak{g}_0,\mathfrak{h}_{0\mathbb{Z}})$, and $\lambda=w(\omega)$ to see that $\chi^{\omega}(t)$ diverges.

The assertions follow. □

**Proposition 6.** (i) $B^{\mathrm{tr}}=B^{>1}\subseteq B^{\mathrm{b}}$,

(ii) $B^{>1}=(B^{\mathrm{hs}})^2$.

**Proof.** (i) It is evident (since unipotent elements are upper triangular) that $b=tu\in B^{\mathrm{tr}}\Leftrightarrow t\in T^{\mathrm{tr}}$ and $B^{\mathrm{tr}}=T^{\mathrm{tr}}U=T^{>1}U=B^{>1}$. The Levi subgroup $L_{\alpha}$ has, by Proposition 2, the Cartan decomposition $L_{\alpha}=K_{\alpha}TK_{\alpha}$ where $K_{\alpha}=L_{\alpha}'\cap K\leqq L_{\alpha}'$ is maximally compact, $\alpha\in\Phi^{\mathrm{re}}$. Then by Lemma 3, we have $L_{\alpha}^{>1}=K_{\alpha}T^{>1}K_{\alpha}\subseteq G^{\mathrm{b}}$ from Theorem 1. An element $b=tu_1\cdots u_m\in B, t\in T^{>1}$, on taking "$m$th root" $t=t_1\cdots t_m$ can be written $b=t_1u_1'\cdots t_mu_m'$ with each $t_ju_j'\in L_{\beta_j}^{>1}$, $\beta_j\in\Phi_+^{re}$, $j\in\{1,\ldots,m\}$. Hence $B^{>1}\subseteq B^{b}$.

(ii) Follows from (i) and Theorem 1 as

$$L_\alpha^{>1}\cap B=T^{>1}(L_\alpha^{>1}\cap B)\subseteq T^{>1}B^{hs}\subseteq(B^{hs})^2$$

Also $(B^{hs})^2\subseteq B^{tr}$. □

**Lemma 4.** $(G^{cl})^{-1}=G^{cl}$.

**Proof.** We know that $T/T_c\cap T_1\subseteq G^{sym}\subseteq G^{cl}$. Also given any $g\in G$, using (1.4) and Proposition 6(i), $\exists t\in T/T_c\cap T_1^{>1}$ with $\mathrm{R}(tg^{-1})$ bounded.

Let $(x_n)$ be a convergent sequence in $V^\omega$ with $\mathrm{R}(g)x_n\to 0$. Then $\mathrm{R}(t)x_n=\mathrm{R}(tg^{-1})\mathrm{R}(g)x_n\to 0$. But $\mathrm{R}(t^{-1})$ is closeable, thus $x_n\to 0$.

Hence we have shown that if $g\in G^{cl}$ then $\overline{\mathrm{R}(g)}$ is injective on $\mathrm{dom}\,\overline{\mathrm{R}(g)}$, which gives the lemma. □

**Corollary.** $G^{cl}=G$.

**Proof.** We know that $G^{cl}G^{b}\subseteq G^{cl}$ and $G^{b}\subseteq G^{cl}$.

Let $g\in G$. So as above $\exists t\in T^{>1}$ with $tg^{-1}\in G^{b}$. Therefore $gt^{-1}\in G^{cl}$ giving $g=(gt^{-1})t\in G^{cl}$. □

**Proposition 7.** (i) $B^{<1}=T^{<1}U\subseteq G\backslash G^{b}$,

(ii) $T^{=1}(U\backslash\{1\})\subseteq G\backslash G^{b}$.

**Proof.** One has $T^{=1}=ZT_cT_0^{sym}$. Taking into account (2.3) Proposition 5 and Theorem 1 (ii) in (3.3), we want to show that $t_0u\in G\backslash G_b$ with $t_0\in T_0^{sym}$, $t_0\neq 1$, $u\in U\backslash\{1\}$.

The formula in (3.3) for the power of an element in $\Upsilon$ and the character formula $\chi^\omega$ give that for $\lambda\in\max(\omega)$, taking a conjugate $\mu=w\tau_{\nu(\theta^\vee)}^{m}w^{-1}(\lambda)$, $w\in W_0$, $t_0=\exp h$ and $w^{-1}(h)$ in the fundamental chamber of $(\mathfrak{g}_0,\mathfrak{h}_{0Z})$, we have

$$\langle R(t_0u)z,z\rangle=\langle \mathrm{R}(t_0)z,z\rangle=e^{\lambda(h)+m\omega(c)\theta(w^{-1}h)/a_0}$$

where $z$ has weight $\mu, \|z\|=1$. □

**Theorem 2.** (0) $G^{b}=KB^{b}, B^{b}=B^{>1}\cup(B^{=1}\cap T^{b})$,

(1) $G^{b}\cap G^{tr}\supseteq G^{>1}=(G^{hs})^2=G^{hs}$,

(2) $G^{cpt}=G^{hs}$.

**Proof.** (0) We have $G=KB$, $KG^{b}=G^{b}$, $B^{>1}\subseteq B^{b}$, $B^{<1}\cap B^{b}=\emptyset$. Also $B^{=1}\cap B^{b}=ZT_c=T^{=1}\cap T^{b}$.

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(1) Follows from (3.2) Lemma 3 and (3.3) Proposition 6.

(2) From (1) and (3.3) Theorem 1, $G^{\text{cpt}}=KB^{\text{cpt}}=KB^{>1}=G^{>1}=G^{\text{hs}}$. □

**3.4. Conjugation invariance.** Let $G$ be of type (1), (2) or (3). Take $G(\emptyset)$ the union of the Borel subgroups of $G$; that is the set of elements of $G$ which are conjugate under $G$ ($\Rightarrow$ under $K$) into the standard Borel subgroup $B$.

**Proposition 8.** *Let $x\in(G^{\text{hs}})^2\cap G(\emptyset)$, and $g\in G$ with $gxg^{-1}\in(G^{\text{hs}})^2$, then*

$$\text{trace}_\omega \mathrm{R}(gxg^{-1})=\text{trace}_\omega \mathrm{R}(x), \forall\omega\in\mathscr{I}\text{nt}_+\cap\mathfrak{h}_{\mathbb{Z}}^*.$$

**Proof.** By definition $\exists k_1\in K$ with $k_1^{-1}xk_1=b\in B$. Also $\exists k\in K$, $b_1\in B$ with $gk_1=kb_1$ giving $gxg^{-1}=kb_1bb_1^{-1}k^{-1}$. Then from (2.2), $\forall\omega\in\mathscr{I}\text{nt}_+\cap\mathfrak{h}_{\mathbb{Z}}^*$,

$$\text{trace}_\omega \mathrm{R}(gxg^{-1})=\text{trace}_\omega \mathrm{R}(b_1bb_1^{-1})=\text{trace}_\omega \mathrm{R}(b)=\text{trace}_\omega \mathrm{R}(x).$$ □

**Lemma 5.**

$$\text{trace}_\omega \mathrm{R}(tgt^{-1})=\text{trace}_\omega \mathrm{R}(g), \forall g\in G, \forall t\in T, \forall\omega\in\mathscr{I}\text{nt}_+\cap\mathfrak{h}_{\mathbb{Z}}^*.$$

**Proof.** In fact writing $t=t_1t_2, t_1\in T\cap K, t_2\in T^{\text{sym}}$ (the polar decomposition), a matrix element

$$\langle R(tgt^{-1})z,z\rangle=\mathrm{e}^\lambda(t^{-1})\langle \mathrm{R}(g)z,\mathrm{R}(t_1^{-1}t_2)z\rangle$$

$$=\mathrm{e}^\lambda(t^{-1})\overline{\mathrm{e}^\lambda(t_1^{-1})}\mathrm{e}^\lambda(t_2)\langle \mathrm{R}(g)z,z\rangle=\langle \mathrm{R}(g)z,z\rangle$$

where $z$ is of weight $\lambda$. □

Now let $G$ be of type (2).

**Theorem 3.**

$$\text{trace}_\omega \mathrm{R}(gxg^{-1})=\text{trace}_\omega \mathrm{R}(x), \forall x\in G^{>1}, \forall g\in G$$

**Proof.** With $g=kb$, $k\in K$, $b\in B$, $b=u \bmod T$, $x=x_1x_2, x_1, x_2\in G^{>1}$ we have from (2.2), Lemma 5 and Theorem 2 that

$$\text{trace}_\omega \mathrm{R}(gxg^{-1})=\text{trace}_\omega \mathrm{R}(uxu^{-1})=\text{trace}_\omega \mathrm{R}((ux_1)(x_2u^{-1}))$$

$$=\text{trace}_\omega \mathrm{R}(x_2x_1)=\text{trace}_\omega \mathrm{R}(x).$$ □

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