The part within the bracket may be written

$$
\begin{aligned}
&\{1 + (n-1)\}\, p_{11} p_{22} \ldots p_{n-1,\,n-1} \\
&+ \{(n-1) + (n-1)(n-2)\}\, p_{11} \ldots p_{n-2,\,n-2}\, q_{n-1,\,n-1} \\
&+ \left\{\binom{n-1}{2} + (n-1)\binom{n-2}{2}\right\} p_{11} \ldots p_{n-3,\,n-3}\, q_{n-2,\,n-2}\, q_{n-1,\,n-2} \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&+ \{(n-1) + (n-1)\}\, p_{11}\, q_{22} \ldots q_{n-1,\,2} \\
&+ \{1 + 0\}\, q_{11} \ldots q_{n-1,\,1}.
\end{aligned}
$$

Adding vertically we get two expressions, exactly similar to the first term, the second expression being multiplied by $(n-1)\,p_{11}$. Hence the part within the brackets is equal to $1 + (n-1)\,p_{11}$, and so the coefficient of $\alpha^2/2!$ in the moment G.F. is

$$np_{00}\,[1 + (n-1)\,p_{11}].$$

Transferring to the mean as origin by subtracting $(np_{00})^2$, we derive the second moment μ_2 about the mean, or squared standard deviation σ^2, as

$$\sigma^2 = np_{00}\,q_{00} - n\,(n-1)\,p_{00}\,(p_{00} - p_{11}).$$

Thus the mean and standard deviation of this particular distribution are given in simple terms.

Example. For the distribution of one suit in a bridge hand, we have

$$p_{00} = \frac{1}{4}, \quad q_{00} = \frac{3}{4}, \quad n = 13, \quad p_{11} = \frac{12}{51} = \frac{4}{17}.$$

$$
\begin{aligned}
\text{Hence} \quad \mu_2 = \sigma^2 &= \frac{39}{16} - \frac{39}{68} \\
&= \frac{507}{272} = 1{\cdot}86. \\
\text{and so} \quad \sigma &= 1{\cdot}36.
\end{aligned}
$$

A Problem in Combinations

By A. C. Aitken.

1. If there are n individuals $A_1, A_2, \ldots, A_n$, in how many ways can they be put into groups? For example, if there are three individuals A, B, C, they may be grouped as

$A + B + C$; $A+(B + C)$, $B+(C + A)$, $C+(A + B)$; $(A + B + C)$,

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that is, in 5 ways, the respective subgroups, 1, 3, 1 in number, corresponding to the partitions $1+1+1$, $1+2$, 3 of the integer 3. Hence $P(3)$, say, is 5.

We shall obtain various expressions for $P(n)$, and shall place the problem in relation to other questions of analysis.

2. Following MacMahon we shall denote partitions, *e.g.* those of 3 above, by 1^3, $1^1 2^1$, 3^1; in general, if the integer n is made up of α integers a, plus β integers b, and so on, $a < b < \ldots.$, we shall write the corresponding partition as

$$a^\alpha b^\beta \ldots. \tag{1}$$

In the example given above, the partition $1^1 2^1$ leads to three subgroups. The 3 here is $3!/(1!\,2!)$, and in general it is easy to see, by the elementary theory of combinations, that the number of subgroups corresponding to the partition $a^\alpha b^\beta \ldots.$ of n is

$$n!/(a!\,b!\ldots.\,\alpha!\,\beta!\ldots.). \tag{2}$$

Hence one answer to our problem is

$$P(n) = \Sigma\, n!/(a!\,b!\ldots.\,\alpha!\,\beta!\ldots.), \tag{3}$$

the summation being over all partitions of the integer n; but this is not a very helpful expression.

3. We seek therefore a generating function, in which $P(n)$ shall appear as coefficient of x^n, or perhaps of $x^n/n!$.

Consider first groups made up of single units. If there are r units in the partition (1) of n, then by (2) a factor $r!$ will be required in the denominator. Hence unit groups will be represented by a generating function

$$\sum_0^\infty x^r/r!,$$

this is, by e^x.

Next, groups of two. By (2), any 2 requires a $2!$ in the denominator for each time it occurs, and if it occurs r times it requires an $r!$ as well. The generating function for groups of two is therefore

$$e^{x^2/2!}.$$

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In the same way the generating function for groups of r is

$$e^{x^r/r!}.$$

Combining in multiplication all such generating functions, since groups of any size may be associated with groups of any other size, we have the required generating function for $P(n)$, namely

$$e^{x+x^2/2!+x^3/3!+\dots} = e^{e^x-1}, \tag{4}$$

and we derive the interesting fact that $P(n)$ is the coefficient of $x^n/n!$ in this expansion, that is, by Maclaurin's theorem,

$$P(n) = [D^n e^{e^x-1}]_{x=0}, \text{ where } D = \frac{d}{dx}. \tag{5}$$

4. This result is connected with the procedure of repeatedly differentiating a function of a function. For example we have

$$\frac{d}{dx} f(u) = f'(u) \frac{du}{dx},$$

$$\left(\frac{d}{dx}\right)^2 f(u) = f''(u) \left(\frac{du}{dx}\right)^2 + f'(u) \frac{d^2u}{dx^2},$$

$$\left(\frac{d}{dx}\right)^3 f(u) = f'''(u) \left(\frac{du}{dx}\right)^3 + 3f''(u) \frac{d^2u}{dx^2} \cdot \frac{du}{dx} + f'(u) \frac{d^3u}{dx^3}, \tag{6}$$

and so on, and we notice that the coefficients 1, 3, 1 in the third of these relations are the same as the numbers of subgroups in our first example.

In the expression for $\left(\frac{d}{dx}\right)^n f(u)$, let us put $f(u) = e^u$, $u = e^x$. Then the left hand side of the general relation of type (6) becomes

$$\left(\frac{d}{dx}\right)^n e^{e^x},$$

while the right hand side is a sum of terms involving

$$e^{nx} e^{e^x}, \; e^{(n-1)x} e^{e^x}, \; \dots, \; e^x e^{e^x}.$$

Putting $x = 0$, we see that the sum of the numerical coefficients in the expansion of $\left(\frac{d}{dx}\right)^n f(u)$ is

$$\left[e^{-1}\left(\frac{d}{dx}\right)^n e^{e^x}\right]_{x=0} = P(n). \tag{7}$$

5. Another set of relations, involving the operator $x\frac{d}{dx}$ or xD which occurs in the theory of homogeneous differential equations, has the same coefficients as the set (6). For example we have

$$(xD)^2 = x^2D^2 + xD,$$
$$(xD)^3 = x^3D^3 + 3x^2D^2 + xD, \text{ etc.} \tag{8}$$

Indeed, if $u = e^x$, the comparison between relations (6) and (8) becomes exact. For example the expression for $D^4 f(e^x)$ is derived by term by term differentiation from that for $D^3 f(e^x)$ by exactly the same formal operations as the expression for $(xD)^4$ is derived from that for $(xD)^3$; and so in general.

Inserting the operand e^x in the expression for $(xD)^n$ corresponding to (8), and then putting $x = 1$, we derive a new expression for $P(n)$,

$$P(n) = [(xD)^n e^{x-1}]_{x=1}, \tag{9}$$

comparison of which with (5) yields the rather peculiar identity

$$[D^n e^{e^x-1}]_{x=0} = [(\overline{x+1}D)^n e^x]_{x=0}. \tag{10}$$

6. Since

$$e^{e^x-1} = e^{-1}(1 + e^x + e^{2x}/2! + e^{3x}/3! + \dots),$$

and $P(n)$ is the coefficient of $x^n/n!$ in the expansion of this, we derive yet another expression,

$$P(n) = e^{-1}\sum_{s=0}^{\infty}(s^n/s!). \tag{11}$$

7. Next, let us write s^n in terms of factorials s, $s(s-1)$, and so on. To do this, let a table of differences be formed from 0^n, 1^n, $\dots$, n^n, the differences of 0^n being denoted by $\Delta^r 0^n$. By the Gregory-Newton interpolation formula we have

$$s^n = 0^n + s\,\Delta 0^n + \frac{s(s-1)}{2!}\Delta^2 0^n + \dots + \binom{s}{n}\Delta^n 0^n.$$

Substituting this in (11) for $s = 0, 1, 2, \dots$, we obtain

$$P(n) = e^{-1}\sum_{s=0}^{\infty}\Big[\sum_{r=0}^{n}\Delta^r 0^n/(\overline{s-r}!\,r!)\Big]. \tag{12}$$

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On summation of expressions like $1/(s-r)!$ we obtain e in each case, and so (12) gives

$$P(n) = e^{-1} \sum_{r=1}^{n} e\, \Delta^r 0^n/r!$$
$$= \sum_{r=1}^{n} \Delta^r 0^n/r!, \qquad (13)$$

which exhibits $P(n)$ as the sum of the "divided differences" of 0^n. As an equivalent for (11) this was given by Herschel.

7. The numbers of *subgroups* also crop up in these divided differences of zero. For example, if $n = 3$, the table of *divided* differences is

$$\begin{array}{cccc} 0 & & & \\ & 1 & & \\ 1 & & 3 & \\ & 7 & & 1 \\ 8 & & 6 & \\ & 19 & & \\ 27 & & & \end{array}$$

the 1, 3, 1 for this case appearing again. The theorem indicated here is a general one. To prove it we may operate on x^n with $(xD)^n$ and its equivalent in (8), the result being an interpolation formula for n^n in terms of 0^n and the differences of 0^n.

8. One of the easiest ways of finding the first dozen or so numerical values of $P(n)$ is by means of the recurrence relation which $P(n)$ satisfies. This relation is

$$P(n+1) = P(n) + nP(n-1) + \binom{n}{2} P(n-2) + \ldots + nP(1) + P(0) \quad (14)$$
$$= (P+1)^n \qquad (15)$$

symbolically if, after expansion, exponents of P are written as arguments. To prove this, we write

$$P_{n+1} = e^{-1} D^{n+1}(e^{e^x}) = e^{-1} D^n [e^x \,.\, e^{e^x}], \qquad x = 0.$$

Expanding the derivative of the product by Leibniz's theorem and then putting $x = 0$, we have the result (14) at once.

Now the right side of (14) is in shape simply a Gregory-Newton interpolation formula. Hence, since $P(0) = P(1) = 1$, we see that if we construct a difference table from $P(1)$, $P(2)$, $P(3)$, ..., then the values of $P(1)$, $\Delta P(1)$, $\Delta^2 P(1)$, ... thereby given are simply $P(0)$, $P(1)$, $P(2)$, ..., and so on. This gives perhaps the easiest way of all for finding the first several values of $P(n)$, namely

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to build up the table, entering each $P(r)$, when found, as a fresh difference $\Delta^r P(1)$ with which to begin a new line of differences. For example we have, for the first few values,

P	Δ	Δ^2	Δ^3	Δ^4	Δ^5
1					
	1				
2		2			
	3		5		
5		7		15	
	10		20		52,
15		27		67	
	37		87		
52		114			
	151				
203					

which puts in evidence the property mentioned.

9. The first ten values of $P(n)$ are

n	1	2	3	4	5	6	7	8	9	10
$P(n)$	1	2	5	15	52	203	877	4140	21147	115975.

Inspection shows that if n is a prime number > 1, p say, then $P(p) - 2$ is divisible by p. For example $877 - 2$ is divisible by 7.

This is a result easily proved. For $\Delta\, 0^p = 1$, and $\Delta^p\, 0^p = p\,!$, so that the p^{th} *divided* difference of 0^p is also 1. As for the differences of 0^p of order r, where $1 < r < p$, it is an instant deduction from Fermat's theorem that

$$\begin{aligned} \Delta^r\, 0^p &\equiv \Delta^r\, 0^1 \qquad \operatorname{mod} p, \\ &= 0, \qquad 1 < r < p. \end{aligned}$$

To obtain the *divided* differences, which must be integers, we divide the ordinary differences $\Delta^r\, 0^p$ by $r\,!$, which does not contain p, since p is a prime greater than r. Hence the divided differences for $1 < r < p$ are also divisible by p. On summing these divided differences in (13), we obtain for $P(p)$ a multiple of p, plus 1 from each end term. Hence, as stated, $P(p) - 2$ is divisible by p.

10. After these various diversions, it would have been pleasing to find an *asymptotic* expression to represent $P(n)$ for large values of n, but this has not so far materialized. The function $n^{\frac{1}{2}n}$ gives a fair representation for small values, up to $n = 8$; for example $P(8) = 4140$, while $8^4 = 4096$. For higher values $P(n)$ increases more rapidly; for example $P(10) = 115975$, while $10^5 = 100000$.