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Universally Overconvergent Power Series via the Riemann Zeta-function

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Abstract. The Riemann zeta-function is employed to generate universally overconvergent power series.

1 Introduction

Universal overconvergence is a generic property of power series on a given disc, but it does not seem easy to find an explicit example of a universally overconvergent power series. The present paper addresses this issue.

For an introduction to universality in general, we recommend the excellent survey by Karl-Goswin Grosse-Erdmann [5]. The two principal types of universal holomorphic functions are functions universal with respect to translations and universally overconvergent power series. The spectacular universality theorem of Sergei Mikhailovich Voronin asserts that the Riemann zeta-function is universal with respect to translations (see Section 2). In this note we shall show that the Riemann zeta-function can be employed to fashion universally overconvergent power series as well.

Every power series $\sum a_n(z-a)^n$ diverges at each point z in the exterior of its disc of convergence. That is, the sequence of partial sums diverges at z. However, in some cases, a subsequence of the sequence of partial sums might converge at z. In this case, the power series is said to be overconvergent at the exterior point z. For $a \in \mathbb{C}$ and $0 \le r < \infty$, let $\overline{D}(a,r)$ be the closed disc $\{z : |z-a| \le r\}$. For $r > 0$, let $\mathcal{F}[a,r]$ be the family of all compact sets K in the complement of the open disc $D(a,r)$ having connected complements and, for $r = 0$, let $\mathcal{F}[a,0]$ be the family of all compact sets K in $\mathbb{C}\setminus\{a\}$ having connected complements. We denote by $A(K)$ the family of functions continuous on K and holomorphic on the interior K^o of K. The following theorem, originally due to Wolfgang Luh [6] and independently to Charles Chui and Milton Parnes [3], was refined by Vassili Nestoridis [7].

Theorem 1.1 ([3, 6, 7]) *Fix $a \in \mathbb{C}$ and $0 \le r < +\infty$. There is a power series $\sum_{n=0}^{\infty} a_n(z-a)^n$, with radius of convergence r, which is universally overconvergent, in the sense that, for every set K in $\mathcal{F}[a,r]$ and every function $f \in A(K)$, there is a sequence of positive integers $\{n_k\}$ such that*

$$\lim_{k\to\infty} \max_{z\in K} \left| \sum_{n=0}^{n_k} a_n(z-a)^n - f(z) \right| \longrightarrow 0,$$

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Definition 1.2 We shall say that a power series satisfying the conclusion of Theorem 1.1 is universally overconvergent in the disc $D(a, r)$.

The original theorem of Luh, Chui, and Parnes was for compact sets K outside of the *closed* disc $\overline{D}(a, r)$. Grosse-Erdmann [4] showed that this property is in fact generic, in the sense that, for $r > 0$, most holomorphic functions in the disc $D(a, r)$ are universal in this sense. Nestoridis [7] improved this theorem by showing that if $r > 0$, the compact sets may be allowed to touch the boundary of the disc.

Although universal overconvergence is generic, I know of no explicit power series that is universally overconvergent. The following two theorems exhibit this sort of universal overconvergence for power series whose coefficients are generated using the Riemann zeta-function in the right half $1/2 < \Re z < 1$ of the fundamental strip.

Theorem 1.3 *Fix $a \in \mathbb{C}$, $0 \le r < +\infty$, and $1/2 < \sigma < 1$. There is a sequence $\{m_n\}$ of integers, such that the power series*

$$\sum_{n=0}^{\infty} \zeta(\sigma + i m_n)(z - a)^n$$

is universally overconvergent in $D(a, r)$.

Theorem 1.4 *Fix $a \in \mathbb{C}$, $0 \le r < +\infty$ and $1/2 < \sigma < 1$. There is a sequence $\{m_n\}$ of integers, such that the power series*

$$\sum_{n=0}^{\infty} \zeta^{(n)}(\sigma + i m_n)(z - a)^n$$

is universally overconvergent in $D(a, r)$.

The existence of a universally overconvergent power series was first established by Selesnev [10], who proved Theorem 1.1 for the case $r = 0$. The existence of universally overconvergent power series in several complex variables was established only recently by my Master's student, Raphaël Clouâtre [2]. Theorem 1.4, for the case $r = 0$, was first obtained by my undergraduate student, Antoine Poirier [8], who wrote the universally overconvergent series in the form

$$\sum_{n=0}^{\infty} \frac{\zeta^{(n)}(\sigma + i\tau_n)}{n!}(z - a)^n$$

and called it a dispersed Taylor series. His sequence $\{\tau_n\}$ consisted of real numbers, not necessarily integers. He also obtained an analog in several complex variables.

In the following section, we present universality theorems with respect to translation, which we shall employ in the final section to prove Theorems 1.3 and 1.4 on universality with respect to overconvergence.

2 Bohr's Universality Theorem and Generalizations by Voronin

The first universality theorem for the Riemann zeta-function was obtained by Harold Bohr in 1915.

Theorem 2.1 (Bohr [1]) *For each* $1/2 < \sigma < 1$, *the curve*

$$\left\{ \zeta(\sigma + it) : -\infty < t < +\infty \right\}$$

is dense in $\mathbb{C}$.

In 1975, Voronin proved a spectacular generalization of Bohr's theorem, known as Voronin's universality theorem. For a compact subset K of the complex plane $\mathbb{C}$, we denote by $A(K)$ the family of functions continuous on K and holomorphic on the interior K^o of K. Let $A_o(K)$ denote the subfamily consisting of *zero-free* functions in $A(K)$. Voronin's theorem was refined successively by Bhaskar Bagchi, Steve M. Gonek, and A. Reich. The following extension of Voronin's universality theorem was obtained by Reich in 1980.

Theorem 2.2 (Reich [9]) *For every compact set K with connected complement lying in the right half* $1/2 < \Re z < 1$ *of the fundamental strip, the sequence* $\{\zeta(z + in)\}$ *of vertical integral translates of the Riemann zeta-function is dense in* $A_o(K)$. *That is, for every function* $f \in A_o(K)$, *there is a sequence* $\{m_n\}$ *of integers, such that*

$$\max_{z \in K} \left| \zeta(z + im_n) - f(z) \right| \longrightarrow 0$$

as $n \to \infty$.

We are unable to prove Theorems 1.3 and 1.4 employing Theorem 2.2, because of the restriction in the hypotheses that f be zero-free. However we may invoke a less powerful but very interesting precursor of Voronin's universality theorem, obtained earlier by Voronin himself in 1972, which generalizes Bohr's Theorem 2.1.

Theorem 2.3 (Voronin [11]) *For each* z_o *in the strip* $1/2 < \sigma < 1$ *and each* $k = 0, 1, 2, \ldots$, *the sequence*

$$\left\{ \left(\zeta(z_o + im), \zeta'(z_o + im), \ldots, \zeta^{(k)}(z_o + im) \right) : m = 1, 2, \ldots \right\}$$

is dense in $\mathbb{C}^{k+1}$.

3 Proof of Theorems 1.3 and 1.4

Our proof is based on the proof of Theorem 1.1. For the convenience of the reader, with the exception of the first lemma, which is due to Nestoridis [7], we shall refer only to the proof in [3] which is more easily accessible than [6]. We shall show that by making several modifications (some of which are not obvious) in the proof in [3] and invoking the lemma of Nestoridis, we obtain Theorem 1.3 without recourse to Baire category.

Lemma 3.1 ([7]) *There exists a countable family $\mathcal{L}$ in $\mathcal{F}[a,r]$ such that each $K \in \mathcal{F}[a,r]$ lies in some $L \in \mathcal{L}$.*

Lemma 3.2 *Fix $a \in \mathbb{C}$, $0 \le r < +\infty$, and $\rho > 1$. Let $K \in \mathcal{F}[a,r]$. Then every $f \in A(K)$ can be uniformly approximated on K by polynomials $p_n(z) = \sum_{j=0}^{n} a_{nj}(z-a)^j$ whose coefficients are bounded by $(\rho/r)^j$.*

For $r = 0$, this is interpreted as meaning that no restriction is imposed on the coefficients.

Proof Let $w = (\rho/r)(z-a)$, $Q = \{w : z \in K\}$, and $g(w) = f(z)$. Then Q is a compact set outside the closed disc $|w| \le 1$ and $g \in A(Q)$. By [3, Lemma 1], the function g can be uniformly approximated by polynomials $q_n(z) = b_{n0} + \cdots + b_{nn}z^n$, whose coefficients are bounded by one. Set $p_n(z) = q_n(w)$. Then

$$p_n(z) = \sum_{j=0}^{n} b_{nj}(\rho/r)^j(z-a)^j = \sum_{j=0}^{n} a_{nj}(z-a)^j,$$

where $|a_{nj}| \le (\rho/r)^j$ for $j = 0, \ldots, n$. ■

Lemma 3.3 *Fix $a \in \mathbb{C}$, $0 \le r < +\infty$, and $\rho > 1$. Let $K \in \mathcal{F}[a,r]$, $f \in A(K)$, and k be a non-negative integer. Then there is a polynomial*

$$p_k(z) = b_k(z-a)^k + b_{k+1}(z-a)^{k+1} + \cdots + b_m(z-a)^m,$$

such that

$$\max_{z \in K} |p_n(z) - f(z)| \le \frac{1}{2^{k+1}},$$

where $m \ge k$ and $|b_\ell| \le (\rho/r)^\ell$ for $\ell = k, \ldots, m$.

Again, for $r = 0$, this is interpreted as meaning that no restriction is imposed on the coefficients.

Proof This follows from [3, Lemma 2] by the same change of variables as in the proof of the previous lemma. ■

In Lemma 3.3, the polynomial p_k depends on f, *i.e.*, for a different f we could have a different p_k. This may seem rather far from universality, which is the key aim of this manuscript. However, Lemma 3.3 allows us to prove Lemma 3.4, which will be used later to prove universality results.

Let $\mathcal{G}$ be the set of all polynomials with rational complex coefficients and $\mathcal{L}$ the family from Lemma 3.1. Let $\mathcal{P} = \{(g_n, L_n)\}$ be a countable listing of the pairs in $\mathcal{G} \times \mathcal{L}$ such that each pair (g, L) appears infinitely often in the list.

Lemma 3.4 *Fix σ with $1/2 < \sigma < 1$ and for $n = 1, 2, 3, \cdots$, choose $\rho_n > 1$. There exists a sequence of integers $\{m_\ell\}$ and an increasing sequence of integers $\{k_n\}$ with $|\zeta(\sigma + im_\ell)| \le (\rho_n^2/r)^\ell$ for $\ell = k_{n-1}+1, \cdots, k_n$ such that the polynomials*

$$s_n(z) = \sum_{\ell=k_{n-1}+1}^{k_n} \zeta(\sigma + im_l)(z-a)^\ell$$

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have the approximation property

$$\max_{z\in L_n}|s_n(z)-(g_n(z)-s_0(z)-\cdots-s_{n-1}(z))|\le\frac{2}{n},$$

where $s_0=0$.

Proof By Lemma 3.3, there is a polynomial $p_1(z)=a_{10}+a_{11}(z-a)+\cdots+a_{1k_1}(z-a)^{k_1}$, with $|a_{1\ell}|\le(\rho_1/r)^\ell$ for $\ell=0,\dots,k_1$, such that

$$\max_{z\in L_1}\Big|\,p_1(z)-\big(g_1(z)-s_0(z)\big)\,\Big|\le\frac{1}{2}.$$

By Theorem 2.3 with $k=0$, we can approximate the coefficients $a_{1\ell}$ by the zeta-function, so there are integers $m_0,m_1,\dots,m_{k_1}$, with $|\zeta(\sigma+im_\ell)|\le(\rho_1^2/r)^\ell$ for $\ell=0,\dots,k_1$ such that the polynomial

$$s_1(z)=\sum_{\ell=0}^{k_1}\zeta(\sigma+im_l)(z-a)^\ell$$

has the approximation property

$$\max_{z\in L_1}\Big|\,s_1(z)-\big(g_1(z)-s_0(z)\big)\,\Big|\le 1.$$

Suppose for $j=1,\dots,n-1$, we have found integers $-1=k_0<\cdots<k_{n-1}$ and integers $m_l, l=0,\dots,k_{n-1}$ with $|\zeta(\sigma+im_\ell)|\le(\rho_j^2/r)^\ell$ for $\ell=k_{j-1}+1,\dots,k_j$ such that the polynomials

$$s_j(z)=\sum_{\ell=k_{j-1}+1}^{k_j}\zeta(\sigma+im_l)(z-a)^\ell$$

have the approximation property

$$\max_{z\in L_j}\Big|\,s_j(z)-\big(g_j(z)-s_0(z)-\cdots-s_{j-1}(z)\big)\,\Big|\le\frac{2}{j}.$$

By Lemma 3.3, there is a polynomial

$$p_n(z)=\sum_{\ell=k_{n-1}+1}^{k_n}a_{n\ell}(z-a)^\ell,$$

with $|a_{n\ell}|\le(\rho_n/r)^\ell$ for $\ell=k_{n-1}+1,\dots,k_n$ such that

$$\max_{z\in L_n}\Big|\,p_n(z)-\big(g_n(z)-s_0(z)-\cdots-s_{n-1}(z)\big)\,\Big|\le\frac{1}{2^{k_{n-1}+1}}\le\frac{1}{n}.$$

By Theorem 2.3 with $k = 0$, we can approximate the coefficients $a_{n\ell}$ by the zeta-function, so there are integers $m_{k_{n-1}}, \dots, m_{k_n}$, with $|\zeta(\sigma + im_\ell)| \le (\rho_n^2/r)^\ell$ for $\ell = m_{k_{n-1}}, \dots, m_{k_n}$, such that the polynomial

$$s_n(z) = \sum_{\ell=k_{n-1}+1}^{k_n} \zeta(\sigma + im_l)(z-a)^\ell$$

has the approximation property

$$\max_{z\in L_n} \Big| s_n(z) - \big(g_n(z) - s_0(z) - \cdots - s_{n-1}(z)\big) \Big| \le \frac{2}{n}. \qquad \blacksquare$$

At last, we are ready to prove Theorems 1.3 and 1.4.

Proof Choose the sequence $\{\rho_n\}$ to be decreasing to 1.

Fix K in $\mathcal{F}[a, r]$, a function $f \in A(K)$ and $\epsilon > 0$. By Mergelyan's theorem, there is a polynomial $p(z)$ such that $\max_{z\in K} |p(z) - f(z)| < \epsilon$. Let $\{s_n\}$ be a sequence of polynomials satisfying the conclusion of Lemma 3.4. For each n, the k_n-th partial sum of the power series

$$\sum_{l=0}^{\infty} \zeta(\sigma + im_l)(z-a)^l \tag{3.1}$$

approximates the polynomial g_n within $2/n$ on L_n. For infinitely many n, the set K is contained in L_n and $|g_n - p| < \epsilon$ on L_n. Thus, for infinitely many n we have

$$\begin{aligned}
&\max_{z\in K}\Big| f(z) - \sum_{\ell=0}^{k_n} \zeta(\sigma + im_l)(z-a)^l \Big| \\
&\quad\le \max_{z\in K}\big| f(z) - p(z)\big| + \max_{z\in L_n}\big| p(z) - g_n(z)\big| \\
&\qquad + \max_{z\in L_n}\Big| s_n(z) - \big(g_n(z) - s_0(z) - \cdots - s_{n-1}(z)\big) \Big| \\
&\quad\le \epsilon + \epsilon + \frac{2}{n}.
\end{aligned}$$

Thus, the series (3.1) has the required approximation property.

Since $\{\rho_n\}$ is decreasing, it follows from the coefficient estimates that, for each n, the radius of convergence of $S(z)$ is at least r/ρ_n, and since $\rho_n \searrow 1$, the radius of convergence is at least r. Since at every point z outside the disc $\overline{D}(a, r)$, different subsequences of partial sums converge to different values, the radius of convergence is at most r. Thus, the radius of convergence is precisely r.

We have shown that the series (3.1) satisfies Theorem 1.3.

Theorem 1.4 follows in the same way, invoking Theorem 2.3 with $k = k_n$. $\blacksquare$

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