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Beurling-Dahlberg-Sjögren Type Theorems for Minimally Thin Sets in a Cone

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Abstract. This paper shows that some characterizations of minimally thin sets connected with a domain having smooth boundary and a half-space in particular also hold for the minimally thin sets at a corner point of a special domain with corners, *i.e.*, the minimally thin set at ∞ of a cone.

1 Introduction

Let $\mathbf{R}$ and $\mathbf{R}_+$ be the set of all real numbers and all positive real numbers, respectively. We denote by $\mathbf{R}^n$ $(n \geq 2)$ the n-dimensional Euclidean space. A point in $\mathbf{R}^n$ is denoted by $P = (X, y)$, $X = (x_1, x_2, \dots, x_{n-1})$. The Euclidean distance of two points P and Q in $\mathbf{R}^n$ is denoted by $|P - Q|$. Also $|P - O|$ with the origin O of $\mathbf{R}^n$ is simply denoted by $|P|$. The boundary and the closure of a set S in $\mathbf{R}^n$ are denoted by ∂S and $\bar{S}$, respectively.

We introduce a system of spherical coordinates (r, Θ), $\Theta = (\theta_1, \theta_2, \dots, \theta_{n-1})$, in $\mathbf{R}^n$ which are related to cartesian coordinates $(x_1, x_2, \dots, x_{n-1}, y)$ by

$$x_1 = r(\Pi_{j=1}^{n-1} \sin\theta_j) \quad (n \geq 2), \qquad y = r\cos\theta_1,$$

and if $n \geq 3$, then

$$x_{n+1-k} = r(\Pi_{j=1}^{k-1} \sin\theta_j)\cos\theta_k \quad (2 \leq k \leq n-1),$$

where $0 \leq r < +\infty$, $-\frac{1}{2}\pi \leq \theta_{n-1} < \frac{3}{2}\pi$, and if $n \geq 3$, then $0 \leq \theta_j \leq \pi$ $(1 \leq j \leq n-2)$.

The unit sphere and the upper half unit sphere are denoted by $\mathbf{S}^{n-1}$ and $\mathbf{S}_+^{n-1}$, respectively. For simplicity, a point $(1, \Theta)$ on $\mathbf{S}^{n-1}$ and the set $\{\Theta \,;\, (1, \Theta) \in \Omega\}$ for a set Ω, $\Omega \subset \mathbf{S}^{n-1}$, are often identified with Θ and Ω, respectively. For two sets $\Lambda \subset \mathbf{R}_+$ and $\Omega \subset \mathbf{S}^{n-1}$, the set

$$\{(r, \Theta) \in \mathbf{R}^n \,;\, r \in \Lambda, (1, \Theta) \in \Omega\}$$

in $\mathbf{R}^n$ is simply denoted by $\Lambda \times \Omega$. In particular, the half-space

$$\mathbf{R}_+ \times \mathbf{S}_+^{n-1} = \{(X, y) \in \mathbf{R}^n \,;\, y > 0\}$$

will be denoted by $\mathbf{T}_n$.

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As an extension of a result of Beurling [6, Lemma 1], Dahlberg proved:

Theorem A **(Dahlberg [9, Theorem 4])** *Suppose that $E \subset \mathbf{T}_n$ is measurable and that*

$$\int_E \frac{dP}{(1+|P|)^n} = \infty.$$

If u is a non-negative superharmonic function in $\mathbf{T}_n$ and m is a positive number such that $u(P) \geq my$ for all $P = (X, y) \in E$, then $u(P) \geq my$ for all $P = (X, y) \in \mathbf{T}_n$.

Sjögren also gave Theorem A in the following form with an ingenious proof of Dahlberg's result.

Theorem B **(Sjögren [16, Theorem 2])** *Let $u(P)$ be a positive superharmonic function on $\mathbf{T}_n$ such that*

$$u(P) = \int_{\mathbf{T}_n} G(P, Q)\, d\mu(Q) + \int_{\partial \mathbf{T}_n} \Pi(P, Q)\, d\lambda(Q)$$

with non-negative measures μ and λ on $\mathbf{T}_n$ and $\partial\mathbf{T}_n$, respectively, where $G(P, Q)$ ($P, Q \in \mathbf{T}_n$) and

$$\Pi(P, Q) = y|P - Q|^{-n} \quad \big(P = (X, y) \in \mathbf{T}_n, Q \in \partial\mathbf{T}_n\big)$$

is the Green function and the Poisson kernel for $\mathbf{T}_n$, respectively. Then

$$\int_{E_u} \frac{dP}{(1+|P|)^n} < \infty,$$

where

$$E_u = \{P = (X, y) \in \mathbf{T}_n\,;\, u(P) > y\}.$$

Let $K(P, Q)$ ($P \in \mathbf{T}_n$, $Q \in \partial\mathbf{T}_n$) be the Martin function with the reference point $(0, 0, \dots, 0, 1) \in \mathbf{T}_n$. Then $K(P, \infty) = y$ for any $P = (X, y) \in \mathbf{T}_n$. A subset E of $\mathbf{T}_n$ is said to be minimally thin at ∞ with respect to $\mathbf{T}_n$, if there exists a point $P = (X, y) \in \mathbf{T}_n$ such that

$$\hat{R}^E_{K(\cdot,\infty)}(P) \neq y,$$

where $\hat{R}^E_{K(\cdot,\infty)}$ is the regularized reduced function of $K(P, \infty) = y$ $\big(P = (X, y) \in \mathbf{T}_n\big)$ relative to E (Helms [13, p. 134]).

We remark that the conclusions of Theorems A and B are equivalent to the facts that E is not minimally thin at ∞ and E_u is minimally thin at ∞, respectively (Theorem 1 in the case where $C_n(\Omega) = \mathbf{T}_n$). Hence Theorems A and B say:

Theorem C *If $E \subset \mathbf{T}_n$ is measurable and minimally thin at ∞ with respect to $\mathbf{T}_n$, then*

$$\int_E \frac{dP}{(1+|P|)^n} < +\infty. \tag{1.1}$$

Further the following Theorem D shows that the characterization of a minimally thin set in Theorem C is sharp.

Theorem D *Let E be a union of cubes from the Whitney cubes of $\mathbf{T}_n$. Then (1.1) is also sufficient for E to be minimally thin at ∞ with respect to $\mathbf{T}_n$.*

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These Theorems A, B, C and D follow from the results of Dahlberg [9, Theorem 2], Sjögren [16, Theorem 2], Aikawa [1, Corollary 7 and Corollary 8], Aikawa and Essén [3, Corollary 7.4.6 in p. 158] which are all connected with a Liapunov-Dini domain in $\mathbf{R}^n$, because $\mathbf{T}_n$ is mapped onto a ball by a suitable Kelvin transformation.

All these results are connected to minimally thin sets at a boundary point of domains with smooth boundary. So we can ask what is a result similar to Theorem C with respect to a minimally thin set at a corner of a domain with corners. In this direction, Aikawa [2, Corollary 4] gave a complicated result with respect to a minimally thin set at a boundary point of an NTA domain which is a mostly irregular domain taken into consideration.

In this paper we shall show that the same type of theorems as Theorems C and D are still true with respect to a minimally thin set at a corner point of a special domain with corners, *i.e.*, a minimally thin set at ∞ of a cone. These theorems are proved by modifying Aikawa's method in [3]. Then we shall generalize Theorems A and B for positive superharmonic functions in a cone one of which is a half-space $\mathbf{T}_n$. In view of our results it is natural to ask whether similar results are valid for Lipshitz domains or more generally, for NTA domains.

2 Statements of Results

Let Ω be a domain on $\mathbf{S}^{n-1}(n \geq 2)$ with smooth boundary. Consider the Dirichlet problem

$$\begin{aligned}(\Lambda_n + \tau) f &= 0 \quad \text{on } \Omega \\ f &= 0 \quad \text{on } \partial\Omega,\end{aligned}$$

where Λ_n is the spherical part of the Laplace operator Δ_n

$$\Delta_n = \frac{n-1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2} + r^{-2}\Lambda_n.$$

We denote the least positive eigenvalue of this boundary value problem by τ_Ω and the normalized positive eigenfunction corresponding to τ_Ω by $f_\Omega(\Theta)$;

$$\int_\Omega f_\Omega^2(\Theta)\, d\sigma_\Theta = 1,$$

where $d\sigma_\Theta$ is the surface element on $\mathbf{S}^{n-1}$. We denote the solutions of the equation

$$t^2 + (n-2)t - \tau_\Omega = 0$$

by $\alpha_\Omega, -\beta_\Omega$ $(\alpha_\Omega, \beta_\Omega > 0)$. If $\Omega = \mathbf{S}_+^{n-1}$, then $\alpha_\Omega = 1, \beta_\Omega = n-1$ and

$$f_\Omega(\Theta) = (2ns_n^{-1})^{1/2}\cos\theta_1,$$

where s_n is the surface area $2\pi^{n/2}\{\Gamma(n/2)\}^{-1}$ of $\mathbf{S}^{n-1}$.

To make simplify our consideration in the following, we shall assume that if $n \geq 3$, then Ω is a $C^{2,\alpha}$-domain $(0 < \alpha < 1)$ on $\mathbf{S}^{n-1}$ (*e.g.* see Gilbarg and Trudinger [12, pp. 88–89] for the definition of $C^{2,\alpha}$-domain).

By $C_n(\Omega)$, we denote the set $\mathbf{R}_+ \times \Omega$ in $\mathbf{R}^n$ with the domain Ω on $\mathbf{S}^{n-1} (n \geq 2)$. We call it a cone. Then $\mathbf{T}_n$ is a special cone obtained by putting $\Omega = \mathbf{S}_+^{n-1}$.

It is known that the Martin boundary of $C_n(\Omega)$ is the set $\partial C_n(\Omega) \cup \{\infty\}$, each of which is a minimal Martin boundary point. When we denote the Martin kernel by $K(P, Q)$ $\big(P \in C_n(\Omega), Q \in \partial C_n(\Omega) \cup \{\infty\}\big)$ with respect to a reference point chosen suitably, we know

$$K(P, \infty) = r^{\alpha_\Omega} f_\Omega(\Theta), \quad K(P, O) = \kappa r^{-\beta_\Omega} f_\Omega(\Theta) \quad \big(P \in C_n(\Omega)\big),$$

where κ is a positive constant.

A subset E of $C_n(\Omega)$ is said to be minimally thin at $Q \in \partial C_n(\Omega) \cup \{\infty\}$ with respect to $C_n(\Omega)$ (Brelot [7, p. 122], Doob [10, p. 208]), if there exists a point $P \in C_n(\Omega)$ such that

$$\hat{R}^E_{K(\cdot, Q)}(P) \neq K(P, Q),$$

where $\hat{R}^E_{K(\cdot, Q)}(P)$ is the regularized reduced function of $K(\cdot, Q)$ relative to E.

Let E be a bounded subset of $C_n(\Omega)$. Then $\hat{R}^E_{K(\cdot, \infty)}$ is bounded on $C_n(\Omega)$ and hence the greatest harmonic minorant of $\hat{R}^E_{K(\cdot, \infty)}$ is zero. When we denote by $G(P, Q)$ $\big(P \in C_n(\Omega), Q \in C_n(\Omega)\big)$ the Green function of $C_n(\Omega)$, we see from the Riesz decomposition theorem that there exists a unique positive measure λ_E on $C_n(\Omega)$ such that

$$\hat{R}^E_{K(\cdot, \infty)}(P) = G\lambda_E(P)$$

for any $P \in C_n(\Omega)$ and λ_E is concentrated on B_E, where

$$B_E = \{P \in C_n(\Omega)\ ;\ E \text{ is not thin at } P\}$$

(see Brelot [7, Theorem VIII, 11] and Doob [10, XI. 14. Theorem (d)]). The (Green) energy $\gamma_\Omega(E)$ of λ_E is defined by

$$\gamma_\Omega(E) = \int_{C_n(\Omega)} (G\lambda_E)\, d\lambda_E$$

(see Helms [13, p. 223]). Let E be a Borel subset of $C_n(\Omega)$ and $E_k = E \cap I_k(\Omega)$ $(k = 0, 1, 2, \dots)$, where

$$I_k(\Omega) = \{(r, \Theta) \in C_n(\Omega)\ ;\ 2^k \leq r < 2^{k+1}\}.$$

First we shall state Theorem 1, essentially due to Miyamoto and Yoshida [15, p. 6, Theorem 1], which, with Theorem 2, gives Corollaries 1 and 2 extending Theorems A and B, respectively.

Theorem 1 *The following statements are equivalent.*

(I) *A subset E of $C_n(\Omega)$ is minimally thin at ∞ with respect to $C_n(\Omega)$.*

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(II) *(Wiener type)* $\sum_{k=0}^{\infty} \gamma_\Omega(E_k) 2^{-k(\alpha_\Omega+\beta_\Omega)} < \infty$.

(III) *(Sjögren type) There exists a positive superharmonic function $v(P)$ on $C_n(\Omega)$ such that*

$$\inf_{P\in C_n(\Omega)} \frac{v(P)}{K(P,\infty)} = 0 \tag{2.1}$$

and

$$E \subset M_v,$$

where

$$M_v = \{P \in C_n(\Omega)\ ;\ v(P) \geq K(P,\infty)\}.$$

(IV) *(Dahlberg type) There exist a positive superharmonic function $v(P)$ on $C_n(\Omega)$ and a positive number m such that even if $v(P) \geq mK(P,\infty)$ ($P \in E$), there exists $P_0 \in C_n(\Omega)$ satisfying $v(P_0) < mK(P_0,\infty)$.*

The following Theorem 2 is the main theorem in this paper.

Theorem 2 *Let a Borel subset E of $C_n(\Omega)$ be minimally thin at ∞ with respect to $C_n(\Omega)$. Then we have*

$$\int_E \frac{dP}{(1+|P|)^n} < \infty. \tag{2.2}$$

When we decompose a positive superharmonic function $v(P)$ on $C_n(\Omega)$ into

$$v(P) = \int_{C_n(\Omega)} G(P,Q)\, d\mu(Q) + \int_{\partial C_n(\Omega)} K(P,Q)\, d\nu(Q) + K(P,\infty)\nu(\{\infty\})$$

with two measures μ and ν on $C_n(\Omega)$ and $\partial C_n(\Omega) \cup \{\infty\}$, respectively, we see that (2.1) is equivalent to $\nu(\{\infty\}) = 0$ (Doob [10, p. 213, Theorem]). This fact shows that the following corollary of Sjögren type generalizes Theorem B.

Corollary 1 *Let $v(P)$ be a positive superharmonic function on $C_n(\Omega)$ such that*

$$\inf_{P\in C_n(\Omega)} \frac{v(P)}{K(P,\infty)} = 0.$$

Then we have

$$\int_{M_v} \frac{dP}{(1+|P|)^n} < \infty.$$

From Theorems 1 and 2 we also obtain the following corollary of Dahlberg type, which generalizes Theorem A.

Corollary 2 *Let E be a Borel measurable subset of $C_n(\Omega)$ satisfying*

$$\int_E \frac{dP}{(1+|P|)^n} = +\infty.$$

If $v(P)$ is a non-negative superharmonic function on $C_n(\Omega)$ and m is a positive number such that $v(P) \geq mK(P,\infty)$ for all $P \in E$, then $v(P) \geq mK(P,\infty)$ for all $P \in C_n(\Omega)$.

In order to state Theorem 3 which shows the sharpness of the characterization of a minimally thin set in Theorem 2, we introduce the Whitney cubes of $C_n(\Omega)$.

A cube is of the form

$$[l_1 2^{-k}, (l_1+1)2^{-k}] \times \cdots \times [l_n 2^{-k}, (l_n+1)2^{-k}]$$

where k, $l_1, \ldots, l_n$ are integers. The Whitney cubes of $C_n(\Omega)$ are a family of cubes having the following properties:

(i) $\bigcup_j W_j = C_n(\Omega)$,

(ii) $\operatorname{int} W_j \cap \operatorname{int} W_k = \varnothing$ $(j \neq k)$,

(iii) $\operatorname{diam} W_j \leq \operatorname{dist}\big(W_j, \mathbf{R}^n \setminus C_n(\Omega)\big) \leq 4 \operatorname{diam} W_j$,

where $\operatorname{int} S$, $\operatorname{diam} S$, $\operatorname{dist}(S_1, S_2)$ stand for the interior of S, the diameter of S, the distance between S_1 and S_2, respectively (Stein [17, p. 167, Theorem 1]).

Theorem 3 *If E is a union of cubes from the Whitney cubes of $C_n(\Omega)$, then (2.2) is also sufficient for E to be minimally thin at ∞ with respect to $C_n(\Omega)$.*

3 Lemmas and Their Proofs

For a function $F(P,Q)$ $\big(P,Q \in C_n(\Omega)\big)$ and a positive measure μ on $C_n(\Omega)$,

$$\int_{C_n(\Omega)} F(P,Q)\, d\mu(Q)$$

is simply denoted by $F\mu(P)$. We shall also write $g_1 \approx g_2$ for two positive functions g_1 and g_2, if and only if there exists a positive constant a such that $a^{-1} g_1 \leq g_2 \leq a g_1$.

Let E be a Borel subset of $C_n(\Omega)$ and let $\delta(P) = \operatorname{dist}\big(P, \partial C_n(\Omega)\big)$ for a point $P \in C_n(\Omega)$. We define a measure σ_Ω on $C_n(\Omega)$ by

$$\sigma_\Omega(E) = \int_E \left(\frac{K(P,\infty)}{\delta(P)} \right)^2 dP.$$

Lemma 1 *Let E be a bounded Borel subset of $C_n(\Omega)$. Then there exists a constant M_1 independent of E such that*

$$\sigma_\Omega(E) \leq M_1 \gamma_\Omega(E).$$

Proof First of all, we remark that $\mathbf{R}^n \setminus C_n(\Omega)$ is $(1,2)$ uniformly fat, *i.e.*, there is a positive constant ι such that at any $P \in \mathbf{R}^n \setminus C_n(\Omega)$

$$\operatorname{Cap}\Big(\big\{ P + r^{-1}(Q-P) \in \mathbf{R}^n \,;\, Q \in B(P,r) \cap \big(\mathbf{R}^n \setminus C_n(\Omega)\big) \big\} \Big) \geq \iota$$

for every positive number r, where $B(P,r) = \{Q \in \mathbf{R}^n : |Q-P| < r\}$ and Cap denotes the Newtonian capacity (see Lewis [14, p. 178]). Then by a result of Lewis [14, Theorem 2], there is a positive constant M_1 depending only on ι and n such that

$$\int_{C_n(\Omega)} \left| \frac{\psi(P)}{\delta(P)} \right|^2 dP \leq M_1 \int_{C_n(\Omega)} |\nabla \psi(P)|^2\, dP \tag{3.1}$$

for every $\psi \in C_0^\infty\big(C_n(\Omega)\big)$ (also see Ancona [4]).

We denote the function $G\lambda_E(P) = \hat{R}^E_{K(\cdot,\infty)}(P)$ on $C_n(\Omega)$ by $\nu_E(P)$. It is well known that the Green energy can be represented as the Dirichlet integral, *i.e.*,

$$\gamma_\Omega(E) = \int_{C_n(\Omega)} |\nabla \nu_E|^2\, dP. \tag{3.2}$$

Since

$$A^{-1} r^{\alpha_\Omega} f_\Omega(\Theta) t^{-\beta_\Omega} f_\Omega(\Phi) \le G(P,Q) \le A r^{\alpha_\Omega} f_\Omega(\Theta) t^{-\beta_\Omega} f_\Omega(\Phi) \tag{3.3}$$

for any $P = (r,\Theta) \in C_n(\Omega)$ and any $Q = (t,\Phi) \in C_n(\Omega)$ satisfying $2r \le t$, where A is a positive constant (see Azarin [5, Lemma 1]) and

$$f_\Omega(\Theta) \approx \delta(P) \tag{3.4}$$

for any $P = (1,\Theta) \in \Omega$ (see Courant and Hilbert [8]), we also see

$$\int_{C_n(\Omega)} \left| \frac{\nu_E(P)}{\delta(P)} \right|^2 dP < +\infty. \tag{3.5}$$

Hence we have $\nu_E \in H\big(C_n(\Omega)\big)$ from (3.2) and (3.5), where

$$H\big(C_n(\Omega)\big) = \big\{ f \in L^2_{\mathrm{loc}}\big(C_n(\Omega)\big) : \nabla f \in L^2\big(C_n(\Omega)\big), \delta^{-1} f \in L^2\big(C_n(\Omega)\big) \big\}$$

equipped with the norm

$$\|f\|_{H(C_n(\Omega))} = (\|\nabla f\|^2_{L^2(C_n(\Omega))} + \|\delta^{-1} f\|^2_{L^2(C_n(\Omega))})^{\frac{1}{2}},$$

and further $\nu_E \in H_0\big(C_n(\Omega)\big)$, where $H_0\big(C_n(\Omega)\big)$ denotes the closure of $C_0^\infty\big(C_n(\Omega)\big)$ in $H\big(C_n(\Omega)\big)$. Thus we obtain from (3.1) that

$$\int_{C_n(\Omega)} \left| \frac{\nu_E(P)}{\delta(P)} \right|^2 dP \le M_1 \int_{C_n(\Omega)} |\nabla \nu_E(P)|^2\, dP$$

(see Ancona [4, p. 288]). Since $\nu_E = K(\cdot,\infty)$ quasi everywhere on E and hence a.e. on E, we have from (3.2)

$$\gamma_\Omega(E) \ge M_1^{-1} \int_{C_n(\Omega)} \left(\frac{\nu_E(P)}{\delta(P)} \right)^2 dP \ge M_1^{-1} \int_E \left(\frac{K(P,\infty)}{\delta(P)} \right)^2 dP = M_1^{-1} \sigma_\Omega(E),$$

which gives the conclusion.

Lemma 2 *Let W_j be a cube from the Whitney cubes of $C_n(\Omega)$. Then there exists a constant M_2 independent of j such that*

$$\gamma_\Omega(W_j) \le M_2 \sigma_\Omega(W_j).$$

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Proof If we apply a result of Aikawa and Essén [3, Theorem 5.6, p. 19] for compact set $\overline{W}_j$, we obtain a measure μ on $C_n(\Omega)$, supp $\mu \subset \overline{W}_j$, $\mu(\overline{W}_j) = 1$ such that

$$\begin{cases} \int_{C_n(\Omega)} |P-Q|^{2-n}\, d\mu(Q) = \{\operatorname{Cap}(\overline{W}_j)\}^{-1} & (n \geq 3), \\ \int_{C_2(\Omega)} \log |P-Q|\, d\mu(Q) = \log \operatorname{Cap}(\overline{W}_j) & (n = 2), \end{cases} \tag{3.6}$$

for any $P \in \overline{W}_j$. Also there exists a positive measure $\lambda_{\overline{W}_j}$ on $C_n(\Omega)$ such that

$$\hat{R}^{\overline{W}_j}_{K(\cdot,\infty)}(P) = G\lambda_{\overline{W}_j}(P) \quad \big(P \in C_n(\Omega)\big). \tag{3.7}$$

Let $P_j = (r_j, \Theta_j)$, ρ_j, t_j be the center of W_j, the diameter of W_j, the distance between W_j and $\partial C_n(\Omega)$, respectively. Then we have $\rho_j \leq t_j \leq 4\rho_j$ and $\rho_j \leq r_j$. Then from (3.4) we can find a positive constant A_1 independent of j such that

$$A_1^{-1} r_j^{\alpha_\Omega - 1} \rho_j \leq K(P, \infty) \leq A_1 r_j^{\alpha_\Omega - 1} \rho_j \tag{3.8}$$

for any $P \in \overline{W}_j$. We can also prove that

$$G(P,Q) \geq \begin{cases} A_2 |P-Q|^{2-n} & (n \geq 3), \\ \log \frac{A_3 \rho_j}{|P-Q|} & (n = 2), \end{cases} \tag{3.9}$$

for any $P \in \overline{W}_j$ and any $Q \in \overline{W}_j$, where A_2 and A_3 are two positive constants independent of j. Hence we obtain

$$\lambda_{\overline{W}_j}\big(C_n(\Omega)\big) \leq \begin{cases} (A_1/A_2) r_j^{\alpha_\Omega - 1} \rho_j \operatorname{Cap}(\overline{W}_j) & (n \geq 3) \\ A_1 r_j^{\alpha_\Omega - 1} \rho_j \left\{ \log \frac{A_3 \rho_j}{\operatorname{Cap}(\overline{W}_j)} \right\}^{-1} & (n = 2) \end{cases} \tag{3.10}$$

from (3.6), (3.7), (3.8) and (3.9). Since

$$\gamma_\Omega(\overline{W}_j) = \int G\lambda_{\overline{W}_j}\, d\lambda_{\overline{W}_j} \leq \int_{\overline{W}_j} K(P,\infty)\, d\lambda_{\overline{W}_j}(P) \leq A_1 r_j^{\alpha_\Omega - 1} \rho_j \lambda_{\overline{W}_j}\big(C_n(\Omega)\big)$$

from (3.7) and (3.8), we have from (3.10)

$$\gamma_\Omega(\overline{W}_j) \leq \begin{cases} A_1^2 A_2^{-1} r_j^{2\alpha_\Omega - 2} \rho_j^2 \operatorname{Cap}(\overline{W}_j) & (n \geq 3), \\ A_1^2 r_j^{2\alpha_\Omega - 2} \rho_j^2 \left\{ \log \frac{A_3 \rho_j}{\operatorname{Cap}(\overline{W}_j)} \right\}^{-1} & (n = 2). \end{cases} \tag{3.11}$$

Since

$$\begin{cases} \operatorname{Cap}(\overline{W}_j) \approx \rho_j^{n-2} & (n \geq 3), \\ \operatorname{Cap}(\overline{W}_j) \approx \rho_j & (n = 2), \end{cases}$$

we obtain from (3.11)

$$\gamma_\Omega(W_j) \leq A_4 r_j^{2\alpha_\Omega - 2} \rho_j^n \tag{3.12}$$

with a positive constant A_4. On the other hand, we have from (3.4) that

$$\sigma_\Omega(W_j) \approx r_j^{2\alpha_\Omega - 2} \rho_j^n \tag{3.13}$$

for any $P = (r, \Theta) \in W_j$. From (3.12) and (3.13) we finally have

$$\gamma_\Omega(W_j) \leq M_2 \sigma_\Omega(W_j),$$

which is the conclusion of Lemma 2.

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4 Proofs of Theorems 1, 2 and 3

Proof of Theorem 1 It is a result of Miyamoto and Yoshida [15, Theorem 1] that (II) follows from (I).

We shall show that (III) follows from (II). Since

$$\hat{R}^{E_k}_{K(\cdot,\infty)}(Q) = K(Q,\infty)$$

for any $Q \in B_{E_k}$ (Brelot [7, p. 61] and Doob [10, p. 169]) and λ_{E_k} is concentrated on B_{E_k}, we have

$$\begin{aligned}\gamma_\Omega(E_k) &= \int_{B_{E_k}} K(Q,\infty)\, d\lambda_{E_k}(Q)\\ &\geq 2^{k\alpha_\Omega} \int_{B_{E_k}} f_\Omega(\Phi)\, d\lambda_{E_k}(t,\Phi) \quad \big(Q=(t,\Phi)\in C_n(\Omega)\big)\end{aligned}$$

and hence from (3.3)

$$\begin{aligned}\hat{R}^{E_k}_{K(\cdot,\infty)}(P) &\leq A r^{\alpha_\Omega} f_\Omega(\Theta) \int_{B_{E_k}} t^{-\beta_\Omega} f_\Omega(\Phi)\, d\lambda_{E_k}(t,\Phi)\\ &\leq A r^{\alpha_\Omega} f_\Omega(\Theta) 2^{-k(\alpha_\Omega+\beta_\Omega)} \gamma_\Omega(E_k)\end{aligned} \tag{4.1}$$

for any $P=(r,\Theta)\in C_n(\Omega)$ and any integer k satisfying $2^k \geq 2r$. If we define a measure μ on $C_n(\Omega)$ by

$$\mu = \sum_{k=0}^{\infty} \lambda_{E_k},$$

then from (II) and (4.1)

$$G\mu(P) = \sum_{k=0}^{\infty} \hat{R}^{E_k}_{K(\cdot,\infty)}(P)$$

is a finite-valued superharmonic function on $C_n(\Omega)$,

$$G\mu(P) \geq \hat{R}^{E_k}_{K(\cdot,\infty)}(P) = r^{\alpha_\Omega} f_\Omega(\Theta)$$

for any $P=(r,\Theta)\in B_{E_k}$ $(k=0,1,2,\dots)$, and from (3.3)

$$G\mu(P) \geq A_5 r^{\alpha_\Omega} f_\Omega(\Theta)$$

for any $P=(r,\Theta)\in \{P=(r,\Theta)\in C_n(\Omega)\,;\, 0<r<1\}$, where

$$A_5 = A^{-1} \int_{\{Q=(t,\Phi)\in C_n(\Omega); 2\leq t\}} t^{-\beta_\Omega} f_\Omega(\Phi)\, d\mu(Q).$$

If we set

$$E' = \bigcup_{k=-1}^{\infty} B_{E_k},$$

where

$$E_{-1} = E \cap \{P = (r, \Theta) \in C_n(\Omega)\ ;\ 0 < r < 1\} \tag{4.2}$$

and $A_6 = \min(A_5, 1)$, then

$$E' \subset \{P = (r, \Theta) \in C_n(\Omega)\ ;\ G\mu(P) \geq A_6 r^{\alpha_\Omega} f_\Omega(\Theta)\}$$

and E' is equal to E except a polar set S (see Brelot [7, p. 57] and Doob [10, p. 177]). If we take a positive measure η on $C_n(\Omega)$ such that $G\eta$ is identically $+\infty$ on S (see Doob [10, p. 58]) and define a measure ν on $C_n(\Omega)$ by

$$\nu = A_6^{-1}(\mu + \eta),$$

then

$$E \subset \{P = (r, \Theta) \in C_n(\Omega)\ ;\ G\nu(P) \geq r^{\alpha_\Omega} f_\Omega(\Theta)\}.$$

If we put $v(P) = G\nu(P)$, then this shows that $v(P)$ is the function required in (III).

Now we shall see that (IV) follows from (III). Let $v(P)$ be the function in (III). It follows that

$$v(P) \geq K(P, \infty)$$

for any $P \in E$. On the other hand from (2.1) we can find a point $P_0 \in C_n(\Omega)$ satisfying

$$v(P_0) < K(P_0, \infty).$$

Therefore $v(P)$ satisfies (IV) with $m = 1$.

Finally we shall prove that (I) follows from (IV). Let $v(P)$ be the function in (IV). If we put

$$\inf_{P \in C_n(\Omega)} \frac{v(P)}{K(P, \infty)} = c_\infty(v)$$

and

$$u(P) = v(P) - c_\infty(v) K(P, \infty),$$

then we have

$$\inf_{P \in C_n(\Omega)} \frac{u(P)}{K(P, \infty)} = 0.$$

Since there exists $P_0 \in C_n(\Omega)$ satisfying $v(P_0) < mK(P_0, \infty)$, we note that

$$c_\infty(v) < m.$$

Now we obtain

$$\begin{aligned} u(P) &\geq mK(P, \infty) - c_\infty(v) K(P, \infty) \\ &= \bigl(m - c_\infty(v)\bigr) K(P, \infty) \end{aligned}$$

for any $P \in E$. Hence by a result of Doob [10, p. 213], E is minimally thin at ∞ with respect to $C_n(\Omega)$, which is the statement of (I).

Proof of Theorem 2 First of all we remark that

$$\int_E \frac{dP}{(1+|P|)^n} = \int_{E_{-1}} \frac{dP}{(1+|P|)^n} + \sum_{k=0}^{\infty} \int_{E_k} \frac{dP}{(1+|P|)^n} \le |E_{-1}| + \sum_{k=0}^{\infty} 2^{-kn}|E_k|, \tag{4.3}$$

where E_{-1} is the set in (4.2) and $|E_k|$ is the n-dimensional Lebesgue measure of E_k. We have from (3.4)

$$A_7\delta(P) \le r f_\Omega(\Theta),$$

for any $P = (r, \Theta) \in C_n(\Omega)$, where A_7 is a positive constant, hence

$$\begin{aligned} \sigma_\Omega(E_k) &= \int_{E_k} \left(\frac{K(P,\infty)}{\delta(P)}\right)^2 dP \ge A_7^2 \int_{E_k} \left(\frac{r^{\alpha_\Omega} f_\Omega(\Theta)}{r f_\Omega(\Theta)}\right)^2 dP \\ &= A_7^2 \int_{E_k} r^{2\alpha_\Omega - 2}\, dP \ge 2^{-2} A_7^2 \int_{E_k} 2^{k(2\alpha_\Omega - 2)}\, dP \\ &= 2^{-2} A_7^2 2^{k(2\alpha_\Omega - 2)} |E_k|. \end{aligned}$$

By using Lemma 1, we obtain

$$\gamma_\Omega(E_k) \ge M_1^{-1} \sigma_\Omega(E_k) \ge A_8 2^{k(2\alpha_\Omega - 2)} |E_k|, \tag{4.4}$$

where A_8 is a positive constant.

If E is minimally thin at ∞ with respect to $C_n(\Omega)$, then from Theorem 1, (4.3) and (4.4), we have

$$\begin{aligned} \int_E \frac{dP}{(1+|P|)^n} &\le |E_{-1}| + \sum_{k=0}^{\infty} 2^{k(2\alpha_\Omega - 2)} |E_k| 2^{-k(\alpha_\Omega + \beta_\Omega)} \\ &\le |E_{-1}| + A_8^{-1} \sum_{k=0}^{\infty} \gamma_\Omega(E_k) 2^{-k(\alpha_\Omega + \beta_\Omega)} < \infty, \end{aligned}$$

which is the conclusion of Theorem 2.

Proof of Theorem 3 Let $\{W_j\}$ be a family of cubes from the Whitney cubes of $C_n(\Omega)$ such that $E = \bigcup_j W_j$. Let $\{W_{k,j}\}$ be a subfamily of $\{W_j\}$ such that $W_{k,j} \subset (E_{k-1} \cup E_k \cup E_{k+1})$ $(k = 1, 2, \dots)$.

Since γ_Ω is a countably subadditive set function (Essén and Jackson [11, Lemma 2.1]), we have

$$\gamma_\Omega(E_k) \le \sum_j \gamma_\Omega(W_{k,j}) \quad (k = 1, 2, \dots). \tag{4.5}$$

Hence we see from Lemma 2

$$\sum_j \gamma_\Omega(W_{k,j}) \le M_2 \sum_j \sigma_\Omega(W_{k,j}) \quad (k = 1, 2, \dots). \tag{4.6}$$

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Since we see from (3.4)

$$r f_\Omega(\Theta) \le A_9 \delta(P)$$

for any $P = (r, \Theta) \in C_n(\Omega)$, where A_9 is a positive constant, we have

(4.7)

$$\begin{aligned}\sum_j \sigma_\Omega(W_{k,j}) &\le A_9^2 \Big\{ \int_{E_{k-1}} r^{2(\alpha_\Omega - 1)}\, dP + \int_{E_k} r^{2(\alpha_\Omega-1)}\, dP + \int_{E_{k+1}} r^{2(\alpha_\Omega-1)}\, dP \Big\} \\ &\le A_{10}\{2^{(k-1)(2\alpha_\Omega-2)}|E_{k-1}| + 2^{k(2\alpha_\Omega-2)}|E_k| + 2^{(k+1)(2\alpha_\Omega-2)}|E_{k+1}|\} \\ &\qquad (k = 1, 2, \dots),\end{aligned}$$

where A_{10} is a positive constant. Thus (4.5), (4.6) and (4.7) give

$$\gamma_\Omega(E_k) \le M_2 \cdot A_{10}\{2^{(k-1)(2\alpha_\Omega-2)}|E_{k-1}| + 2^{k(2\alpha_\Omega-2)}|E_k| + 2^{(k+1)(2\alpha_\Omega-2)}|E_{k+1}|\}$$

for $k = 1, 2, \dots$. Finally we obtain

$$\begin{aligned}\sum_{k=0}^{\infty} \gamma_\Omega(E_k) 2^{-k(\alpha_\Omega+\beta_\Omega)} &\le \gamma_\Omega(E_0) + A_{11} \cdot 2^{-2n} \sum_{k=0}^{\infty} 2^{k(2\alpha_\Omega-2)}|E_k| 2^{-k(\alpha_\Omega+\beta_\Omega)} \\ &\le \gamma_\Omega(E_0) + A_{11} \int_E \frac{dP}{(1+|P|)^n} < \infty,\end{aligned}$$

where A_{11} is a positive constant, which shows with Theorem 1 that E is minimally thin at ∞ with respect to $C_n(\Omega)$.

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