Bull. Aust. Math. Soc. **80** (2009), 233–236
doi:10.1017/S0004972709000215

THE *D*-PROPERTY AND THE SORGENFREY LINE

YIN-ZHU GAO ✉ and WEI-XUE SHI

(Received 6 October 2008)

Abstract

We show that for the Sorgenfrey line S, the minimal dense linearly ordered extension of S is a D-space, but not a monotone D-space; the minimal closed linearly ordered extension of S is not a monotone D-space; the monotone D-property is inversely preserved by finite-to-one closed mappings, but cannot be inversely preserved by perfect mappings.

2000 *Mathematics subject classification*: primary 54D20; secondary 54F05, 54G15, 54C25.
Keywords and phrases: Sorgenfrey line, real line, linearly ordered topological space, monotone D -space.

1. Introduction

The notion of D-spaces was introduced by van Douwen and interesting results for the D-property and the Sorgenfrey line were demonstrated in [7].

A neighborhood assignment for a space X is a function φ from X to the topology of X such that $x \in \varphi(x)$ for all $x \in X$. A space X is a D-space if, for each neighborhood assignment φ for the space X, there exists a closed discrete subset F of X satisfying $X = \bigcup\{\varphi(x) \mid x \in F\}$.

A space X is a monotone D-space ([5]) if, for each neighborhood assignment φ for X, we can pick a closed discrete subset $F(\varphi)$ of X with $X = \bigcup\{\varphi(x) \mid x \in F(\varphi)\}$ such that if ψ is a neighborhood assignment for X and $\varphi(x) \subset \psi(x)$ for all x, then $F(\psi) \subset F(\varphi)$. Monotone D-spaces are D-spaces, but the converse is not true (see [5]).

The Sorgenfrey line S (that is, the set of all real numbers topologized by letting all half-open intervals $[a, b)$ be a base) is one of the most important elementary examples in general topology. In [7], it is shown that the Sorgenfrey line S is a D-space. However, the Sorgenfrey line S is not a monotone D-space [5].

The main result of this note is as follows:

(1) the minimal dense linearly ordered extension of the Sorgenfrey line is a D-space, but not monotonically D;

The project is supported by NSFC (No.10571081).

(2) the minimal closed linearly ordered extension of Sorgenfrey line is not a monotone D-space;
(3) the monotone D-property is inversely preserved by finite-to-one closed mappings, but cannot be inversely preserved by perfect mappings.

Throughout the note, spaces are topological spaces and are Hausdorff. Mappings are continuous. We reserve the symbols $\mathbb{R}$ and $\mathbb{Z}$ for the sets of all real numbers and all integers, respectively. For a neighborhood assignment φ for the space X and $F \subset X$, we denote $\bigcup\{\varphi(x) \mid x \in F\}$ by $\varphi(F)$. Undefined terminology and symbols will be found in [2].

2. Main results

Let $\ell(S) = \mathbb{R} \times \{0, -1\}$ be with the linearly ordered topology generated by the lexicographical order $\preceq$ on $\ell(S)$.

Note that the Sorgenfrey line S is homeomorphic to the dense subspace $\mathbb{R} \times \{0\}$ of the space $\ell(S)$. By [4, Theorem 2.1], the space $\ell(S)$ is the minimal dense linearly ordered extension of S.

THEOREM 1. *The minimal dense linearly ordered extension $\ell(S)$ of the Sorgenfrey line S is a D-space.*

PROOF. Note that the subset $\mathbb{R} \times \{0\}$ of $\ell(S)$ with the restricted order $\preceq |_{\mathbb{R}\times\{0\}}$ is a linearly ordered set. By the linearly ordered topological space $\mathbb{R} \times \{0\}$ we mean the subset $\mathbb{R} \times \{0\}$ of $\ell(S)$ with the open interval topology generated by the linear order $\preceq |_{\mathbb{R}\times\{0\}}$. Obviously the linearly ordered topological space $\mathbb{R} \times \{0\}$ is homeomorphic to the real line $\mathbb{R}$ (the set $\mathbb{R}$ with the Euclidean topology).

Let φ' be a neighborhood assignment for $\ell(S)$. We now define a neighborhood assignment φ for the linearly ordered topological space $\mathbb{R} \times \{0\}$ as follows. For any $x \in \mathbb{R}$, take an $s_x \in \mathbb{R}$ such that $x < s_x$ and the open interval $(\langle x, 0\rangle, \langle s_x, 0\rangle) \subset \varphi'(\langle x, 0\rangle)$. We can also take an $a_x \in \mathbb{R}$ such that $a_x < x$ and $(\langle a_x, 0\rangle, \langle x, 0\rangle) \subset \varphi'(\langle x, -1\rangle)$. Define $\varphi(\langle x, 0\rangle) = (a_x, s_x) \times \{0\}$.

Since metrizablity implies the D-property, the real line $\mathbb{R}$ is a D-space. So for φ there exists a closed discrete subset F of the real line $\mathbb{R}$ such that $\varphi(F \times \{0\}) = \mathbb{R} \times \{0\}$.

Put $F' = F \times \{0, -1\}$. Then F' is closed in $\ell(S)$.

In fact, for any $x' = \langle x, i\rangle \in \ell(S) \setminus F'$, since $x \notin F$ and F is closed in the real line $\mathbb{R}$ there exist real numbers a_x and s_x with $a_x < x < s_x$ such that $(a_x, s_x) \cap F = \emptyset$. Then the open neighborhood $I_{x'} = (\langle a_x, 0\rangle, \langle s_x, -1\rangle)$ of x' satisfies $I_{x'} \cap F' = \emptyset$.

To show that F' is discrete, let $x' = \langle x, i\rangle \in F'$. Then $x \in F$ and thus there exists an open interval (c_x, d_x) containing x such that $(c_x, d_x) \cap F = \{x\}$ since F is discrete in the real line $\mathbb{R}$. If $i = 0$, put $U_{x'} = (\langle x, -1\rangle, \langle d_x, -1\rangle)$. If $i = -1$, put $U_{x'} = (\langle c_x, 0\rangle, \langle x, 0\rangle)$. Then the open neighborhood $U_{x'}$ of x' satisfies $U_{x'} \cap F' = \{x'\}$.

Finally, we will show that $\{\varphi'(x') \mid x' \in F'\}$ covers $\ell(S)$. For any $y' = \langle y, i\rangle \in \ell(S) \setminus F'$, since $\varphi(F \times \{0\}) = \mathbb{R} \times \{0\}$ there exists an $x \in F$ such that

https://doi.org/10.1017/S0004972709000215 Published online by Cambridge University Press

$\langle y, 0\rangle \in \varphi(\langle x, 0\rangle)$. Since $y \neq x$, by the definition of φ we have the following: if $x < y$, then $y' \in \varphi'(\langle x, 0\rangle)$; if $y < x$, then $y' \in \varphi'(\langle x, -1\rangle)$. So $\varphi'(F') = \ell(S)$ and thus $\ell(S)$ is a D-space. □

The minimal closed linearly ordered extension S^* of S is defined as follows. Put

$$S^* = \mathbb{R} \times \{k \in \mathbb{Z} \mid k \leq 0\}.$$

Let the linear order $\preceq$ be the lexicographic order on S^*. Equip S^* with the linearly ordered topology generated by the order $\preceq$ on S^* (that is, the topology on S^* is generated by $\{(a, \rightarrow) \mid a \in S^*\} \cup \{(\leftarrow, a) \mid a \in S^*\}$ as a subbase), where $(a, \rightarrow) = \{x \in S^* \mid a \prec x\}$ and $(\leftarrow, a) = \{x \in S^* \mid x \prec a\}$.

The Sorgenfrey line S is homeomorphic to the closed subspace $\mathbb{R} \times \{0\}$ of the linearly ordered topological space S^*. The space S^* is called a closed linearly ordered extension of S (see [3]). By [6, Theorem 9], the space S^* is the minimal closed linearly ordered extension of S.

THEOREM 2. *The following are true:*
(1) *the space S^* is not a monotone D-space;*
(2) *the space $\ell(S)$ is not a monotone D-space.*

PROOF. (1) Assume that S^* is a monotone D-space. Since the monotone D-property is hereditary with respect to closed subspaces (see [5, Theorem 1.7]) and S is homeomorphic to the closed subspace $\mathbb{R} \times \{0\}$ of S^*, S is a monotone D-space. By [5, Theorem 2.4] S is not a monotone D-space, which is a contradiction.

(2) Assume that the space $\ell(S)$ is a monotone D-space. Define a mapping $f : \ell(S) \rightarrow \mathbb{R}$, where $\mathbb{R}$ is the real line, as follows. For each $x' = \langle x, i\rangle \in \ell(S)$, $f(x') = x$. Then f is continuous and closed surjective mapping. In fact, for an open interval (a, b) of the real line $\mathbb{R}$, $f^{-1}((a, b))$ is obviously open in $\ell(S)$, so f is continuous. Let F' be a closed subset of $\ell(S)$ and $x \notin f(F')$. Then $f^{-1}(x) = \{\langle x, 0\rangle, \langle x, -1\rangle\}$ and $f^{-1}(x) \cap F' = \emptyset$. Thus there exist open intervals $U = (\langle a_x, 0\rangle, \langle x, 0\rangle)$ and $V = (\langle x, -1\rangle, \langle b_x, 0\rangle)$ of $\ell(S)$ with $\langle x, -1\rangle \in U$, $U \cap F' = \emptyset$ and $\langle x, 0\rangle \in V$, $V \cap F' = \emptyset$, where $a_x, b_x \in \mathbb{R}$. Thus $x \in (a_x, b_x)$ and $(a_x, b_x) \cap f(F') = \emptyset$. Hence $f(F')$ is closed.

Since the image of a monotone D-space under a continuous closed mapping is monotonically D ([5, Theorem 1.7]), the real line $\mathbb{R}$ is a monotone D-space. Thus the closed subspace $[0, 1]$ of $\mathbb{R}$ is monotonically D, which contradicts the fact that closed unit interval $[0, 1]$ is not monotonically D (see [5, Theorem 2.3]). □

It is shown that the closed image of a D-space is a D-space, and the perfect inverse image of a D-space is a D-space (see [1]), For the monotone D-property, although it is also preserved by closed mappings (see [5]), it cannot be inversely preserved by perfect mappings.

EXAMPLE 3. There exists a perfect mapping f from X onto Y with Y a monotone D-space, but where X not a monotone D-space.

https://doi.org/10.1017/S0004972709000215 Published online by Cambridge University Press

PROOF. Let S_0 be a countable subspace of the Sorgenfrey line S. Put $X = S_0 \times [0, 1]$ and $Y = S_0$, where $[0, 1]$ is the usual unit closed interval. Define $f : X \to Y$ such that, for each $x = \langle s, t\rangle \in X$, $f(x) = s$. Clearly f is perfect. By [5, Theorem 2.4], the countable subspace Y of the Sorgenfrey line S is a monotone D-space. Take an $s \in S_0$. Since the closed subspace $\{s\} \times [0, 1]$ of X is homeomorphic to $[0, 1]$ and $[0, 1]$ is not a monotone D-space (see [5, Theorem 2.3]), X is not monotonically D. □

Recall that a mapping $f : X \to Y$ is called finite-to-one if, for each $y \in Y$, $f^{-1}(y)$ is finite.

THEOREM 4. *Let a closed mapping $f : X \to Y$ be finite-to-one and surjective. If Y is a monotone D-space, then so is X.*

PROOF. Let φ be a neighborhood assignment for X. For each $y \in Y$, put $U_y = \bigcup\{\varphi(x) \mid x \in f^{-1}(y)\}$ and $\varphi'(y) = Y \setminus f(X \setminus U_y)$. Then φ' is a neighborhood assignment for Y. Since Y is a monotone D-space, there exists a closed discrete subset $D_{\varphi'}$ of Y such that $Y = \bigcup\{\varphi'(t) \mid t \in D_{\varphi'}\}$. Then $D_\varphi = \bigcup\{f^{-1}(t) \mid t \in D_{\varphi'}\}$ is a closed discrete subset of X and $X = \bigcup\{\varphi(x) \mid x \in D_\varphi\}$. Hence X is a monotone D-space. □

Acknowledgement

We wish to express our thanks to the referee for helpful suggestions.

References

[1] C. R. Borges and A. C. Wehrly, 'A study of D-spaces', *Topology Proc.* **16** (1991), 7–15.
[2] R. R. Engelking, *General Topology*, revised and completed edition (Heldermann Verlag, Berlin, 1989).
[3] D. J. Lutzer, 'On generalized ordered spaces', *Dissertationes Math.* **89** (1971).
[4] T. Miwa and N. Kemoto, 'Linearly ordered extensions of GO-spaces', *Top. Appl.* **54** (1993), 133–140.
[5] S. G. Popvassilev and J. E. Porter, 'Monotonically D-spaces', *Topology Proc.* **30** (2006), 355–365.
[6] W.-X. Shi, 'Perfect GO-spaces which have a perfect linearly ordered extension', *Top. Appl.* **81** (1997), 23–33.
[7] E. K. van Douwen and W. F. Pfeffer, 'Some properties of the Sorgenfrey line and related spaces', *Pacific J. Math.* **81** (1979), 371–377.

YIN-ZHU GAO, Department of Mathematics, Nanjing University, Nanjing 210093, PR China
e-mail: yzgao@jsmail.com.cn

WEI-XUE SHI, Department of Mathematics, Nanjing University, Nanjing 210093, PR China
e-mail: wxshi@nju.edu.cn