ON THE COMMUTATIVITY OF CERTAIN DIVISION RINGS

TADASI NAKAYAMA

Formerly Hua [**1**] proved that if A is a division ring with centre Z and if there exists a natural number n such that $a^n \in Z$ for every $a \in A$, then A is commutative; this generalizes Wedderburn's theorem on finite division rings. Another generalization of Wedderburn's theorem, due to Jacobson [**3**], asserts that every algebraic division algebra over a finite field is commutative. On the other hand, a theorem of Noether and Jacboson [**3**] states that every non-commutative algebraic division algebra contains an element which is not contained in the centre Z and is separable over Z. These results have been successfully unified by Kaplansky [**4**] into one theorem, which asserts that if there exists for every element a in a division ring A with centre Z a natural number $n(a)$ (depending, perhaps, on a) such that $a^{n(a)} \in Z$, then A is commutative. He also proved that if there exists a (fixed) non-zero polynomial f with coefficients in Z and without constant term, such that $f(a) \in Z$ for every $a \in A$, then A is commutative. Recently Ikeda [**2**] obtained a certain generalization of the former of these theorems of Kaplansky, which deals with polynomials with coefficients from the prime field, instead of single powers, and which includes a particular case of the latter of Kaplansky's theorems. In the present note we prove the following theorem[1] which includes all these results:

THEOREM. *Let A be a division ring and Z be its centre. Let r be a natural number and $\alpha_1, \alpha_2, \ldots, \alpha_r$ be r (fixed) non-zero elements in Z. Suppose that there exist, for each element a of A, r natural numbers $n_1(a), n_2(a), \ldots, n_r(a)$ such that*

$$(1) \qquad n_1(a) < n_i(a) \qquad (i = 2, \ldots, r),$$

$$(2) \qquad a^{n_1(a)}\alpha_1 + a^{n_2(a)}\alpha_2 + \ldots + a^{n_r(a)}\alpha_r \in Z.$$

Then necessarily $A = Z$, that is, A is commutative.

Our proof is somewhat arithmetical (in a weak sense), while the approaches of the former authors have all been algebraic. We need

LEMMA 1. *Let Z be a field which is either*

(i) *of characteristic* 0, *or*

(ii) *of characteristic $p \neq 0$ and non-algebraic over its prime field, and let L be an algebraic proper extension of Z which is not purely inseparable over Z. Then*

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[1]Cf. I. N. Herstein, *A generalization of a theorem of Jacobson* III, Amer. J. Math., *75* (1953), 105–111.

there exists a pair of distinct (special) exponential valuations ρ_1, ρ_2 in L which coincide on Z.

This lemma is perhaps more or less known; anyway we shall come back to it elsewhere.

LEMMA 2. *Let Z and L be as in Lemma 1. There cannot exist a natural number r and a set of r non-zero elements $\alpha_1, \alpha_2, \ldots, \alpha_r$ in L such that for each element a in L there exist r natural numbers $n_1(a), n_2(a), \ldots, n_r(a)$ satisfying the conditions (1) and (2) in our Theorem.*

Proof. Let r be a natural number, and $\alpha_1, \alpha_2, \ldots, \alpha_r$ be r non-zero elements in L. Let ρ_1, ρ_2 be as in Lemma 1. Take two elements a_1, a_2 in L such that

$$\rho_1(a_1) \geqslant 1, \quad \rho_2(a_1) = 0, \quad \rho_1(a_2) = 0, \quad \rho_2(a_2) \geqslant 1.$$

Let k be a natural number larger than all of $2|\rho_j(\alpha_i)|$ $(i = 1, 2, \ldots, r; j = 1, 2)$, and let m be a natural number such that $m\rho_2(a_2) - \rho_1(a_1) > 1$. Put $a = a_1^k a_2^{mk}$.

Let, now, $n_1(a), n_2(a), \ldots, n_r(a)$ be r natural numbers satisfying (1), and consider the sum $\sum a^{n_i(a)}\alpha_i$. Observing (1) and $\rho_1(a) \geqslant k$, we see readily that the ρ_1-value of the sum is simply the ρ_1-value of its first term, i.e.

$$\rho_1(a^{n_1(a)}\alpha_1) = n_1(a)k\rho_1(a_1) + \rho_1(\alpha_1). \tag{3}$$

Similarly the ρ_2-value of the same sum is equal to

$$\rho_2(a^{n_1(a)}\alpha_1) = n_1(a)mk\rho_2(a_2) + \rho_2(\alpha_1). \tag{4}$$

These two numbers (3) and (4) are not equal. For, if they were equal, then

$$m\rho_2(a_2) - \rho_1(a_1) = (\rho_1(\alpha_1) - \rho_2(\alpha_1))/n_1(a)k < 1$$

contrary to our choice of m. Thus our sum

$$\sum a^{n_i(a)}\alpha_i$$

cannot belong to Z. The lemma is thus proved.[2]

Now we can derive our Theorem exactly as in Kaplansky [**4**]. Thus suppose that $A \neq Z$, and let a be an element of A not contained in Z and separable over Z (Theorem of Noether and Jacboson). Let L be the field generated by a over Z. It follows from Lemma 2 that Z must be of characteristic $p \neq 0$ and algebraic over its prime field. But this is a contradiction, by virtue of the first cited theorem of Wedderburn-Jacobson.

Theorem 7 of Hua [**1**] actually states that a non-commutative division ring is generated by the nth powers of its elements, n being an arbitrary natural number. Also Kaplansky [**4**] gives a corresponding modification of his result. by means of a theorem of Cartan-Brauer-Hua [**1**, Theorem 2]. Our theorem too may be combined with the Cartan-Brauer-Hua theorem, to yield

[2]Cf. [**5**], setting $\alpha_1 = -1$, $\alpha_2 = 1$, $\alpha_3 = \ldots = \alpha_r = 0$..

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COROLLARY. *Let A be a non-commutative division ring and Z be its centre. Let r be a natural number, and $\alpha_1, \alpha_2, \ldots, \alpha_r$ be a set of r non-zero elements in Z. Let there be given for each element a in A a set of r natural numbers $n_1(a), n_2(a), \ldots, n_r(a)$ such that $n_1(a) < n_i(a)$ $(i = 2, \ldots, r)$ and*

$$n_i(a) = n_i(c^{-1}ac) \qquad (i = 1, 2, \ldots, r),$$

for every non-zero element c in A. Then A is generated, as a division ring, by the elements

$$\sum a^{n_i(a)}\alpha_i ,$$

where a runs over A.

Added March 28, 1953. After the submission of the present note for publication I obtained access to the papers by Herstein (referred to in footnote 1) and Krasner [**5**] where a valuation-theoretical approach, analogous to ours, is made in similar context. Krasner's theorem is a particular case of our Lemma 2, while the division ring case of Herstein's result is a special case of our Theorem. As to our Lemma 1, a simple proof (which yields in fact a little more) will be given in M. Nagata, T. Nakayama, and T. Tuzuku, *An existence lemma in valuation theory*, to appear in the Nagoya Mathematical Journal.

REFERENCES

1. L. K. Hua, *Some properties of a sfield*, Proc. Nat. Acad. Sci. U.S.A., *35* (1949), 533–537.

2. M. Ikeda, *On a theorem of Kaplansky*, Osaka Math. J., *4* (1952), 235–240.

3. N. Jacobson, *Structure theory of algebraic algebras of bounded degree*, Ann. Math., *46* (1945), 695–707.

4. I. Kaplansky, *A theorem on division rings*, Can. J. Math., *3* (1951), 290–292.

5. M. Krasner, *On the non-existence of certain extensions*, Amer. J. Math., *75* (1953), 112–116.

Nagoya University